数理解析研究所講究録
692巻 1989年 88-112

RWA－振動子に対する森のランジュバン方程式

北大・理　広川　真男（Masao Hirokawa）

§1. INTRODUCTION.

We are concerned with Mori's Langevin equation for a model of a quantum harmonic oscillator coupled to infinitely many scalar bosons whose Hamiltonian H is given formally by

$$(1.1)\quad \begin{cases} H = H_0 + H_I \\ H_0 = H_S + H_B \\ H_S = \hbar\omega_0 a^+ a & (0<\omega_0) \\ H_B = \Sigma_{k=1}^{\infty}\hbar\omega_k b_k^+ b_k & (0<\omega_k<\omega_{k+1}, k\in\mathbb{N}) \\ H_I = \Sigma_{k=1}^{\infty}(\Gamma_k a^+ b_k + \overline{\Gamma}_k b_k^+ a) & (\Gamma_k\in\mathbb{C}, k\in\mathbb{N}) . \end{cases}$$

Here a and a^+ (resp. b_k and b_k^+) denote annihilation and creation operators of a quantum harmonic oscillator (resp. boson), respectively, which act in the symmetric Boson Fock space $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ over $\mathbb{C}\oplus\ell^2(\mathbb{N})$. Operators H_S and H_B denote a Hamiltonian of a quantum harmonic oscillator and the one of infinitely many scalar bosons, respectively. The operator

H_I which represents the interaction between the quantum harmonic oscillator and infinitely many scalar bosons is said to be the coupling Hamiltonian for the rotating wave approximation (abbr.RWA)-oscillator . We shall simply call the operator H the Hamiltonian for the RWA-oscillator .

The behavior of the Heisenberg picture $e^{itH/\hbar}a e^{-itH/\hbar} = a(t)$ has been studied in H.Haken([9]),K.Lindenberg and B.J. West([11]) , and E.Braun([4]). They considered their own equations of Langevin type for $a(t)$ whose form are dependent on their consideration . In particular , K.Lindenberg and B.J.West rewrite the total Hamiltonian H given by (1.1) into

$$
(1.2) \quad \begin{cases} H = H_S^{(m)} + H_B + H_I^{(m)} \\ H_S^{(m)} = \hbar(\omega_0 - \Delta)a^+ a \\ H_B + H_I^{(m)} = \Sigma_{k=1}^{\infty}\hbar\omega_k B_k^+ B_k \\ B_k = b_k + \dfrac{\overline{\Gamma}_k}{\hbar\omega_k}a \ . \end{cases}
$$

where $\Delta = \Sigma_{k=1}^{\infty}\dfrac{|\Gamma_k|^2}{\hbar^2\omega_k}$.

By solving a simultaneous system of differential equations for $a(t)$ and $B_k(t)$:

$$(1.3)\quad \begin{cases} \dfrac{d}{dt}a(t) = \dfrac{i}{\hbar}[H, a(t)] \\[2ex] \dfrac{d}{dt}B_k(t) = \dfrac{i}{\hbar}[H, B_k(t)] \end{cases} ,$$

they derived an equation of Langevin type for $a(t)$. However, the physical meaning of the quantities $\omega_0-\Delta$ and Δ are not clear in [11]. While, in his theory of generalized Brownian motion in statistical physics, H.Mori derived the so-called Mori's Langevin equation which consists of the Mori's frequency , Mori's memory function and Mori noise ([12],[13]). Mathematicaly, Mori's Langevin equation can be derived if a Mori-Okabe (abbr. an MO)-structure is given. Here an MO-structure consists of the triplet of a Hilbert space X, a self-adjoint operator L on X and a non-zero A_0 in the domain of L ([15],[16],[17]) .

The purpose is to show that the Hamiltonian H in (1.1) has an MO-structure and investigate Mori's Langevin equation for the Heisenberg picture $a(t)$ in detail . In order to carry out it , we shall construct a Hilbert space $X_C(H)$ of unbounded operators on $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ containing the annihilation and creation operators, where the inner product of $X_C(H)$ is introduced from Bogoliubov scalar product (Kubo-

Mori scalar product,or the canonical correlation) (see [5, p96],[13]). Furtheremore,we shall construct a self-adjoint operator L (Liouville operator) on $X_C(H)$ in such a way that the Heisenberg picture of α by the operator H coincides with the time evolution of α by the operator L . Since the triplet $(X_C(H),L,\alpha)$ satisfies an MO-structure , we can develop Mori's theory of generalized Brownian motion on $X_C(H)$. The main point is to express the Mori's frequency , the complex mobility of the Mori's memory function and the canonical correlation function of $\alpha(t)$ in terms of the parameters in the Hamiltonian H in (1.1), by obtaining the Bogoliubov transformation of H .Furthermore, we shall give a physical meaning of two constants $\omega_0-\Delta$ and Δ in (1.2) , and show that the canonical correlation function is almost periodic , and does not converges as the time tends to infinity .

§2. MO-STRUCTURES.

By an MO-structure,we mean a triplet(X,L,A_0) such that X is a Hilbert space with an inner product $(\, , \,)_X$, L a self-adjoint operator on X with domain D(L), and A_0 a non-zero element in D(L) , where the inner product $(\, , \,)_X$ is linear in the right vector . For any MO-structure , we consider a stationary curve $A=\{A(t);t\in\mathbb{R}\}$ defined by

(2.1) $\quad A(t) := e^{itL}A_0 \quad (t \in \mathbb{R})$

and the correlation function R_A of A is given by

(2.2) $\quad R_A(t) := (A(0),A(t))_X$.

