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SIEGEL MODULAR FORMS AND QUATERNION ALGEBRAS

(On a construction of H. Yoshida)

by

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In two interesting papers [Y 1, Y 2] H. Yoshida constructed a lifting from pairs of automorphic forms on a quaternion algebra to Siegel modular forms of degree two. However the non-vanishing of his construction was proved only in a weak form [Y 2, Theorem 6.7].

In this paper we describe two approaches to Conjecture 7.6 of [Y 1] (= Conjecture B of [Y 2]).

Our first appoach was arithmetical in nature : We express (a certain average of) the Fourier coefficients of our Siegel modular form in terms of the Fourier coefficients of two modular forms of weight $\frac{3}{2}$. In this way we do not get a definite result concerning the non-vanishing, but we get some insight into the arithmetic of our Siegel modular forms.

The second approach uses properties of automorphic L-functions and leads to a full proof of Yoshida's conjecture for weight 2. We shall describe - without any technical details - both approaches and some applications. This exposition does not reflect the chronological order of our research, e.g. the "first" approach appears in the last chapter !

For details we refer to [Bö - Sp 2]; some of our results were announced in [Bö - Sp 1].

Chapter I : Yoshida's lift and some problems related to it

Let $N = q_1 \dots q_t$ be a square-free number (fixed throughout) and let D be the quaternion algebra over $\mathbb{Q}$ ramified in $\infty, q_1, \dots, q_t$. We denote by R some maximal order of D and by R_p its localization. For the adelization $D_A^\times$ of $D^\times$ we have a double coset decomposition

$$D_A^\times = \bigcup_{i=1}^{H} D^\times y_i K$$

where H is the class number of D, K is defined as $\prod_{p,\infty} R_p^\times$ and we assume that the y_i have reduced norm 1.

We define 4-dimensional lattices L_{ij} in the $\mathbb{Q}$-vector space D by $L_{ij} := D \cap y_i \left(\prod_p R_p \right) y_j^{-1}$; these L_{ij} correspond to integral quadratic forms Q_{ij} which we identify with half-integral positive definite symmetric matrices of size 4 with $\det(2 Q_{ij}) = N^2$. We consider theta series of type

$$\theta_{ij}^n(Z) = \sum_{X \in \mathbb{Z}^{(4,n)}} e^{2\pi i \,\mathrm{trace}(X^t Q_{ij} X Z)}$$

with $Z \in \mathbb{H}_n$ (= Siegel's upper half space of degree n).

We shall denote by Θ^n the $\mathbb{C}$-vector space generated by all the θ_{ij}^n ; this is known to be a subspace of $\mathcal{M}^n$, by which we mean the space of all Siegel modular forms of degree n and weight 2 with respect to $\Gamma_0^n(N)$. By Θ_{cusp}^n and $\mathcal{M}_{cusp}^n$ we mean the corresponding subspaces of cusp forms.

Now let $\mathcal{A}$ be the the space of right K-invariant automorphic forms for D, that is the space of all functions $\varphi : D_A^\times \longrightarrow \mathbb{C}$ satisfying $\varphi(\gamma g k) = \varphi(g)$ for all $\gamma \in D^\times$, $g \in D_A^\times$, $k \in K$.

Yoshida's construction can now be described easily (we do it for degree n instead of degree 2) :

For any $n \geq 1$ we define

$$Y^n : \begin{cases} \mathcal{A} \times \mathcal{A} \longrightarrow \Theta^n \subset \mathcal{M}^n \\ (\varphi, \psi) \longmapsto \sum_{i,j=1}^{H} \frac{1}{e_i e_j} \varphi(y_i) \psi(y_j) \, \theta^n_{ij} \end{cases}$$

Here e_i denotes the order of $R_i^\times = L_{ii}^\times$.

The main problem is to study the (non-) vanishing properties of those mappings Y^n. Yoshida mentions two obstructions to non-vanishing :

1. Obstruction : If φ and ψ are eigenforms which are not proportional to each other, then $Y^1(\varphi, \psi) = 0$.

To describe the second obstruction, we recall that each $q|N$ gives rise to an involution on $\mathcal{A}$. For any map $\varepsilon : \{q_1, \ldots, q_t\} \to \{\pm 1\}$ let $\mathcal{A}^\varepsilon$ be the corresponding eigenspace for these involutions.

