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# Notes on $C^0$ determinancy of analytic functions related to weights

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Let $\mathcal{E}_{[k]}(n,1)$ be the set of $C^k$ function germs : $(\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$ for $k = 1, 2, \cdots, \infty, \omega$, and let $\mathcal{H}(n,1)$ be the set of holomorphic function germs : $(\mathbb{C}^n,0) \longrightarrow (\mathbb{C},0)$ . If for two function germs $f$ , $g \in \mathcal{E}_{[k]}(n,1)$ ( resp. $\mathcal{H}(n,1)$ ) there exists a local homeomorphism $\sigma : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ ( resp. $\sigma : (\mathbb{C}^n,0) \longrightarrow (\mathbb{C}^n,0)$ ) such that $f = g \circ \sigma$ , we say that $f$ is $C^0$-equivalent to $g$ and write $f \sim g$ . We shall not distinguish between germs and their representatives.

Consider the polynomial function $f : (\mathbb{R}^2,0) \longrightarrow (\mathbb{R},0)$ defined by

$$f(x,y) = x^3 + 3xy^{20} + y^{29} .$$

Then we see that

$$x^3 + 3xy^{20} \overset{(i)}{\sim} x^3 + 3xy^{20} + y^{29} \overset{(ii)}{\sim} x^3 + y^{29} .$$

Here we interpret the above equivalences as follows ( see [6] Example 4.3 also ) :

(i) Put $w = j^{21}f(0) = x^3 + 3xy^{20}$ . We identify r-jets with their polynomial representatives of degree not exceeding $r$ . Then there exist $C$ , $\alpha > 0$ such that

$$|\operatorname{grad} w(x,y)| \geq C\, |(x,y)|^{20} \quad \text{for} \quad |(x,y)| < \alpha .$$

Therefore it follows from Kuiper-Kuo theorem that $w$ is $C^0$-equivalent to $f$ .

(ii) Put $z = x^3 + y^{29}$ . Then $z$ is a weighted homogeneous polynomial of type $(\frac{1}{3}, \frac{1}{29})$ with a finite codimension and the weight of the term $3\,x\,y^{20}$ is $\frac{1}{3} + \frac{20}{29} > 1$ . Therefore $z$ is $C^0$-equivalent to $f$ ( see V. I. Arnol'd [1] ).

In the complex case, the equivalence (i) does not hold. For $w$ is a weighted homogeneous polynomial of type $(\frac{1}{3}, \frac{1}{30})$ with an isolated singularity and the weight of the term $y^{29}$ is $\frac{29}{30} < 1$ . Furthermore $y^{29} \notin \mathfrak{M}(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$ . Therefore $w$ is not $C^0$-equivalent to $w + y^{29} = f$ ( see M. Suzuki [16] or A. N. Varčenko [18] ). ( Of course, we can see this fact directly by considering the $C^0$-type of $w^{-1}(0)$ and $f^{-1}(0)$ , as germs at $0 \in \mathbb{C}^2$ . ) Even in the real case, the equivalence (i) does not hold, if we replace plus by minus ( i.e. $w = x^3 - 3\,x\,y^{20}$ ). This difference between them on $C^0$-type of functions is one of special qualities in the real case. Thus we have the following

PROBLEM. Is there a discription explaining the above interpretations simultaneously ?

The purpose in this paper is to give Kuiper-Kuo type's theorem for real analytic functions of at most three variables related to weights as the answer to the above problem, and the corresponding result for holomorphic functions of general n variables ( $n \neq 3$ ). We shall describe the results and corollaries in §1. In §2, we shall apply the real result for the above example, and explain our results.

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## §1. Main results.

Let $\mathbb{Q}^+$ ( resp. $\mathbb{R}^+$ ) denote the set of positive rational numbers ( resp. positive real numbers ). For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$ with $\min_{1 \leq j \leq n} \alpha_j = 1$ , we define the subset of $\mathbb{Q}^+$ by

$$I(\alpha) = \{ \sum_{j=1}^{n} \alpha_j \beta_j \mid \beta_j \in \mathbb{N} \cup \{0\} \ ( 1 \leqq j \leqq n ), \beta_1 + \cdots + \beta_n \geqq 1 \} .$$

Then we can express

$$I(\alpha) = \{ i(N) \mid 0 < i(1) < i(2) < \cdots < i(N) < \cdots \ ( N \in \mathbb{N} ) \} .$$

We put $\delta_N = i(N+1) - i(N) > 0$ . Note that $\delta_N$ is constant for properly large $N$ .