Let X_0 be the closed subspace generated by A_0, and P_0 and X_1 the orthogonal projection operator on X_0 and the complementary subspace of X_0 in X, respectively. Then we define a linear operator L_1 on the Hilbert space X_1 by

(2.3) $$\begin{cases} D(L_1) := X_1 \cap D(L) \\ L_1 x := (1 - P_0)Lx \quad (x \in D(L_1)) \end{cases}$$.

Lemma 2.1. ([12],[15],[16]) L_1 is self-adjoint on the Hilbert space X_1 .

We define a real number $\omega=\omega(A_0)$, a stationary curve $I_M=\{I_M(t);t\in\mathbb{R}\}$ in X_1 and a non-negative definite function ϕ on $\mathbb{R}$ by

(2.4) $\quad \omega = \omega(A_0) := (A(0),LA(0))_X \cdot (A(0),A(0))_X^{-1}$,

(2.5) $\quad I_M(t) := i \cdot e^{itL_1} \cdot (1 - P_0)LA_0$,

(2.6) $\quad \phi(t) := (I_M(0),I_M(t))_X \cdot (A(0),A(0))_X^{-1}$.

Concerning the correlation function R_A , we have

Theorem 2.2.([7,§6.2],[12],[15],[16]) (a) For all $t\in\mathbb{R}$,

$$\frac{d}{dt}R_A(t) = i\omega\cdot R_A(t) - \int_0^t ds\ \phi(t-s)R_A(s) .$$

(b) For all $z\in\mathbb{C}^+ := \{z\in\mathbb{C};\ \mathrm{Im}z > 0\}$,

$$\int_0^\infty dt\ e^{itz}R_A(t) = R_A(0)\cdot\frac{1}{i\omega - iz + \int_0^\infty dt\ e^{itz}\phi(t)} .$$

Furtheremore, the equation of motion describing the time evolution of stationary curve $A=\{A(t);t\in\mathbb{R}\}$ is given in the following

Theorem 2.3.([7,§6.2],[12],[15],[16]) For all $t\in\mathbb{R}$,

$$(2.7)\qquad \frac{d}{dt}A(t) = i\omega\cdot A(t) - \int_0^t ds\ \phi(t-s)A(s) + I_M(t) .$$

The quantities $\omega=\omega(A_0)$, ϕ and I_M are called the Mori's frequency, Mori's memory function and Mori noise , respectively .

The equation (2.7) is said to be the Mori's Langevin equation associated with the MO-structure (X,L,A_0) .

§3.CONSTRUCTION OF A HILBERT SPACE ASSOCIATED WITH A CLASS OF CLOSABLE OPERATORS.

Let $\mathcal{J}$ be a separable Hilbert space with an inner product (,),which is linear in the right vector. Let H be a non-negative self-adjoint operator acting in $\mathfrak{H}$ with the following properties (H.1) and (H.2) :

(H.1) There exists a complete orthonormal basis $\{\varphi_n ; n\in\mathbb{N}^* = \{0\}\cup\mathbb{N}\}$ in D(H) such that $H\varphi_n=\lambda_n\varphi_n$ with $0\le\lambda_n\le\lambda_{n+1}$ $(n\in\mathbb{N}^*)$.

(H.2) $e^{-\tau H} \in \mathcal{J}_1$ for all $\tau\in(0,\beta]$, where $\beta>0$ is the inverse temperature and $\mathcal{J}_1$ denotes the family of all trace class operators on $\mathfrak{H}$.

Let

(3.1) $\quad D := \{\Sigma_{k=0}^{n} \alpha_k\varphi_k ; n\in\mathbb{N}^*, \alpha_k\in\mathbb{C}, k=0,1,\cdots,n\}$.

Obviously D is dense in $\mathfrak{H}$. We denote by $\mathcal{L}(D,\mathfrak{H})$ the space of bounded linear operators from D into $\mathfrak{H}$. Every element A in $\mathcal{L}(D,\mathfrak{H})$ has a unique extension to an element in $\mathcal{L}(\mathfrak{H})$, the space of bounded linear operators on $\mathfrak{H}$. We denote the extension of A by $\bar{A}$ or A^- .

Let $\mathcal{J}(D,H)$ be the set of linear operators A acting in $\mathfrak{H}$ with the following properties ($\mathcal{J}$.1) - ($\mathcal{J}$.3) :

($\mathcal{J}$.1) $\quad D \subset D(A) \cap D(A^*)$.

($\mathcal{J}$.2) $\quad$ For all $x\in D$, Ax and A^*x are in D .

($\mathcal{J}$.3) $\quad$ For all $\tau\in(0,\beta]$, $e^{-\tau H}A$ (resp. $Ae^{-\tau H}$) is in $\mathcal{L}(D,\mathfrak{H})$ and $(e^{-\tau H}A)^-$ (resp. $(Ae^{-\tau H})^-$) is in $\mathcal{J}_2$, where $\mathcal{J}_2$ denotes the family of all Hilbert-Schmidt operators with the Hilbert-Schmidt norm $\| \ \|_2$.

In this section, we shall construct a Hilbert space $X_C(H)$ and a Liouville operator L associated with the Hamiltonian H and show that the triplet $(X_C(H), L, A_0)$ satisfies an MO-structure for any non-zero $A_0 \in D(L)$.

If two operators A and B in $\mathcal{J}(D,H)$ satisfy Ax=Bx for all $x \in D$, then we write as $A \sim B$, which gives an equivalence relation in $\mathcal{J}(D,H)$. We denote by [A] the equivalence class of $A \in \mathcal{J}(D,H)$ and by $\mathcal{J}(D,H)/\sim$ the set of all the equivalence classes .

