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A STRONG RELATIVE VERSION OF WEIERSTRASS APPROXIMATION THEOREM

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I want to expose a strong relative version of Weierstrass approximation theorem, which seems very usefull in real algebraic geometry.

Problem: Let $\phi:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^∞ function such that $Y = \phi^{-1}(0)$ has a good structure (i.e. is an algebraic or an analytic subvariety of $\mathbb{R}^n$) When is it possible to approximate ϕ by good functions, i.e. rational regular, polynomial or analytic functions $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = \phi^{-1}(0)$?

In general the answer is negative: for instance consider the following function which vanishes on $x \cdot y = 0$

$$\phi(x,y) = x^2y^2\left[x + y - \sqrt{x^2+ y^2}\right] \cdot \exp\left[- \frac{1}{x^2+ y^2}\right]$$

The sign of ϕ is as in the figure below:

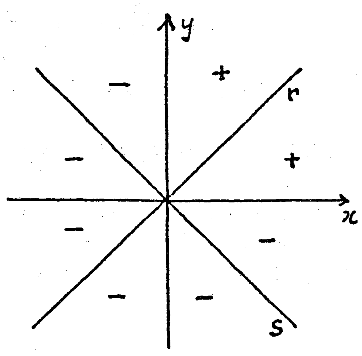


figure 1

$$\phi^{-1}(0) = \{x \cdot y = 0\}$$

$\phi(x,y) > 0$ if $x > 0$ and $y > 0$

$\phi(x,y) < 0$ if $x < 0$ or $y < 0$

If f is an analytic function and approximates ϕ with the same zero set, then f has the same distribution of signs but this is impossible for an analytic function: indeed take the Taylor series of f at the origin and call q the degree of the first non zero coefficient: we deduce that q is odd by taking $f_{|r}$ where $r = \left\{ x = \lambda y \text{ , } \lambda > 0 \right\}$ and that q is even by taking $f_{|s}$ where $s = \left\{ x = \lambda y \text{ , } \lambda < 0 \right\}$. This is a contradiction.

The above problem has several solutions that can be summarised in the:

<u>Theorem</u> : ϕ can be approximated in $C^0_w(\mathbb{R}^n)$ by polynomials f such that $f^{-1}(0) = \phi^{-1}(0)$ if and only if there exists a polynomial function $p:\mathbb{R}^n \longrightarrow \mathbb{R}$ with the same zero set such that $p(x)\phi(x) \geq 0 \quad \forall x \in \mathbb{R}^n$.
Moreover, if Y is "almost regular" and of pure codimension 1 then the approximation is in $C^s_w(M)$ $\forall s \leq \infty$.
The statement in the analytic case is very similar: when Y has codimension one we must suppose that it has only a finite number of irreducible components; the approximation is in the strong topology; when Y has codimension one and it is coherent, the approximation is also with derivatives.

The meaning of "almost regular" will be explained later.

As one can see, solutions are different following the different cases.

The first distinction is wether the following statement is true or not.

(*) If $\phi \in C^{\infty}(\mathbb{R}^n)$ vanishes on Y , then one can write

$$\phi = \sum \alpha_i p_i$$

where $\{p_i\}$ is a system of generators for $I(Y)$ and $\alpha_i \in C^{\infty}(\mathbb{R}^n)$.

α) The statement (*) is true.

This is the case if Y is an almost regular algebraic variety, or it is a coherent analytic space.

Almost regular means that:

$$\mathcal{I}_{Y,y} = I(Y)\cdot\mathcal{A}_{\mathbb{R}^n,y}$$

(The stalk of the ideal sheaf of germs of analytic functions vanishing on Y is generated as $\mathcal{A}_{\mathbb{R}^n,y}$- module by the polynomials vanishing on Y, for any y in Y.)

One can show that almost regularity implies coherence, see [T]; by Malgrange theorem, coherence implies that

$$\mathcal{I}^{\mathcal{E}(U)}_{Y,y} = \mathcal{I}_{Y,y}\cdot\mathcal{E}(U)$$

where U is an open neighborhood of Y and $\mathcal{I}^{\mathcal{E}(U)}_{Y,y}$ is the stalk of the sheaf of germs of smooth functions vanishing on Y. So one has:

$$\mathcal{I}^{\mathcal{E}(U)}_{Y,y} = I(Y)\cdot\mathcal{E}(U).$$

By a suitable partition of unity one obtains (*).

Now we have two possibilities:

A) If ϕ does not change sign: we can suppose $\phi \geq 0$.

Step 1. Consider the function $\sqrt{\phi} \in C^0(\mathbb{R}^n)$. It is easy to approximate it by a function $\psi \in C^\infty(\mathbb{R}^n)$ vanishing on Y (see for instance [Hi]).

Step 2. Apply (*) to ψ and write $\psi = \sum \alpha_i p_i$ where p_i are generators of I(Y).