REMARK 1. (1) When we consider the concept of the weights, we are not interested in their quantity, but the ratio of them. Therefore, as the representative of the equivalence class

$\{ ( \frac{\alpha_1}{r}, \cdots, \frac{\alpha_n}{r} ) \mid r \in \mathbb{R}^+ \} / \sim$ , we adopt $\alpha = ( \alpha_1, \cdots, \alpha_n )$ $\in \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$ with $\min_{1 \leq j \leq n} \alpha_j = 1$ for convinience' sake.

For this adoption is the most desirable for our results which we shall state below. Namely, the case $\min_{1 \leq j \leq n} \alpha_j > 1$ is covered by this one, and the case $\min_{1 \leq j \leq n} \alpha_j < 1$ is not valid.

(2) In this paper, we think the ratio of weights only in rational numbers. Because, if the theorems hold for an irrational ratio of weights, then they hold for sharper rational one ( refer to J. Bochnak - J.-J. Risler [2], T. C. Kuo [12] ).

Let $i(N) \in I(\alpha)$ , and $f$ , $g$ be $C^\omega$ functions where

$$f(x) = \sum_{\beta \in \mathcal{B}} A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{and} \quad g(x) = \sum_{\gamma \in \Gamma} B_\gamma x_1^{\gamma_1} \cdots x_n^{\gamma_n} .$$

We say that $f$ and $g$ are i(N)-equivalent, if

$$\overline{\sum_{\beta \in \mathcal{B},\ \alpha_1\beta_1 + \cdots + \alpha_n\beta_n \leq i(N)}} A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}$$

$$= \overline{\sum_{\gamma \in \Gamma,\ \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n \leq i(N)}} B_\gamma x_1^{\gamma_1} \cdots x_n^{\gamma_n} .$$

This relation $\widetilde{\ _{i(N)}\ }$ is an equivalence relation. Then we denote by $J_\alpha^{i(N)}(n,1)$ the quotient set of $\mathcal{E}_{[\omega]}(n,1)$ and by $j_\alpha^{i(N)} f(0)$ the equivalence class of $f$ by the relation $\widetilde{\ _{i(N)}\ }$ .

We identify $j_\alpha^{i(N)}f(0)$ with its polynomial representative

$$\sum_{\beta \in \mathcal{B},\ \alpha_1\beta_1+\cdots+\alpha_n\beta_n \leqq i(N)} A_\beta\, x_1^{\beta_1} \cdots x_n^{\beta_n} .$$

Let $f \in \mathcal{E}_{[\omega]}(n,1)$ such that $0 \in \mathbb{R}^n$ is a singular point of $f$ . For $i(N) \in I(\alpha)$ with $\alpha_j < i(N)$ ( $1 \leqq j \leqq n$ ) we define

$$\tilde{\partial}_{i(N)}f(x) = \left( \left|\frac{\partial f}{\partial x_1}(x)\right|^{(i(N)-1)/(i(N)-\alpha_1)} ,\right.$$

$$\left. \cdots ,\ \left|\frac{\partial f}{\partial x_n}(x)\right|^{(i(N)-1)/(i(N)-\alpha_n)} \right) .$$

near $0 \in \mathbb{R}^n$ .

REMARK 2. If $0 \in \mathbb{R}^n$ is an isolated singular point of $H = j_\alpha^{i(N)}f(0)$ , then we have $\alpha_j < i(N)$ ( $1 \leqq j \leqq n$ ) . Therefore $\tilde{\partial}_{i(N)}H(x)$ and $\tilde{\partial}_{i(N)}f(x)$ are defined.

NOTATION. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$ . We put

$$|x|_\alpha = \sqrt{|x_1|^{2/\alpha_1} + \cdots + |x_n|^{2/\alpha_n}}$$

for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ .

Let $J$ ( $\supset [0, 1]$ ) be an open interval. Consider the $C^\omega$ function $F : (\mathbb{R}^n \times J, \{0\} \times J) \longrightarrow (\mathbb{R},0)$ . We shall denote a family of germs by $f_s : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ , $s \in J$ where $f_s(x) = F(x,s)$ . A family of germs $\{ f_s \mid s \in J \}$ or $F : (\mathbb{R}^n \times J, \{0\} \times J) \longrightarrow (\mathbb{R}, 0)$ is said to have no coalescing of critical points, if there exists $a > 0$ such that

$$|\mathrm{grad}\, f_s(x)| \neq 0 \quad \text{for} \quad 0 < |x| < a \quad \text{and} \quad s \in J .$$

Put $\Lambda = \{0\} \times J \subset \mathbb{R}^n \times J$ . Let $F$ be non-singular outside $\Lambda$ . Then, the pair $( \mathbb{R}^n \times J - \Lambda , \Lambda )$ is called $(a_F)$-regular at $(0,s) \in \Lambda$ , if for any sequence of points $\{p_n\}$ in $\mathbb{R}^n \times J - \Lambda$ tending to $(0,s) \in \Lambda$ such that the plane $T_{p_n} F^{-1}(F(p_n))$ tends to $\tau$ , we have $\tau \supset \Lambda$ . The pair $( \mathbb{R}^n \times J - \Lambda , \Lambda )$ is called $(a_F)$-regular , if it is $(a_F)$-regular at any $(0,s) \in \Lambda$ . In the complex case, we can define the concept of no coalescing of critical points and $(a_F)$-regularity similarly.

Now we state the main results in this paper.

PROPOSITION. Let $H \in J_\alpha^{i(N)}(n,1)$ , $G \in \mathcal{E}_{[\omega]}(n,1)$ with $j_\alpha^{i(N)} G(0) = 0$ , and let $F(x,s) = H(x) + s\, G(x)$ for $s \in J$ . If there exist $C$ , $a > 0$ and $\varepsilon > 1 - \delta_N$ such that

$$|\widetilde{\partial}_{i(N)} H(x)| \geqq C\, |x|_\alpha^{\,i(N)-\varepsilon} \quad \text{for} \quad |x| < a ,$$

then $F$ has no coalescing of critical points. Furthermore the pair $( \mathbb{R}^n \times J - \Lambda , \Lambda )$ is $(a_F)$-regular.

THEOREM 1. Let $H \in J_\alpha^{i(N)}(n,1)$ ( $n \leqq 3$ ). If there exist $C$ , $a > 0$ and $\varepsilon > 1 - \delta_N$ such that

$$|\tilde{\partial}_{i(N)}H(x)| \geqq C\,|x|_\alpha^{\,i(N)-\varepsilon} \quad \text{for} \quad |x| < a\ ,$$

then for any $G \in \mathcal{E}_{[\omega]}(n,1)$ with $j_\alpha^{i(N)}G(0) = 0$ , $H + G$ is $C^0$-equivalent to $H$ .

REMARK 3. (1) If $0 \in \mathbb{R}^n$ is a regular point of $H$ , then for any $G \in \mathcal{E}_{[\omega]}(n,1)$ with $j_\alpha^{i(N)}G(0) = 0$ , $H + G$ is $C^0$-equivalent to $H$ ( any $\alpha \in \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$ and $i(N) \in I(\alpha)$ ) This follows from the implicit function theorem.

(2) Since $1 \geqq \delta_N > 0$ , we have $\varepsilon > 1 - \delta_N \geqq 0$ .

(3) In [7], the author proposed a partition problem on real analytic functions. Concerning an attempt to it, we have obtained this theorem.

PROBLEM. Does the above theorem for $n \geqq 4$ hold ? ( See Remark 6 in §3 also. )

Let [ ] denote the Gauss symbol.

COROLLARY 1. Suppose $H \in J_\alpha^{i(N)}(n,1)$ ( $n \leqq 3$ ) satisfies the hypothesis of Theorem 1. Then $H + G$ is $C^0$-equivalent to $H$ for any $G \in \mathcal{E}_{[[i(N)]+1]}(n,1)$ with $j_\alpha^{i(N)}G(0) = 0$ .

REMARK 4. If $\alpha_1\beta_1 + \cdots + \alpha_n\beta_n \leqq i(N)$ , then $\beta_1 + \cdots + \beta_n \leqq [i(N)]$ . Therefore we can define $J_\alpha^{i(N)}(n,1)$ in $\mathcal{E}_{[k]}(n,1)$ for $k = [i(N)]$ , $[i(N)] + 1$ , $\cdots$.