We can introduce in $\mathcal{J}(D,H)/\sim$ the operation of addition , scalar multiplication and involution * as follows :

(3.2) $\quad [A]+[B] := [A+B]$.

(3.3) $\quad \alpha[A] := [\alpha A] \quad (\alpha \in \mathbb{C})$.

(3.4) $\quad [A]^* := [A^*]$.

We can define a correlation ; Bogoliubov scalar product (Kubo-Mori scalar product , or the canonical correlation) $< ; >_H$ on $\mathcal{J}(D,H)/\sim$ as follows: For any $[A],[B] \in \mathcal{J}(D,H)/\sim$,

(3.5) $$<[A];[B]>_H := \frac{1}{\beta} \frac{\int_0^\beta d\lambda \, \Sigma_n (Ae^{-(\beta-\lambda)H} \varphi_n , e^{-\lambda H} B\varphi_n)}{tr(e^{-\beta H})}$$

It can be easily seen to prove that $< ; >_H$ is inner products on $\mathcal{J}(D,H)/\sim$. We denote the norm of [A] by

(3.6) $\quad \|[A]\|_H := <[A];[A]>_H^{1/2}$.

We can define an element A(t) in $\mathcal{J}(D,H)$ and its equivalence class [A](t) by

(3.7) $\quad A(t) := e^{itH/\hbar} A e^{-itH/\hbar} \qquad (A \in \mathcal{J}(D,H) , t \in \mathbb{R})$,

(3.8) $\quad [A](t) := [A(t)] \qquad (A \in \mathcal{J}(D,H) , t \in \mathbb{R})$,

where $\hbar>0$ is a parameter denoting the Planck constant divided by 2π .

Lemma 3.1. For all $A \in \mathcal{J}(D,H)$, $\|[A]\|_H = \|[A^*]\|_H$.

We obtain a Hilbert space $X_C(H)$ as the completion of $\mathcal{J}(D,H)/\sim$ by the norm $\| \ \|_H$.

Remark 3.1. We can define an involution $[A] \longrightarrow [A]^+$ on $X_C(H)$ such that $[A]^+=[A]^*$ for all $[A] \in \mathcal{J}(D,H)/\sim$.

Let

(3.9) $\quad \mathcal{J}_\delta := \{A \in \mathcal{J}(D,H) \ ; \ AH , HA \in \mathcal{J}(D,H)\}$,

(3.10) $\quad D(\delta) := \{[A] \in X_C(H) \ ; \ A \in \mathcal{J}_\delta\}$.

$D(\delta)$ is a dense subspace in $X_C(H)$.

We define a linear operator $\delta : D(\delta) \longrightarrow X_C(H)$ by

(3.11) $\quad \delta[A] := [\frac{1}{\hbar}[H,A]] \quad ([A] \in D(\delta))$, where $[H,A] := HA - AH$. Then , δ is a symmetric operator .

We define for each $t \in \mathbb{R}$ the operator $U(t) : \mathcal{J}(D,H)/\sim \longrightarrow X_C(H)$ by

(3.12) $\quad U(t)[A] := [A](t) \quad (A \in \mathcal{J}(D,H))$.

Proposition 3.2. For any $[A] \in \mathcal{J}(D,H)/\sim$, and $t,s \in \mathbb{R}$,

(a) $U(t)$ is unitary on $\mathcal{J}(D,H)/\sim$,

(b) $U(0)[A] = [A]$,

(c) $U(t+s)[A] = U(t)U(s)[A]$,

(d) $\text{s-lim}_{t\to 0}U(t)[A] = [A]$.

Since it follows from Proposition 3.2 that $\{U(t);t\in\mathbb{R}\}$ is uniquely extended to a strongly continuous unitary group on $X_C(H)$, we denote its extension by the same symbol . By Stone's theorem , there exists a unique self-adjoint operator L on $X_C(H)$ such that

(3.13) $\quad U(t) = e^{itL}$.

Proposition 3.3. $L \supset \delta$.

Remark 3.2. Proposition 3.3 means that the time evolution by Liouville operator L coincides with the Heisenberg picture on $D(\delta)$,i.e.,

(3.14) $\quad e^{itL}[A] = [e^{itH/\hbar} A\, e^{-itH/\hbar}] \quad ([A] \in D(\delta))$.

Definition 3.4. We say that $A \in \mathcal{J}(D,H)$ is in $M(D,H)$ if it satisfies the following condition (C.1) :

(C.1) For all τ,τ' with $0<\tau'\leq\tau\leq\beta$, there exist non-negative functions $f_A(\tau,\tau')$, $g_A(\tau,\tau')$, $f_A{}^*(\tau,\tau')$ and $g_A{}^*(\tau,\tau')\geq 0$ such that , for all $x\in D$,

$$\begin{cases} \|e^{-\tau H}Ax\|\leq f_A(\tau,\tau')\|e^{-(\tau-\tau')H}Ae^{-\tau' H}x\|+g_A(\tau,\tau')\|e^{-\tau' H}x\| \\ \|e^{-\tau H}A^*x\|\leq f_A{}^*(\tau,\tau')\|e^{-(\tau-\tau')H}A^*e^{-\tau' H}x\| \\ \qquad\qquad +g_A{}^*(\tau,\tau')\|e^{-\tau' H}x\| . \end{cases}$$

If A and B $\in \mathcal{J}(D,H)$ are in M(D,H) , we have AB $\in \mathcal{J}(D,H)$. Then , for any A,B $\in$ M(D,H) , we define the product of [A] and [B] $\in \mathcal{J}(D,H)/\sim$ by

(3.15) $\quad [A][B] := [AB]$.

This definition is independent of the choice of the representatives of [A] and [B] .