2. Obstruction : For $\varphi \in \mathcal{A}^\varepsilon$, $\psi \in \mathcal{A}^{\tilde{\varepsilon}}$ with $\varepsilon \neq \tilde{\varepsilon}$ we have $Y^n(\varphi, \psi) = 0$ for all n.

Roughly speaking Yoshida's conjecture says that these two obstructions are the only obstructions to non-vanishing. We shall see below that this is almost true, we shall however discover a third (more subtle) obstruction.

For later purposes it is helpful to divide the vanishing problem into three different problems.

(A) "Stable non-vanishing" : Is there any $n \geq 1$ with $Y^n(\varphi, \psi) \neq 0$?

(B) Which is the smallest n with $Y^n(\varphi, \psi) \neq 0$?

(C) Let n_o be the smallest n with $Y^n(\varphi, \psi) \neq 0$; can we describe $Y^n(\varphi, \psi)$ for $n > n_o$ by some kind of (Klingen type) Eisenstein series attached to $Y^{n_o}(\varphi, \psi)$?

Concerning Ⓑ and Ⓒ we should mention here (and we shall use this tacitly in the sequel) that $F \in \Theta^n$ is a cusp form iff $\Phi F = 0$, where Φ is the Siegel Φ - operator; so we do not have to care about "several cusps".

There are some more problems related to the Yoshida - lift :

Ⓓ "scalar product formulas"

Ⓔ Relations to modular forms of weight $\frac{3}{2}$

Ⓕ Yoshida has shown that $Y^2(1, \psi)$ satisfies the Maaß - relations. We may ask more generally whether the Fourier coefficients of $Y^n(\varphi, \psi)$ have some special properties.

In chapter II we shall describe our proof of Yoshida's conjecture. Chapters III and IV will deal with Ⓔ and Ⓕ (respectively).

Chapter II : Non-vanishing properties of Y^n and applications (The method of L-functions)

To prove the conjecture of Yoshida, we make extensive use of properties of automorphic (standard-) L - functions. This should not be surprising because the relevance of these L-functions for problems related to theta series is now well known (e.g. [Bö_1], [Bö_4], [Gr], [We]). Our proof of Yoshida's conjecture has essentially three ingredients :

-- Solution of problem A ("stable non-vanishing")

-- A characterization of Θ^3_{cusp} inside Θ^3 in terms of automorphic L - functions

-- A theorem of A. Ogg

The first ingredient is the easiest one :

<u>Proposition</u> : For $0 \neq \varphi \in \mathcal{A}^{\varepsilon}$, $0 \neq \psi \in \mathcal{A}^{\tilde{\varepsilon}}$ we have

$$Y^n(\varphi,\psi) = 0 \text{ for all } n \iff Y^3(\varphi,\psi) = 0 \iff \varepsilon \neq \tilde{\varepsilon}$$

The first equivalence follows from a result of Kitaoka [Kit] on the linear independance of theta series. To prove the second assertion, one has to understand precisely under which conditions two lattices L_{ij} and $L_{i'j'}$ are isometric.

Let $\mathcal{H}^n_N$ be the "N-integral" Heckealgebra (spanned by double cosets $\Gamma^n_o(N)\begin{pmatrix} M^{-1} & 0 \\ 0 & M^t\end{pmatrix}\Gamma^n_o(N)$ with M integral, det(M) coprime to N). It is known that $\mathcal{M}^n$ has a basis consisting of eigenforms of $\mathcal{H}^n_N$; to such an eigenform F we attach the standard L-function

$$D^N(F,s) = \prod_{p \nmid N} \frac{1}{1-p^{-s}} \prod_{i=1}^{n} \frac{1}{(1-\alpha_{ip}\, p^{-s})(1-\alpha_{ip}^{-1}\, p^{-s})}$$

where the α_{ip} are the Satake - parameters of F.

Our second ingredient is the crucial

<u>Theorem</u> : Let $0 \neq F \in \Theta^3$ be an eigenform of $\mathcal{H}^3_N$ with $\Phi^3 F=0$; then

$$F \in \Theta^3_{cusp} \iff \operatorname*{ord}_{s=1} D^N(F,s) \geq t$$

Indication of proof:

" $\Longrightarrow$" : We use an integral representation for $D^N(F,s)$ which involves a (pullback of a) degree 6 Eisenstein series. Then the claim follows from the results of Feit [Fe] on the poles of such Eisenstein series and by a careful analysis of the "bad primes" (for this analysis we need that F is in Θ^3).