In this paper, a polynomial $H(x_1,\cdots,x_n) = \sum_{\beta\in\mathcal{B}} C_\beta x_1^{\beta_1}\cdots x_n^{\beta_n}$ is called weighted homogeneous of type $(\frac{\alpha_1}{r},\cdots,\frac{\alpha_n}{r})$ where $r$, $\alpha_j$ $(1\leqq j\leqq n)\in\mathbb{Q}^+$, if

$$\min_{1\leqq j\leqq n}\alpha_j = 1 \quad\text{and}\quad \frac{\alpha_1}{r}\beta_1+\cdots+\frac{\alpha_n}{r}\beta_n = 1$$

for any multiindex $\beta = (\beta_1,\cdots,\beta_n)\in\mathcal{B}$.

COROLLARY 2. Let $H:(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$ $(n\leqq 3)$ be a weighted homogeneous polynomial of type $(\frac{\alpha_1}{r},\cdots,\frac{\alpha_n}{r})$ with an isolated singularity, and let $G\in\mathcal{E}_{[[r]+1]}(n,1)$ such that $j^{[r]+1}G(0) = \sum_{\beta\in\mathcal{B}} A_\beta x_1^{\beta_1}\cdots x_n^{\beta_n}$. If $\sum_{j=1}^{n}\frac{\alpha_j}{r}\beta_j > 1$ (resp. $\sum_{j=1}^{n}\frac{\alpha_j}{r}\beta_j \geqq 1$) for any $\beta\in\mathcal{B}$, then $H+G$ is $C^0$-equivalent to $H$ (resp. then there exists $\varepsilon > 0$ such that $H+sG$ is $C^0$-equivalent to $H$ for $|s|<\varepsilon$).

REMARK 5. ( J. Damon - T. Gaffney [4] Corollary 5 ) If $H$ has an algebraically isolated singularity, then the above corollary holds for general n variables case. ( See the proof of Corollary 1 in §3 also. )

In the complex case, we define $I(\alpha)$, $\delta_N$, $J_\alpha^{i(N)}(n,1)$, $\tilde{\partial}_{i(N)}$, and $|\ |_\alpha$ for a given system of the weights $\alpha$

$= (\alpha_1, \cdots, \alpha_n)$ with $\min_{1 \leq j \leq n} \alpha_j = 1$, similarly as the real case. Then we have the corresponding complex result to Theorem 1.

THEOREM 2. Let $H \in J_\alpha^{i(N)}(n,1)$ ( $n \neq 3$ ). If there exist $C$, $a > 0$ and $\varepsilon > 1 - \delta_N$ such that

$$|\tilde{\partial}_{i(N)} H(x)| \geq C\, |x|_\alpha^{\,i(N)-\varepsilon} \quad \text{for} \quad |x| < a\,,$$

then for any $G \in \mathcal{H}(n,1)$ with $j_\alpha^{i(N)} G(0) = 0$, $H + G$ is $C^0$-equivalent to $H$.

§2. Applications.

Let $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ be a $C^\omega$ function with an isolated singularity. Then, for any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^+ \times \cdots \times \mathbb{Q}^+$ with $\min_{1 \leq j \leq n} \alpha_j = 1$, there exist $i(N) \in I(\alpha)$, $C$, $a > 0$, and $\varepsilon > 1 - \delta_N$ such that

$$|\tilde{\partial}_{i(N)} H(x)| \geq C\, |x|_\alpha^{\,i(N)-\varepsilon} \quad \text{for} \quad |x| < a\,,$$

where $H(x) = j_\alpha^{i(N)} f(0)$. Therefore Theorem 1 is saying that every system of weights given, a $C^\omega$ function ( $n \leq 3$ ) with an isolated singularity has the $C^0$ property related to it.

( The figure is inserted here. )

Let us go back to the example in the introduction. Then we can interpret the equivalence (i) by applying Theorem 1 with the system of weights $\alpha = (1, 1)$ ( $i(N) = 21$ ), and the equivalence (ii) by applying one with $\alpha = (\frac{29}{3}, 1)$ ( $i(N) = 29$ ). Furthermore the theorem holds as $\varepsilon = 1$ in both cases. Therefore, by Kuiper-Kuo theorem and Corollary 2, $f$ is controling not only $C^0$-type of terms inside the shaded region in Figure, but also one of local terms ( their coefficients are sufficiently small ) on the boundary.