Proposition 3.5. Suppose that $\{B_0,B_1,\cdots,B_N\}\subset\mathcal{J}(D,H)$ $(N \in \mathbb{N}^*)$ satisfy the following conditions :

$$B_kB_\ell^*\ ,B_\ell^*B_k\in\mathcal{J}(D,H) \text{ and } [B_k,B_\ell]x\delta_{k\ell}x \quad (k,\ell=0,1,\cdots,N,x\in D)$$

Then , $\{[B_0],[B_1],\cdots,[B_N]\}$ is linearly independent .

Definition 3.6. We say that A $\in \mathcal{J}(D,H)$ is H-diagonal if there exists $\sigma_A \in \mathbb{R}$ such that

(3.16) $\quad [H,A]x = -\hbar\sigma_A Ax \quad , x \in D$.

Remark 3.3. If A is H-diagonal with $[A] \neq 0$,then σ_A is uniquely determined .

Lemma 3.7. For any H-diagonal $A\in\mathcal{J}(D,H)$, $x\in D$, $\tau>0$ and $t\in\mathbb{R}$,

(a) $Ae^{-\tau H}x = \exp[-\tau\hbar\sigma_A]e^{-\tau H}Ax$,

(b) $Ae^{itH/\hbar}x = \exp[it\sigma_A]e^{itH/\hbar}Ax$.

Proposition 3.8. If $A,B\in\mathcal{J}(D,H)$ are H-diagonal and $\sigma_A\neq\sigma_B$, then $<[A];[B]>_H=0$.

Lemma 3.9. If $A\in\mathcal{J}(D,H)$ is H-diagonal, then A is in M(D,H).

Proposition 3.10. (a) If $A\in\mathcal{J}(D,H)$ is H-diagonal, then A^* $\in\mathcal{J}(D,H)$ is also H-diagonal. Moreover, if $[A]\neq 0$, then $\sigma_{A^*}=-\sigma_A$.

(b) If $A,B\in\mathcal{J}(D,H)$ are H-diagonal , then $AB\in\mathcal{J}(D,H)$ is also H-diagonal . Moreover, if $[AB]\neq 0$, then $\sigma_{AB}=\sigma_A+\sigma_B$.

Lemma 3.11. If $A\in\mathcal{J}(D,H)$ is H-diagonal , then $A\in\mathcal{J}_\delta$ and so $[A]\in D(\delta)$.

We define a subset $\mathfrak{A}_f(H)$ of $X_C(H)$ as follows :

(3.17) $\mathfrak{A}_f(H):=\{u_0[A_0]+u_1[A_1]+\cdots+u_N[A_N]\in X_C(H);N\in\mathbb{N}^*,\ u_0,u_1,\cdots,$ $u_N\in\mathbb{C}$, and $A_0,A_1,\cdots,A_N\in\mathcal{J}(D,H)$ are H-diagonal$\}$.

Proposition 3.12. $\mathfrak{A}_f(H)$ is a *-algebra and $\mathfrak{A}_f(H)\subset D(\delta)$.

Remark 3.4. $X_C(H)$ is a partial *-algebra with a unity (see ,[1,Definition 2.1. and Definition 2.2.])

Lemma 3.13. For all $[A]$, $[B]\in\mathfrak{A}_f(H)$ and $t\in\mathbb{R}$.

(a) $L[A]$ and $e^{itL}[A] \in \mathfrak{A}_f(H)$,

(b) $L([A][B]) = (L[A])[B] + [A](L[B])$,

(c) $L[A]^* = -(L[A])^*$.

Lemma 3.14. Suppose that $A \in \mathcal{J}(D,H)$ is H-diagonal with $\sigma_A>0$ and $[A,A^*]x=x$ for all $x\in D$. Then ,

(a) $$\frac{tr((e^{-\beta H}A^*A)^-)}{tr(e^{-\beta H})} = \frac{1}{\exp[\beta\hbar\sigma_A]-1} .$$

(b) $$\|[A]\|_H^2 = \|[A]^*\|_H^2 = \frac{1}{\beta\hbar\sigma_A} .$$

Deffinition 3.15. We say that the linear operator A is in $\mathfrak{C}_\eta(H)$ $(\eta\in(0,\beta))$ if A satisfies the following conditions $(\mathfrak{C}.1)$ and $(\mathfrak{C}.2)_\eta$:

$(\mathfrak{C}.1)$ $(=(\mathcal{J}.1))$ $D(A)$, $D(A^*) \supset D$.

$(\mathfrak{C}.2)_\eta$ $(e^{-\eta H/2}A)^-$, $(Ae^{-(\beta-\eta)H/2})^- \in \mathcal{J}_2$.

Lemma 3.16. For all $\eta \in (0,\beta)$,

(a) If A is in $\mathfrak{C}_\eta(H)$, then A^* is in $\mathfrak{C}_{(\beta-\eta)}(H)$,

(b) $\mathfrak{C}_\eta(H)$ is a complex vector space ,

(c) $\mathcal{J}(D,H) \subset \mathfrak{C}_\eta(H)$.

Lemma 3.17. For each $\eta\in(0,\beta)$ and $A\in\mathfrak{C}_\eta(H)$, there exists a Cauchy sequence $\{[A_N];N\in\mathbb{N}^*\}\subset\mathcal{J}(D,H)/\sim$ such that

(a) $A_N \in\mathcal{J}_\delta$,

(b) $\lim_{N\to\infty}A_Nx = Ax$ in $\mathfrak{J}$ for all $x \in D$,

(c) As a function of $\lambda\in[0,\beta]$,

$\Sigma_n((A-A_N)e^{-(\beta-\lambda)H}\varphi_n, e^{-\lambda H}(A-A_N)\varphi_n)$ converges uniformly to 0 as $N\longrightarrow\infty$.

