Algebraic Riemann manifolds

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We give a criterion by which we decide whether two given Riemann manifolds $M, \overline{M}$ are isometric or not. We recall the following classical theorem.

**Theorem ( $C^\omega$ Isometry Theorem ).** Let $M, \overline{M}$ be real analytic Riemann manifolds of dimension $n$. Let $p \in M$, $\overline{p} \in \overline{M}$. Suppose that there exists a linear isometry $I : T_p(M) \rightarrow T_{\overline{p}}(\overline{M})$ which preserves the curvature tensors $R, \overline{R}$, and their covariant differentials $\nabla^k R, \nabla^k \overline{R}$ of any order $k$. Then the mapping $I$ can be extended to an isometry $h$ between neighborhoods of $p, \overline{p}$. Hence in particular if $M, \overline{M}$ are complete, connected, and simply connected, then $M, \overline{M}$ are isometric.

By replacing $C^\omega$ with the Nash category $C^\Omega$, and introducing the notion "minimal differential polynomial" $\phi_M$ of a $C^\Omega$ Riemann manifold $M$, we observe that the proof of this theorem implies the following criterion.

**Theorem 1.** Let $M, \overline{M}$ be $C^\Omega$ Riemann manifolds of dimension $n$. Let $p \in M$, $\overline{p} \in \overline{M}$. Suppose that

1. the minimal differential polynomials $\phi_M, \phi_{\overline{M}}$ coincide,
2. the two point $p, \overline{p}$ are "nonsingular" with respect to $\phi_M, \phi_{\overline{M}}$, respectively, and
3. there exists a linear isometry $I : T_p(M) \rightarrow T_{\overline{p}}(\overline{M})$ which preserves the curvature tensors $R, \overline{R}$, and their first $4n - 5$ covariant differentials $\nabla^k R, \nabla^k \overline{R}$. 
Then the mapping $I$ can be extended to an isometry $h$ between neighborhoods of $p, \bar{p}$.

As an application we obtain

**Theorem 2.** Let $M$ be a compact $C^\Omega$ Riemann manifold of dimension $n$. Suppose that $M$ is nowhere homogeneous, i.e. for any distinct points $p, q$ of $M$, there exists no isometry $h$, $h(p) = q$, between neighborhoods of $p, q$. Then $M$ is $C^\Omega$ embeddable, and the embedding is given by means of general scalar curvatures. If any point of $M$ is nonsingular with respect to $\phi_M$, then some finite number of general scalar curvatures of order at most $4n-5$ give a one to one mapping of $M$ into a vector space.

**Reference**