For the problem on topological triviality of deformations of a complex function germ, there are interesting works using the conditions on the Newton boundary ( e.g. V.I. Arnol'd [1], A.G. Kouchnirenko [8], M. Oka [15], J. Damon - T. Gaffney [4], M. Buchner - W. Kucharz [3] ). The works [3] and [4] contain the corresponding real results, too. On the other hand, E. Yoshinaga has shown $\pi$-MAT ( which is stronger than topological triviality ) under the assumptions on the Newton boundary in the real case ( [19] ). The approaches based on the Newton boundary are very effective for the problem of topological triviality in the complex case. In fact, this is intrinsic in the semi-quasihomogeneous case ( V. I. Arnol'd [1], M. Suzuki [16], A. N. Varčenko [18] ). But the above example shows that this is not neccesarily so in the real case. Therefore we formulate our results on $C^0$ determinancy of analytic functions by using an arbitary system of weights, apparting from the Newton boundary.

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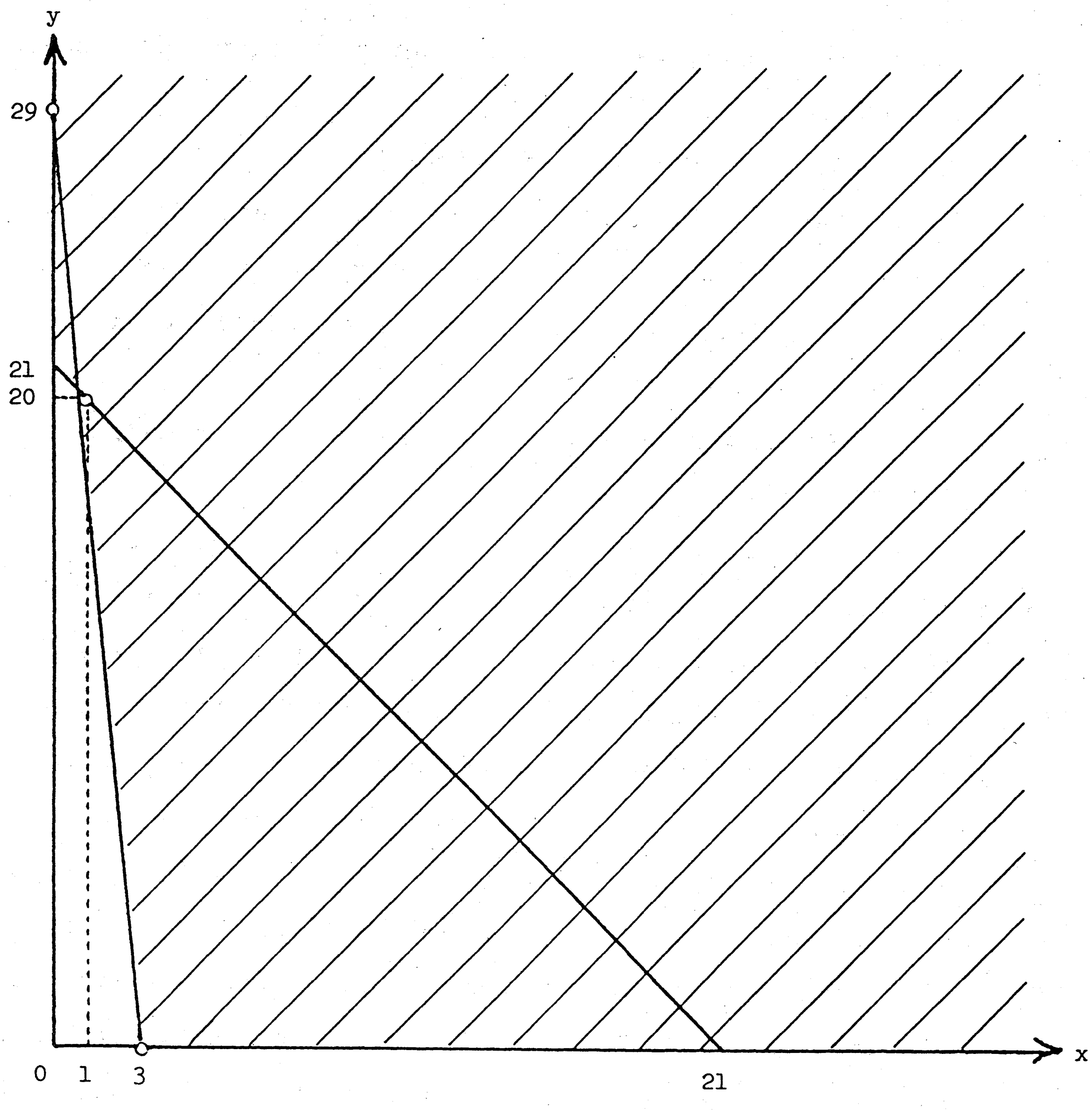


( Figure )