"$\Longleftarrow$" : Let us assume that F is not a cusp form. The case $\Phi^2 F \neq 0$ reduces everything to elliptic cusp forms - this is easy. So we suppose that ΦF is cuspidal; for all $G \in \Theta^2_{cusp}$

which are eigenfunctions of $\mathcal{H}_N^2$ we can prove an identity (analogous to the one in [Bö4] for level 1) *)

$$\sum_{i,j} \frac{\langle G , \theta^2_{ij} \rangle}{e_i e_j} \theta^2_{ij} = c \operatorname*{Res}_{s=1}(D^N(G,s) G \qquad (*)$$

This shows that $D^N(F,s)$ has a pole in s=1 and therefore

$$D^N(F,s) = \zeta^N(s-1)\, \zeta^N(s+1)\, D^N(F,s)$$

cannot be of order $\geq t$ in s=1 (Here $\zeta^N(s)$ denotes the N-restricted Riemann zeta function).

Now let $0 \neq \varphi \in \mathcal{A}^{\varepsilon}$, $0 \neq \psi \in \mathcal{A}^{\varepsilon}$ be eigenfunctions of the Hecke algebra. To make the theorem above applicable to our problem, we should first determine the standard L - function of $F := Y^3(\varphi,\psi)$ in terms of data attached to φ, ψ ; by the results obtained so far it is clear that F is non-zero !

Let $\hat{\varphi}$, $\hat{\psi}$ be elliptic modular forms of weight two corresponding (via Eichler-Shimizu-Jacquet-Langlands) to φ,ψ and let a(p),b(p) be their eigenvalues for the usual Hecke operator T(p), $p \nmid N$.

We define $\alpha(p)$, $\bar{\alpha}(p)$, $\beta(p)$, $\bar{\beta}(p)$ by

$$\alpha(p) + \overline{\alpha(p)} = a(p), \quad \alpha(p)\cdot\overline{\alpha(p)} = p$$

$$\beta(p) + \overline{\beta(p)} = b(p) \quad \beta(p)\cdot\overline{\beta(p)} = p$$

By some local computations we get

$$D^N(F,s) = \zeta^N(s)\, \zeta^N(s-1)\, \zeta^N(s+1)\, L^N_{sym}(\tilde{\varphi},\tilde{\psi},s+1)$$

with $L^N_{sym}(\tilde{\varphi},\tilde{\psi},s) =$

$$\prod_{p\nmid N} \frac{1}{(1-\alpha(p)\beta(p)p^{-s})(1-\alpha(p)\overline{\beta(p)}p^{-s})(1-\overline{\alpha(p)}\beta(p)p^{-s})(1-\overline{\alpha(p)}\,\overline{\beta(p)}p^{-s})}$$

*) We shall use the symbol "c" several times in the sequel to to indicate constants $\neq 0$; of course these constants do not coincide in general. $\langle\ ,\ \rangle$ is the Petersson scalar product.

If φ is not cuspidal - this means that φ is just a constant - we have

$$L^N_{sym}(\text{const},\psi,s) = L^N(\tilde{\psi},s)\; L^N(\tilde{\psi},s-1)$$

where $L^N(\tilde{\psi},s)$ is just the ordinary L-series, defined by

$$L^N(\tilde{\psi},s) = \prod_{p \nmid N} \frac{1}{(1-\beta(p)p^{-s})(1-\bar{\beta}(p)p^{-s})} \;.$$

If we summarize all these informations, we see that we have indeed obtained a third obstruction for non-vanishing, now in terms of an analytic condition on an automorphic L-function :

For φ, ψ as above - but not both constant - we have $Y^3(\varphi,\psi)$ cuspidal iff $Y^2(\varphi,\psi) = 0$ iff $L^N_{sym}(\tilde{\varphi},\tilde{\psi},2) = 0$.