Approximate α_i by a polynomial q_i; then $q = \left[\sum q_i p_i\right]^2$ is a polynomial such that $q^{-1}(0) \supset Y$, $q \geq 0$ and q approximates ϕ. So $q + \eta \sum p_i^2$ is the required one for a suitable positive constant η.

If codim Y = 1 this approximation can be taken in $C^s_W(\mathbb{R}^n)$; in fact, always by (*), $\phi = \phi' p_Y$, where p_Y is a generator of I(Y); but p_Y changes sign in every point of maximal dimension while ϕ has constant sign, so we have $\phi' = \psi\, p_Y$ with sign ψ = sign ϕ.

We can approximate the smooth function $\sqrt{\psi + \delta}$, with small positive δ, by a positive polynomial function P, in the C^s-topology. The function $P^2 \cdot p_Y^2$ gives the desired approximation of ϕ.

B) If ϕ changes sign: this implies that Y has codimension one.

As we have seen, in this case ϕ may do not have a good approximation.

I need some definitions. Remember that if Y is analytic, we suppose that it has only a finite number of irreducible components.

A continous function $\sigma:\mathbb{R}^n - Y \longrightarrow \mathbb{Z}_2$ will be called a signature on $\mathbb{R}^n - Y$.

<u>Definition</u> :A signature σ on $\mathbb{R}^n - Y$ is called <u>admissible</u>:

1-if Y is <u>irreducible</u>, when it is induced by one of the following polynomials: p, $-p$, p^2, $-p^2$, where p is a generator of the ideal $I(Y)$ of polynomials vanishing on Y.

2-if Y is <u>reducible</u> and $\{Y_i\}_{i=1,...,k}$ are the irreducible components, when $\sigma = \prod \sigma_i$ where σ_i is an admissible signature on $\mathbb{R}^n - Y_i$, $i = 1,...,k$,

<u>Definition</u>:

-A point $P \in Y$ is called a <u>change point</u> with respect to σ (we say also that σ changes sign at P) if for any neighborhood U of P there exist $P_1, P_2 \in U - Y$ such that $\sigma(P_1) \neq \sigma(P_2)$.

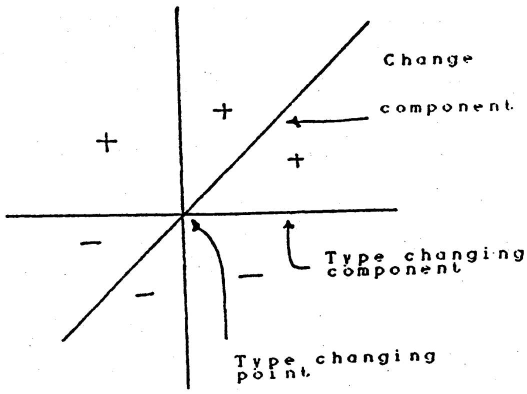
-An irreducible component Y_j of Y is called a <u>change component</u> with respect to a signature σ if any point $P \in Y_j$ such that $\dim_P Y_j = n - 1$ is a change point.

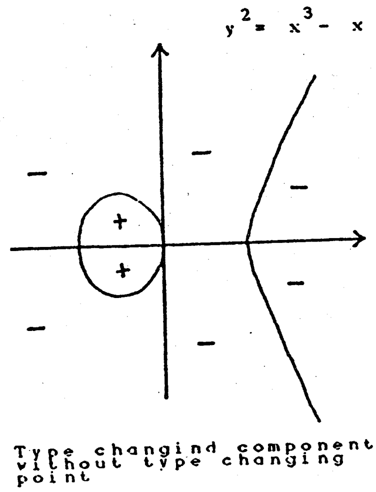
-An irreducible component Y_j of Y is called a <u>type changing component</u> (with respect to a signature σ) if both changing and not changing points belong to Y_j.

-A type changing component Y_j <u>changes type</u> at $P \in Y_j$ if in any neighborhood U of P there are both change points and not change points of Y_j.

-A point $P \in Y$ is <u>type changing</u> if some component of Y changes type at P.

<u>Exemples</u>





One can show that ([B.T.]):

1-A signature σ is admissible if and only if it is induced by a polynomial.

2-A signature σ on $\mathbb{R}^n$-Y is admissible if and only if no irreducible component Y_j of Y is type changing with respect to σ.

3-If the signature of ϕ is not admissible then ϕ is flat at any type changing point.

I will come back to the third statement at the end.

We come back now to the original question.

Since ϕ changes sign it is necessary to suppose that σ is an admissible signature.

If $Y_1,\dots,Y_n$ are the irreducible components of Y, no one of them can be type changing with respect to the signature σ, because it is admissible. Let $Y_1,\dots,Y_k$ be the change components of Y; if p_i is the generator of $I(Y_i)$, $i = 1,\dots,n$, then the function $\psi = \phi\cdot p_1\cdot\dots\cdot p_k$ has constant sign. So we can apply the previous result and approximate ψ in $C^s_W(\mathbb{R}^n)$ by a good function q such that $q^{-1}(0) = Y$:, so q is divisible by $p_1,\dots,p_k$ and $q/_{p_1\dots p_k}$ is a good approximation of ϕ.