For every $A\in\mathfrak{C}_\eta(H)$ $(\eta\in(0,\beta))$, we can define an element of $X_C(H)$ $[A]$ by

(3.18) $[A] := \lim_{N\to\infty}[A_N]$.

Remark 3.5. If we can take another convergent sequence $\{[B_N];N\in\mathbb{N}^*\}\subset\mathcal{J}(D,H)/\sim$ to the operator A in the sense of (b) and (c) of Lemma 3.17 , we can show that $\lim_{N\to\infty}[B_N] = [A]$ $\in X_C(H)$. Furthermore , we can show that there exists an injective mapping ι : $\mathfrak{C}_\eta(H)\lceil D \longrightarrow X_C(H)$ defined by

$\iota(A\lceil D) := [A]$,

where $\mathfrak{C}_\eta(H)\lceil D := \{A\lceil D \; ; \; A \in \mathfrak{C}_\eta(H)\}$.

Definition 3.18. For each $\eta \in (0,\beta)$,

$\mathfrak{C}_\eta(H)/\sim := \iota(\mathfrak{C}_\eta(H)\lceil D)$.

Lemma 3.19. For each $A \in \mathfrak{C}_\eta(H)$, $\eta \in (0,\beta)$ and $t \in \mathbb{R}$,

(a) $e^{itH/\hbar} A e^{-itH/\hbar}$ is in $\mathfrak{C}_\eta(H)$,

(b) $[e^{itH/\hbar} A e^{-itH/\hbar}] = e^{itL} [A]$,

(c) $[A]^+ = [A^*]$, and $[A]^+ \in \mathfrak{C}_{(\beta-\eta)}(H)$.

Remark 3.6. For $[A]\in\mathfrak{C}_\eta(H)/\sim$ $(\eta\neq\beta/2)$, it does not always hold that $[A]^+$ is in $\mathfrak{C}_\eta(H)/\sim$.

§4. MORI'S LANGEVIN EQUATION FOR THE RWA-OSCILLATOR

Let a complex Hilbert space $\ell^2(\mathbb{N})$ be given by

(4.1) $\ell^2(\mathbb{N}) := \{(c_k; k\in\mathbb{N}) \in \mathbb{C}^{\mathbb{N}} ; \Sigma_{k=1}^{\infty} |c_k|^2 < \infty\}$.

For each $f \in \mathbb{C}\oplus\ell^2(\mathbb{N})$, we denote f by

(4.2) $f = (f_0, f_1, f_2, \cdots)$,

where $f_0 \in \mathbb{C}$ and $(f_1, f_2, \cdots) \in \ell^2(\mathbb{N})$.

An inner product $< | >$ of $\mathbb{C}\oplus\ell^2(\mathbb{N})$ is given by

(4.3) $<f|g> := \Sigma_{k=0}^{\infty} \overline{f}_k g_k \quad (f, g \in \mathbb{C}\oplus\ell^2(\mathbb{N}))$.

Let $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ be the symmetric Boson Fock space over $\mathbb{C}\oplus\ell^2(\mathbb{N})$, i.e.,

(4.4) $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N})) = \oplus_{n=0}^{\infty} S_n(\mathbb{C}\oplus\ell^2(\mathbb{N}))^n$,

where , for all $n\in\mathbb{N}$, $S_n(\mathbb{C}\oplus\ell^2(\mathbb{N}))^n$ is the n-fold symmetric tensor product of $\mathbb{C}\oplus\ell^2(\mathbb{N})$, $S_0(\mathbb{C}\oplus\ell^2(\mathbb{N}))^0 := \mathbb{C}$ (see, e.g., [20, p.53]) .

For any $f \in \mathbb{C}\oplus\ell^2(\mathbb{N})$, we define $B^+(f) : S_n(\mathbb{C}\oplus\ell^2(\mathbb{N}))^n \longrightarrow S_{n+1}(\mathbb{C}\oplus\ell^2(\mathbb{N}))^{n+1}$ by

(4.5) $B^+(f)\Psi := \sqrt{n+1}\, S_{n+1}(f\otimes\Psi) \quad (\Psi \in S_n(\mathbb{C}\oplus\ell^2(\mathbb{N}))^n)$.

Let

(4.6) $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N})) := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$; there exists $n_0 \in \mathbb{N}^*$ such that, for all $n \geq n_0$, $\Psi^{(n)} = 0\}$.

The subspace $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ is dense in $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ (see, e.g., [6, p.68]). We denote the Fock vacuum Ω_0 by

(4.7) $\quad \Omega_0 := \{1, 0, 0, \cdots\}$.

For each $f \in \mathbb{C}\oplus\ell^2(\mathbb{N})$, we define a linear operator $A^+(f)$ on $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ by

(4.8) $$\begin{cases} (A^+(f)\Psi)^{(n)} := B^+(f)\Psi^{(n-1)} & (n \in \mathbb{N}) \\ (A^+(f)\Psi)^{(0)} := 0 & . \end{cases}$$

Then, $A^+(f)$ is densely defined, and we put

(4.9) $$A(f) := \overline{A^+(f)^* \restriction \mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))} \quad .$$

$A(f)$ and $A^+(f)$ are called the annihilation and creation operator, respectively (see, e.g., [6, §.3.1]). Operators $A^+(f)$ and $A(f)$ are closable and the closure of them are be denoted by the same notation.