Surprisingly the latter condition is possible only in very few cases :

a) If φ and ψ are proportional to each other we may apply a classical result of Rankin, which says that $L^N_{sym}(\tilde{\varphi},\tilde{\psi},s)$ has a first order pole in s=2 with residue beeing essentially equal to the Petersson scalar product $\langle\varphi,\psi\rangle$.

b) If φ and ψ are both non-constant and not proportional to each other, we can apply a theorem of Ogg [O], which says that $L^N_{sym}(\tilde{\varphi},\tilde{\psi},2) \neq 0$ - in other words $Y^2(\varphi,\psi) \neq 0$.

c) It remains the case where precisely one of the automorphic forms - let us say φ - is constant. Since $L^N(\psi,2)$ is different from zero (convergent Euler product !) we get

$$Y^3(\varphi,\psi) \text{ cuspidal iff } L^N(\tilde{\psi},1) = 0$$

Since the case φ, ψ both constant is somewhat trivial (it just produces a Siegel Eisenstein series) we omit it from the formulation of the

Final result : If $0 \neq \varphi$, $0 \neq \psi$ are eigenfunctions in $\mathcal{A}^{\varepsilon}$ with ψ not constant, then

a) $Y^3(\varphi,\psi) \neq 0$

b) $Y^2(\varphi,\psi) = 0$ iff φ = const and $L^N(\tilde{\psi},1) = 0$

c) $Y^1(\varphi,\psi) = 0$ unless φ and ψ are proportional

Complementary remarks

1) Linear independence of theta series. According to a conjecture of Andrianov [An] and Yoshida [Y 2] on the linear independence of theta series we should have $\Theta^3_{cusp} = \{0\}$. Our results show that (via the map $\psi \mapsto Y^3(1,\psi)$) Θ^3_{cusp} is isomorphic to $\mathcal{A}^0$:= linear span of those eigenforms ψ in $\mathcal{A}^{\varepsilon_0}_{cusp}$ with

$$L^N(\tilde{\psi},1) = 0,$$

where ε_0 is the constant map $\varepsilon_0 : \{q_1, \dots, q_t\} \longrightarrow \{1\}$.
In general, $\mathcal{A}^0 \neq \{0\}$, as the example $N = q = 389$ shows (see [SP], [Gr] and [Ha]). Anyway, the space $\mathcal{A}^0$ describes precisely up to which amount the conjecture of Andrianov-Yoshida is (not) true!

2) Scalar products. Take $\varphi, \psi \in \mathcal{A}^{\varepsilon}$ with $0 \neq Y^2(\varphi,\psi)$ cuspidal. Then we know two representations of $Y^2(\varphi,\psi)$ as linear combinations of theta series - one involving the values of φ and ψ, the other one involving the scalar products $\langle F, \Theta^2_{ij} \rangle$ (φ, ψ eigenforms, see (*)). Actually these representations are the same

Theorem: If φ, ψ and F are as above then

a) $\langle F, \Theta^2_{ij} \rangle = c \operatorname{Res}_{s=1} D^N(F,s) \{\varphi(y_i)\psi(y_j) + \varphi(y_j)\psi(y_i)\}$

b) $\langle F, F \rangle = c \operatorname{Res}_{s=1} D^N(F,s) \{\varphi,\varphi\} \cdot \{\psi,\psi\}$

where $\{\ ,\ \}$ is the canonical scalar product on $\mathcal{A}$.
We have similar formulas also for the degree 1 and degree 3 cusp forms produced by Yoshida - lifts. We sketch a proof of the theorem above : We consider the degree 3 cusp form

$$c \operatorname*{Res}_{s=1} D^N(F,s)\ Y^3(\varphi,\psi) - \sum_{i,j} \frac{1}{e_i e_j} \langle F, \theta^2_{ij} \rangle\ \theta^3_{ij}$$

All we have to show is that this function is identically zero ! But this follows from the fact that this function is again an eigenform of $\mathcal{H}^3_N$ with the same eigenvalues as those of the non-cusp form $Y^3(\varphi,\psi)$; by the results above, it must be zero.

3) <u>Eisenstein series of Klingen type</u>. From general properties of pullbacks of Eisenstein series (see [Bö$_2$] or [Ga]), combined with a version of Siegel's theorem, we see that for $F \in \mathcal{M}^2_{cusp}$ indeed

$$\sum_{i,j} \frac{1}{e_i e_j} \langle F, \theta^2_{ij} \rangle\ \theta^n_{ij}$$

is essentially (a residue of) an Eisenstein series of Klingen type attached to F. From this (and the scalar product formulas above) we can get a solution of Problem (C) ; similar arguments also work for the degree 1 and degree 3 cusp forms.

4) In a letter [Y 3] Yoshida kindly informed us that he has also made some progress towards his conjectures [Y1, Y2]. His methods are different from ours. In particular - using results of Waldspurger - he has also obtained our result on $Y^2(1,\psi)$ and an unconditional proof of Theorem 6.7 of [Y2].

Chapter III : Modular forms of weight $\frac{3}{2}$

In this chapter we restrict ourselves to t=1, so N=q is a prime; we write $\mathcal{A}^{\pm}$ instead of $\mathcal{A}^{\pm\varepsilon_0}$. The basic facts which we need in this chapter can be found in [Gr], [Ko], [Kr], [Y 2].

To each maximal order $R_i = L_{ii}$ of D we attach a ternary lattice

$$R_i^o := \{ x \in 2R_i + \mathbb{Z} \mid \mathrm{trace}(x) = 0 \}$$

and a ternary theta series

$$\vartheta_i(\tau) = \sum_{x \in R_i^o} e^{2\pi i\, \mathrm{norm}(x)} \quad .$$

Following [Ko] we define a space M of those modular forms g of weight $\frac{3}{2}$ with respect to $\Gamma_0(4q)$ which satisfy in addition

$$g(\tau) = \sum_{D \geq 0} e^{2\pi iD\tau} \quad \text{with} \quad a(D)=0 \quad \text{unless} \quad -D \equiv 0,1 \mod 4$$

Even more important for us is the subspace $M^- \subset M$ of those forms g whose Fourier coefficients a(D) vanish unless $\left(\frac{-D}{q}\right) \neq 1$. Via the Shimura-correspondence M_{cusp} is isomorphic to $\mathcal{M}^1_{cusp}$ and M^- corresponds to those forms f in $\mathcal{M}^1$ with $f|\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = -f$ (see [Ko]).

Now we define two mappings, both Hecke-equivariant

$$W : \begin{cases} \mathcal{A} \longrightarrow M^- \\ \varphi \longmapsto \sum \frac{\varphi(y_i)}{e_i} \vartheta_i \end{cases}$$

$$\widetilde{W} : \begin{cases} M^- \longrightarrow \mathcal{A} \\ g \longmapsto \varphi_g \end{cases} \quad \text{with} \quad \varphi_g(y_i) = \langle g, \vartheta_i \rangle .$$

These mappings are adjoint to each other with respect to the Petersson scalar products on $\mathcal{A}$ and M^- (we may extend the scalar product from M^-_{cusp} to M^-).

Clearly $W(\varphi)$ can also be obtained in the following way : From [Y 2, Thm 4.3] we know that $Y^2(1,\varphi)$ is in the Maaß space; so let $JY^2(1,\varphi)$ be the corresponding Jacobi form of index 1; a theorem of Kramer [Kr] asserts that the modular form of weight $\frac{3}{2}$ which corresponds to $JY^2(1,\varphi)$ is just $W(\varphi)$, so we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{Y^2(1,\_)} & \Theta^2 & \xrightarrow{J} & \text{Jacobi theta series of index 1} \\ & \searrow^{W} & & & \downarrow \\ & & & & M^- \end{array}$$

Combining this with the results obtained in chapter II we get

<u>Proposition</u>: Let $0 \neq \mathcal{A}_{cusp}$ be an eigenform, then

$$W(\varphi) \neq 0 \quad \text{iff} \quad L^q(\hat{\varphi},1) \neq 0$$

(Actually we only proved this for $\varphi \in \mathcal{A}^+$, but for $\varphi \in \mathcal{A}^-$ both $W(\varphi)$ and $L^q(\hat{\varphi},1)$ are automatically equal to zero).

To proceed further, we need an analogue of (*) for our ternary theta series :

<u>Theorem</u>: Let $0 \neq g \in M_{cusp}$ be an eigenform of all Hecke operators, then

$$\sum \frac{\langle g, \vartheta_i \rangle}{e_i} \vartheta_i = c\, L(g,1)\, g \ ,$$

in particular, g is a linear combination of the ϑ_i iff $L(g,1)$ is different from zero.