Let $N = d\Gamma(I)$ be the second quantization of the identity I (the number operator). It is well-known that

(4.10) $\quad D(A^+(f)),\ D(A(f)) \supset D(N^{1/2})$

with estimates

(4.11) $$\begin{cases} \|A^+(f)\Psi\| \leq \|f\|\cdot\|(N+1)^{1/2}\Psi\| \\ \|A(f)\Psi\| \leq \|f\|\cdot\|N^{1/2}\Psi\| \ . \end{cases} \qquad (\Psi \in D(N^{1/2}))$$

Futhermore, $A^+(f)$ and $A(f)$ leave $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ invariant which satisfy the canonical commutation relations

$$(4.12)\quad \begin{cases} [A(f),A(g)]=0,\ [A^{+}(f),A^{+}(g)]=0 \\ [A(f),A^{+}(g)]=\langle f|g\rangle\ , \end{cases} \qquad (f,g\in\mathbb{C}\oplus\ell^{2}(\mathbb{N}))$$

on $\mathcal{F}_{F}(\mathbb{C}\oplus\ell^{2}(\mathbb{N}))$, where $[A,B] := AB - BA$.

Let $\{e_k; k\in\mathbb{N}^{*}\}$ be a complete orthonormal system of $\mathbb{C}\oplus\ell^{2}(\mathbb{N})$ given by

$$(4.13)\quad e_k := (0,0,\cdots,0,\overset{(k+1)\text{-th}}{\overset{\smile}{1}},0,0,\cdots)\ .$$

We put

$$(4.14)\quad \begin{cases} a := A(e_0)\ , \\ a^{+} := A^{+}(e_0)\ , \\ b_k := A(e_k)\ , \\ b_k^{+} := A^{+}(e_k)\ . \end{cases} \qquad (k\in\mathbb{N})$$

Let $D^{(0)}$ denote the algebraic span of a complete orthonormal basis of $\mathcal{F}_{S}(\mathbb{C}\oplus\ell^{2}(\mathbb{N}))$ (see [3]) ,

$$\frac{1}{\sqrt{n_0!\,n_1!\cdots n_M!}}(a^{+})^{n_0}(b_1^{+})^{n_1}\cdots(b_M^{+})^{n_M}\Omega_0\ ,\ M\in\mathbb{N}^{*},\ n_0,n_1,\cdots,n_M\in\mathbb{N}^{*}\ .$$

Operators a and a^{+} physically denote the annihilation and creation operators of a quantum harmonic oscillator , respectively . On the other hand, the operators b_k and b_k^{+} denote the annihilation and creation operators of a heat bath , respectively .

Let $\{\omega_k;k\in\mathbb{N}^*\}$ and $\{\Gamma_k;k\in\mathbb{N}\}$ be sequences satisfying the following conditions (A.1) , (A.2) and (A.3) :

$$\text{(A.1)}\quad \begin{cases} 0<\omega_0\ ;\quad 0<\omega_1<\omega_2<\cdots , \\ \Sigma_{k=1}^{\infty}\dfrac{1}{\omega_k^2}<\infty . \end{cases}$$

$$\text{(A.2)}\qquad \Sigma_{k=1}^{\infty}\omega_k^2|\Gamma_k|^2<\infty .$$

$$\text{(A.3)}\qquad \hbar\omega_0>\Sigma_{k=1}^{\infty}\frac{|\Gamma_k|^2}{\hbar\omega_k} .$$

Example: We have two examples , as the frequency $\{\omega_k;k\in\mathbb{N}\}$ satisfying (A.1) ;

$\omega_k=(k^2+m^2)^{1/2}$, $m>0$ (the relativistic case) ,

$\omega_k=k^2(2M)^{-1}$, $M>0$ (the non-relativistic case) .

We can define a positive self-adjoint operator $\boldsymbol{\omega}$ on $\mathbb{C}\oplus\ell^2(\mathbb{N})$ by

$$\text{(4.15)}\quad \begin{cases} \boldsymbol{\omega}\, e_0 := \hbar\omega_0 e_0 , \\ \boldsymbol{\omega}\, e_k := \hbar\omega_k e_k \quad (k\in\mathbb{N}) . \end{cases}$$

Then we get the free Hamiltonian H_0 associated with $\boldsymbol{\omega}$ defined by

$$\text{(4.16)}\qquad H_0 := d\Gamma(\boldsymbol{\omega}) ,$$

where $d\Gamma(\boldsymbol{\omega})$ is the second quantization of $\boldsymbol{\omega}$.

Let

(4.17) $\quad \gamma := (0, \overline{\Gamma_1}, \overline{\Gamma_2}, \cdots) \in \mathbb{C}\oplus\ell^2(\mathbb{N})$,

and let

(4.18) $\quad H_I := A^+(\gamma)A(e_0) + A^+(e_0)A(\gamma)$.

The operator H_I describes the Hamiltonian of the oscillator interacting with infinitely many scalar bosons . We note that

(4.19) $\quad H_I = \Sigma_{k=1}^{\infty}(\overline{\Gamma}_k b_k^+ a + \Gamma_k a^+ b_k) \quad$ on $D^{(0)}$.

Since the operator H_I is well-defined on $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ and symmetric , we denote the closure of $H_I \lceil \mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ by the same symbol . Then , the total Hamiltonian H is given by

(4.20) $\quad H := H_0 + H_I$.

We have $D(H_0) \subset D(H)$, and the closure of $H\lceil D(H_0)$ is essentially self-adjoint on any core for H_0. We shall denote the closure of $H\lceil D(H_0)$ by the same symbol H .