Here we mean by $L(g,s)$ the Dirichlet series $\sum \lambda(m)\, m^{-s}$ where $\lambda(m)$ is the eigenvalue of g for the Hecke operator $T(m^2)$.

This theorem, combined with the fact that $\dim M^- = T$ = type number of D, gives a new proof of a result of Gross [Gr] which says that we have T linear independant theta series ϑ_i iff there is no $g \in M^-_{cusp}$ with $L(g,1) = 0$.

Again we can obtain scalar product formulas :

<u>Proposition</u> : Let $0 \neq \varphi \in \mathcal{A}_{cusp}$ be an eigenform.

a) $c\, L^q(\tilde{\varphi},1)\, \varphi(y_i) = \langle W(\varphi), \vartheta_i \rangle$

b) $c\, L^q(\hat{\varphi},1) \{ \varphi,\varphi \} = \langle W(\varphi), W(\varphi) \rangle$

<u>Corollary</u>:

a) For φ as above $\quad \tilde{W}\, W(\varphi) = c\, L^q(\varphi,1)\, \varphi$

b) For $g \in M^-$, g eigenform $\quad W\, \hat{W}(g) = c\, L(g,1)\, g$

It is reasonable now to introduce a <u>modified Yoshida - lift</u> by

$$\hat{Y}^n : \begin{cases} M^- \times M^- & \longrightarrow \quad \Theta^n \\ (g,h) & \longmapsto \sum\limits_{i,j} \dfrac{1}{e_i e_j} \langle g, \vartheta_i \rangle \langle h, \vartheta_j \rangle \; \Theta^n_{ij} \end{cases}$$

From the above it is clear that

$$\hat{Y}^n(g,h) = Y^n(\tilde{W}(g), \hat{W}(h))$$

$$\hat{Y}^n(W(\varphi), W(\psi)) = c\, L^q(\hat{\varphi},1)\, L^q(\hat{\psi},1)\, Y^n(\varphi,\psi).$$

So there is not much difference between looking at Y^n or at $\tilde{Y}^n$ as long as we are only interested in those $\varphi \in \mathcal{A}_{cusp}$ with $L^q(\hat{\varphi},1) \neq 0$. The striking point about $\tilde{Y}^n$ is that we have a beautiful <u>kernel function</u> to describe it :

<u>Theorem</u> :

$$\mathfrak{K}^n(\tau,\tau', Z) := \sum_{i,j} \frac{1}{e_i e_j} \vartheta_i(\tau)\, \vartheta_j(\tau')\, \Theta^n_{ij}(Z)$$

is a kernel function for $\tilde{Y}^n$.

Chapter IV : The Fourier coefficients

In this chapter $N = q$ is again a prime.

The most ambitious programme would of course be to look for explicit formulas for the Fourier coefficients of $Y^n(\varphi,\psi)$ in terms of some data attached to φ and ψ.

Our results are more modest; we consider the case $n=2$ and compute a certain mean value of Fourier coefficients : For $Y^2(\varphi,\psi)$ with Fourier expansion $\sum_T a(T)\, e^{2\pi i \,\mathrm{trace}(TZ)}$ we study (for any discriminant $-D<0$) the weighted average

$$a_D := \sum_T \frac{a(T)}{\varepsilon(T)} \quad ,$$

where T runs over all $Sl_2(\mathbb{Z})$ - classes of binary integral quadratic forms with $\mathrm{disc}(T) = -D$ and $\varepsilon(T) = \#$ proper automorphisms of T ($=1$ in general).

In analogy to the results in [Bö$_3$] we may expect here also some relations to modular forms of weight $\frac{3}{2}$. Indeed, put

$$g = W(\varphi) = \sum_{D\geq 0} b(D) e^{2\pi i D\tau}$$

$$h = W(\psi) = \sum_{D\geq 0} c(D) e^{2\pi i D\tau} .$$

Then we get (at least for fundamental discriminants $-D<0$) a very simple identity :

$$\boxed{a_D = \gamma_D\, b(D)\, c(D)}$$

where $\gamma_D = 2$ if $q \mid D$ and $\gamma_D = 1$ otherwise.