We define a linear operator $\mathbb{L}$ on $\mathbb{C}\oplus\ell^2(\mathbb{N})$ as follows :

(4.21) $\quad D(\mathbb{L}) := E = \mathrm{L.h.}[\{e_k ; k\in\mathbb{N}^*\}]$

(4.22) $$\mathbb{L}\, e_k := \begin{cases} \omega_0 e_0 + \Sigma_{\ell=1}^{\infty} \dfrac{\overline{\Gamma_\ell}}{\hbar} e_\ell & (k=0) \\ \dfrac{\Gamma_k}{\hbar} e_0 + \omega_k e_k & (k=1,2,\cdots) . \end{cases}$$

It is easy to see that the operator $\mathbb{L}$ can be extended to a closed symmetric operator on $\mathbb{C}\oplus\ell^2(\mathbb{N})$, and E is a core for its extension .

Lemma 4.1. (a) $\mathbb{L}$ is self-adjoint , and there exists a unitary operator $\mathbb{U}$ on $\mathbb{C}\oplus\ell^2(\mathbb{N})$ such that $\mathbb{U}^*\mathbb{L}\mathbb{U}e_p = \epsilon_p e_p$, where $\epsilon_p>0$ for any $p\in\mathbb{N}^*$ and $\{\epsilon_p;p\in\mathbb{N}^*\}$ is the all zeros of $D(z):=z - \omega_0 + \Sigma_{k=1}^{\infty}\frac{|\Gamma_k|^2}{\hbar^2(\omega_k-z)}$.

(b) Put $\langle e_k|\mathbb{U}e_p\rangle=:u_{kp}$ $(k,p\in\mathbb{N}^*)$, then

$$u_{kp}=-\frac{\overline{\Gamma_k}}{\hbar(\omega_k-\epsilon_p)}u_{0p} \qquad (k\in\mathbb{N};p\in\mathbb{N}^*) ,$$

$$\epsilon_p - \omega_0 + \Sigma_{k=1}^{\infty}\frac{|\Gamma_k|^2}{\hbar^2(\omega_k-\epsilon_p)} = 0 \qquad (p\in\mathbb{N}^*) ,$$

$$|u_{0p}|^2 = \{1+\Sigma_{k=1}^{\infty}\frac{|\Gamma_k|^2}{\hbar^2(\omega_k-\epsilon_p)^2}\}^{-1} \qquad (p\in\mathbb{N}^*) .$$

Let $\Gamma(\mathbb{U})$ be a unitary operator on $\mathcal{F}_S(\mathbb{C}\oplus\ell^2(\mathbb{N}))$ defined by

$$(4.23) \qquad \Gamma(\mathbb{U}) \upharpoonright S_n(\mathbb{C}\oplus\ell^2(\mathbb{N}))^n = \otimes_{k=1}^{n}\mathbb{U} \quad (n \in \mathbb{N}) ,$$

$$(4.24) \qquad \Gamma(\mathbb{U}) \upharpoonright S_0(\mathbb{C} \oplus \ell^2(\mathbb{N}))^0 = I \quad \text{(the identity)} ,$$

and define for each $f \in \mathbb{C}\oplus\ell^2(\mathbb{N})$

$$(4.25) \qquad \begin{cases} \mathcal{B}^+(f) := \Gamma(\mathbb{U})A^+(f)\Gamma(\mathbb{U})^{-1} , \\ \mathcal{B}\ (f) := \Gamma(\mathbb{U})A\ (f)\Gamma(\mathbb{U})^{-1} . \end{cases}$$

We put

$$(4.26)\quad \begin{cases} \beta_k^+ := \beta^+(e_k) \ , \\ \beta_k := \beta\,(e_k) \ . \end{cases} \qquad (k \in \mathbb{N}^*)$$

Since $\mathbb{U}$ is unitary on $\mathbb{C}\oplus\ell^2(\mathbb{N})$, the following commutation relations hold on $\mathcal{F}_F(\mathbb{C}\oplus\ell^2(\mathbb{N}))$, for all $f,g \in \mathbb{C}\oplus\ell^2(\mathbb{N})$.

$$(4.27)\quad [\beta(f),\beta^+(g)] = \langle f|g\rangle \ ,$$

$$(4.28)\quad [\beta(f),\beta(g)] = 0 \ , \ [\beta^+(f),\beta^+(g)] = 0 \ .$$

Lemma 4.2. $\sigma(H)=\{\hbar\epsilon_0 n_0+\cdots+\hbar\epsilon_N n_N ; N\in\mathbb{N}^*, n_0,\cdots,n_N\in\mathbb{N}^*\}$.

Corollary: H satisfies conditions (H.1) and (H.2) .

Lemma 4.3. (a) β_p, $\beta_p^+\in\mathcal{J}(D,H)$ are H-diagonal .

(b) $$\sigma_{\beta_p} = -\epsilon_p = -\sigma_{\beta_p^+} \qquad (p \in \mathbb{N}^*) \ .$$

Lemma 4.4. a and $a^+\in\mathfrak{C}_\eta(H)$ $\quad (\eta\in(0,\beta))$.

Lemma 4.5. (a) $\omega(a)=-(\mathbb{L}^{-1})_{00}^{-1}$

where $\omega(a)$ denotes the Mori's frequency of a .

(b) For $z\in\mathbb{C}^+$, $$\int_0^\infty dt\ \phi(t)e^{itz} = i\omega(a)\cdot\frac{F(z)}{1+F(z)} \ ,$$

where ϕ is the Mori's memory function of a ,

$$F(z) := \Sigma_{\substack{p=0\\p\neq k}}^{\infty} \frac{\overline{(\mathbb{L}^{-1})_{kp}}((z-\mathbb{L})^{-1})_{kp}}{\overline{(\mathbb{L}^{-1})_{kk}}((z-\mathbb{L})^{-1})_{kk}} \ ,$$

and , for any operator T on $\mathbb{C}\oplus\ell^2(\mathbb{N})$, $T_{kp}=\langle e_k|Te_p\rangle$.