We may reformulate this result as an identity for Dirichlet series (now for general discriminants) as follows :

Recall that for any degree 2 Siegel modular form $F(Z) = \sum_T a(T)\, e^{2\pi i\, tr(TZ)}$ we have the Koecher - Maaß - Dirichlet series $\zeta_F(s) = \sum_D a_D D^{-s}$ and for modular forms $g, h \in M^-$ as above we define a (modified) Rankin - Konvolution

$$\mathcal{R}(g,h,s) := \sum_D \gamma_D\, b(D)\, c(D)\, D^{-s}.$$

Theorem : For any $F = Y^2(\varphi, \psi)$, $g = W(\varphi)$, $h = W(\psi)$ we have

$$\zeta_F(s) = \zeta(2s-1)\,\mathcal{R}(g,h,s).$$

Remarks.

1) If $L^q(\varphi,1) = 0$, then $g = 0$ (and the same for ψ); in other words, the formula above together with the results of the preceeding chapters prove the existence of many degree 2 Siegel modular forms with Koecher-Maaß series vanishing identically.

2) Our first attempt to prove the non-vanishing of $Y^2(\varphi, \psi)$ was by means of the theorem above (if $W(\varphi) \neq 0$, $W(\psi) \neq 0$). However it seems to be a very difficult problem to get a reasonable criterion for the (non-) vanishing of the Rankin-convolution attached to two modular forms of half-integral weight. We can however prove directly (i.e. by the theorem above, not using the results of chapter II) a version of Theorem 6.7 of [Y 2] :

Corollary. For $\varphi \in \mathcal{A}$ with $W(\varphi) \neq 0$ we have

$$Y^2(\varphi, \varphi^\sigma) \neq 0 \quad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}).$$

The assertion of the theorem above will easily follow from a purely arithmetical statement on representation numbers (representations of binary quadratic forms by quaternary forms).

Let us start with a numerical example :

We take $q = 11$ - this also occurs in [He, p.900, Beispiel 2], [Y 1, Example 1] and [Gr, §13]. We have 3 inequivalent integral quaternary quadratic forms of determinant $\frac{1}{16} q^2$:

$$Q_{11} \sim x_1^2 + x_1x_2 + 3x_2^2 + x_3^2 + x_3x_4 + 3x_4^2$$
$$Q_{12} \sim 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2x_1x_3 + x_1x_4 + x_2x_3 - 2x_2x_4$$
$$Q_{22} \sim x_1^2 + 4(x_2^2 + x_3^2 + x_4^2) + x_1x_3 + 4x_2x_3 + 3x_2x_4 + 7x_3x_4$$

The ternary forms corresponding to Q_{11} and Q_{22} are

$$R_1^o \sim 12x^2 + 44xy + 44y^2 + 11z^2$$
$$R_2^o \sim 3x^2 + 2xy + 15y^2 + 44yz + 44z^2$$

The adjoint forms of R_1^o and R_2^o are equivalent to

$$\hat{R}_1 \sim x^2 + xy + 3y^2 + z^2$$
$$\hat{R}_2 \sim x^2 + xy + y^2 + xz + 4z^2$$

For two positive definite quadratic forms S and T we denote by A(S,T) the number of integral representations of T by S. We claim that there should be some relation between $A(R_i^o,D)$ and $A(R_j^o,D)$ on one hand and $A(Q_{ij},T)$ with T binary of discriminant $-D$ on the other hand.

D	$A(R_1^o,D)$	$A(R_2^o,D)$	$A(R_1^o,D)^2$	$A(R_1^o,D)A(R_2^o,D)$	$A(R_2^o,D)^2$
3	0	2	0	0	4
4	2	0	4	0	0
11	2	0	4	0	0
15	4	6	16	24	36
31	8	6	64	48	36

The quadratic form $ax^2+bxy+y^2$ will be denoted by $[a,b,c]$.