Theorem 4.6. (a) Mori's Langevin equation for $[a](t)$ is

(4.29) $$\frac{d}{dt}[a](t)=iD(0)[a](t)-\int_0^t ds\ \phi(t-s)[a](s)+[I_M](t)\ .$$

Here the Mori's memory function ϕ is characterized by

(4.30) $$\frac{1}{2\pi}\int_0^\infty ds\ e^{itz}\phi(t) = \frac{iD(0)F(z)}{2\pi(1+F(z))} \qquad (z\in\mathbb{C}^+)\ ,$$

and $\phi(0)=-D(0)\Delta=-\omega(a)\Delta$, where

(4.31) $$F(z):=\Sigma_{p=1}^\infty\frac{|\Gamma_p|^2}{\hbar^2\omega_p(\omega_p-z)} \quad \text{and} \quad \Delta=\Sigma_{k=1}^\infty\frac{|\Gamma_k|^2}{\hbar^2\omega_k}\ ,$$

and Mori noise $[I_M](t)$ satisfies

(4.32) $$<[I_M](0);[I_M](0)>_H=-\frac{\phi(0)}{\beta\hbar\omega(a)}=\frac{1}{\beta\hbar}\Delta\ .$$

(b) $$R_a(t) = <[a];[a](t)>_H$$
$$= \frac{1}{\beta\hbar}\Sigma_{p=0}^\infty\frac{1}{|D'(\epsilon_p)|}\cdot\frac{e^{-it\epsilon_p}}{\epsilon_p}$$

is an almost periodic function .

Remark 4.1. The equality $\omega(a)=D(0)$ physically means that Mori's frequency is equal to the difference between the frequency ω_0 of a quantum harmonic oscillator and the initial value of the canonical correlation function of Mori noise multiplied by $\beta\hbar$. In [11], K.Lindenberg and B.J.West gave attention to the quantity Δ . (4.32) gives such a physical meaning that Δ is the initial value of the canonical correlation function of Mori noise multiplied by $\beta\hbar$.

We note that $R_a(t)$ does not converge as $t \longrightarrow \infty$.

References

[1] J.-P.Antoine and W.Karwowski , Partial *-algebras of closed linear operators in Hilbert space ,Publ.RIMS,**21**, (1985) 205-236 .

[2] A.Arai , Spectral analysis of a quantum harmonic oscillator coupled to infinitely many scalar bosons , to be published in J.Math.Anal.Appl . .

[3] F.A.Berezin , *The method of second quantization* , Academic Press , New York ,(1966)

[4] E.Braun , Irreversible behavior of a quantum a quantum harmonic oscillator coupled to a heat bath,Physica 129A, (1985) 262-301 .

[5] O.Bratteli and D.W.Robinson , *Operator Algebras and Quantum Statistical Mechanics* II , Springer-Verlag , New York Inc .,(1981)

[6] H.Ezawa and A.Arai, *Quantum Field Theory and Statistical Mechanics* , Nihon Hyōron Shya ,(1988) (in Japanese)

[7] D.Forster , *Hydrodynamic fluctuations ,broken symmety , and correlation functions* , Benjamin ,(1975)

[8] J.Glimm and A.Jaffe , *Quantum physics*, Springer-Verlag, (1981)

[9] H.Haken , Laser Theory , in **Vol.XXV/2C** of Encyclopedia of Physics , Springer , Berlin ,(1970)

[10] R.Kubo, Statistical-mechanical theory of irreversible processes .I : General theory and simple applications to magnetic and conduction problems , J.Phys.Soc.Japan, **12** .6 ,(1957) , 570-586 .

[11] K.Lindenberg and J.West , Statistical properties of quantum systems : The linear oscillator , Phys.Rev.,**30**, 1 ,(1984), 568-582 .

[12] H.Mori , Transport,collective motion , and Brownian motion,Prog.Theo.Phys., Vol.**33** ,No.3 ,(1965), 423-455 .

[13] ——— , A continued - fraction representation of the time-correlation functions ,Prog.Theo.Phys. ,**34** , 3 , (1965), 399-416 .

[14] H.Nakazawa , in proceedings of the third university of California conference on statistical mechanics , (1988), to appear in Nucl.Phys. B supplement .

[15] Y.Okabe , On a Langevin equation ,
Sugaku , 33 ,(1981), 306-324 (in Japanese) .

[16] ―――― , KMO - Langevin equation and fluctuation-dissipation theorem (I), Hokkaido Mathematical Journal, 15 ,2 ,(1986), 163-216 .

[17] ―――― , KMO - Langevin equation and fluctuation-dissipation theorem (II), Hokkaido Mathematical Journal , 15 ,3 ,(1986), 317-355 .

[18] E.Prugovečki , *Quantum Mechanics in Hilbert Space* , (second edition) , Acadmic Press ,New York ,(1981)

[19] ―――――― and A.Tip,Semi-groups of rank-preserving transformers on minimal norm ideals in $\mathcal{B}(\mathcal{H})$, Comp.Math. 30 ,(1975), 113-136 .

[20] M.Reed and B.Simon , *Methods of Modern Mathematical Phisics* Vol.I :Acadmic Press, New York ,(1972)

[21] ―――――――――― , *Methods of Modern Mathematical Phisics* Vol.II : Acadmic Press ,New York ,(1975)

[22] ―――――――――― , *Methods of Modern Mathematical Phisics* Vol.III: Acadmic Press , New York ,(1978)