D	T	$\frac{A(Q_{11},T)}{\varepsilon(T)}$	$\frac{A(Q_{12},T)}{\varepsilon(T)}$	$\frac{A(Q_{22},T)}{\varepsilon(T)}$
3	[1,1,1]	0	0	4
4	[1,0,1]	4	0	0
11	[1,1,3]	8	0	0
15	[1,1,4]	16	0	0
15	[2,1,2]	0	24	36
31	[1,1,8]	32	0	36
31	[2,±1,4]	16	24	0

Everything in these tables becomes very smooth if we look at the weighted average

$$A_{ij}(D) := \sum_{\{T\}} \frac{A(Q_{ij},T)}{\varepsilon(T)} ,$$

where T runs over all (properly) inequivalent integral binary quadratic forms of discriminant $-D$; in fact we have (not only for the numerical example above but for arbitrary primes q) the following

Theorem : $A_{ij}(D) = \gamma_D \cdot A(R_i^o,D) \cdot A(R_j^o,D)$

for $1 \leq i,j \leq H$ and $-D$ a fundamental discriminant – and a similar statement for non-fundamental discriminants.

Remark. We can reformulate the statement above as follows; It is elementary that (for $-D$ fundamental)

$$A(R_i^o,D) = \frac{1}{2} \sum_{\{T\}} \frac{A(\hat{R}_i,T)}{\varepsilon(T)} .$$

Therefore the theorem above can be written in a more symmetric way as

$$\sum_{\{T\}} \frac{A(Q_{ij},T)}{(T)} = \frac{1}{4}\,\gamma_D \left(\sum_{T} \frac{A(\hat{R}_i,T)}{(T)}\right)\left(\sum_{T} \frac{A(\hat{R}_j,T)}{(T)}\right) \qquad (***)$$

One might try to make this statement stronger by putting in everywhere a character of the class group of $\mathbb{Q}(\sqrt{-D})$. In our example the cases $D = 3, 4, 11$ are of course trivial.

D	T	$A(\hat{R}_1,T)$	$A(\hat{R}_2,T)$
15	[1,1,4]	8	12
15	[2,1,2]	0	0
31	[1,1,8]	0	12
31	[2,±1,4]	8	0

The example $D = 15$ shows that we have been too optimistic :

$$\pm 24 \overset{?}{=} \frac{1}{4}(8 \pm 0)(12 \pm 0)$$

For a correct strengthening of (***) we refer to [Bö-Sp 2].

<u>Some applications</u>.

1) In [$Bö_3$] we conjectured (for Siegel modular forms of degree 2 and level 1) that the square of the average a_D should be related to a special value of the twisted spinor-L-function. The result of this chapter shows that $Y^2(f,\psi)$ satisfies that conjecture.

2) We may obtain a new proof of Waldspurger's formulas for the square of the Fourier coefficients of modular forms of half-integral weight (in the case of weight $\frac{3}{2}$) as follows: For $f \in \mathcal{A}$, f an eigenform, and a fundamental discriminant $-D$, we can compute the average a_D for $Y^2(f,f)$ in two ways:

First of all, according to the formulas above,

$$a_D = \gamma_D \, b(D)^2 ,$$

with $b(D)$ = D-th Fourier coefficient of $g := W(\varphi)$.

Secondly, by interpretating $Y^2(\varphi,\varphi)$ as a kind of Eisenstein series attached to $Y^1(\varphi,\varphi)$, we may get (by some analytic considerations) a formula of type

$$a_D = A(\varphi)\, D^{\frac{1}{2}}\, L(g,1)\, L(g,D,1)$$

where $A(\varphi)$ is some (explicitely known) constant depending on φ and $L(g,D,s)$ denotes the twist of $L(g,s)$ by the quadratic character $\left(\frac{-D}{\cdot}\right)$. Combining these two formulas we get that $b(D)^2$ is proportional to $\gamma_D\, D^{1/2}\, L(g,D,1)$; unfortunately this proof does not give any information for those φ with $W(\varphi) = 0$ (i.e. for those φ with $L^q(\hat{\varphi},1) = 0$). For details we refer to a paper in preparation.

Final remarks.

In some sense the results presented here are not complete.

-- We should consider Eichler orders instead of maximal orders; this will indeed be done in [Bö-Sp 2].

-- We should include the case of theta series with harmonic coefficients as in [Y 1, Y 2] and in [Ta].

-- The results of chapters III and IV should be extended to arbitrary quaternion algebras (not just those ramified only in q, ∞).

Our results on these more general problems are not complete at present, but we are working on them. We hope to treat them in future papers.

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