β) The statement (*) is not true.

In any case we need global equations for Y: In the algebraic case this is always true; in the analytic we must suppose that Y is the <u>support of a coherent sheaf</u>.

The approximation is in C^0_W in the first case, in C^0_S in the second one.

We consider only the case codim $Y = 1$: if Y has greater codimension the proof is more or less the same.

<u>Theorem</u>: Let ϕ be a smooth function, $Y = \phi^{-1}(0)$ be an algebraic variety or the support of a coherent analytic sheaf; suppose codim $Y = 1$, and that the signature of ϕ is admissible. Then there exists a polynomial or an analytic function f such that $f^{-1}(0) = Y$ and f approximates ϕ in $C^0_W(\mathbb{R}^n)$ or in $C^0_S(\mathbb{R}^n)$.

<u>Proof</u>: Let p be a generator for $\mathcal{I}_Y$. Consider the map $j:\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ defined by: $j(x) = (x,p(x))$.

This is an embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ and $j(Y) = j(\mathbb{R}^n) \cap H$ where $H = \{x_{n+1} = 0\}$.

Consider the function ψ defined by $\phi \circ j^{-1}$ on $j(\mathbb{R}^n)$.

Step 1 Extend ψ by 0 to a function $\psi_1 : j(\mathbb{R}^n) \cup H \longrightarrow \mathbb{R}$.

Step 2 Extend ψ_1 to a C^0 function on $\mathbb{R}^{n+1}$ with the same zero set: take for instance a Tietze extension T of ψ_1 and then take

$$\psi_2(x) = \pm(|T(x)| + d(x, j(\mathbb{R}^n) \cup H))$$

(+ if $x \in (\mathbb{R}^{n+1})^+$ and - if $x \in (\mathbb{R}^{n+1})^-$).

This is possible since the signature is admissible.

Step 3 ψ_2 can be approximated by a C^∞-function ψ_3 without changing the zero-set: this is an easy tecnical lemma.

Step 4 Now we have a C^∞-function which vanishes over H which is a coherent space.

Then we can find a polynomial q' such that $q'^{-1} = H$ and q' approximates ψ_3

So $q' \circ j$ is the required approximation, but only in C^0 because of steps 1 and 2.

I want to remark that all these results generalize to the case $\phi : M \longrightarrow \mathbb{R}$ where M is a compact algebraic variety.

The most relevant difficulty is in defining admissible signatures, since in general irreducible comnponents do not have global equations.

For this reason we need to add the condition $[Y] = 0$ in $H_{m-1}(M,\mathbb{Z}_2)$ and to give a more complicated definition of admissible signature (see [B.T.]).

I come back to non admissible signatures.

<u>Theorem</u>: If the signature of ϕ is not admissible then ϕ is flat at any type changing point.

<u>Remark</u> : Unlike the analytic case, in the algebraic case the set of type changing points of Y with respect to a non admissible signature σ may be empty. Take for instance a disconnected irreducible real algebraic hypersurface Y. Take analytic equations $f_1,\ldots,f_k$ for the connected components. Then $f_1^2 \cdot f_2 \cdot \ldots \cdot f_k$ is an analytic function vanishing only at Y, with non admissible algebraic signature.

<u>Sketch of the proof</u>: Let P be a changing point. Assume first that there are only two components Y_1, Y_2 of Y passing through P, smooth at P and crossing normally at P. Take the tangent spaces $T(Y_1)$ and $T(Y_2)$ at P: we have the same situation as in exemple 1, so one can find two non empty disjoint open sets A and B in the linear space of lines through P such that $\phi_{|\ell}$ changes sign at P if $\ell \in A$ and does not if $\ell \in B$. This is enough to prove that ϕ is flat at P by the same argument as in the exemple 1.

If this is not true, one can reduce to this case by a suitable suite of (global) blowing-ups.

Let U be a neighborhood of P and Y_j, $j = 1,\dots,k$ be the components through P.

One can find a smooth algebraic (or analytic) subspace Z of Y, $P \in Z$, an algebraic (or analytic) manifold M and a map $\pi : M \longrightarrow U$ such that:

1) $\pi : M \longrightarrow U$ is surjective.
2) $\pi_{|M - \pi^{-1}(Z)} : M - \pi^{-1}(Z) \longrightarrow U - Z$ is an isomorfism.
3) if $E = \pi^{-1}(Z)$ and $E_j = \pi^{-1}(Y_j)$ E_j and E cross normally (see [H]).

So $\psi = \pi \circ \phi$ is a C^{∞}-function. If Y_i is type changing, it easy to verify that E is type changing at every point of $E_i \cap E$ and so every point $Q \in \pi^{-1}(P) \cap E \cap E_i$ is a type changing point for E. But now only two components of $\pi^{-1}(Y)$ cross transversally at Q.

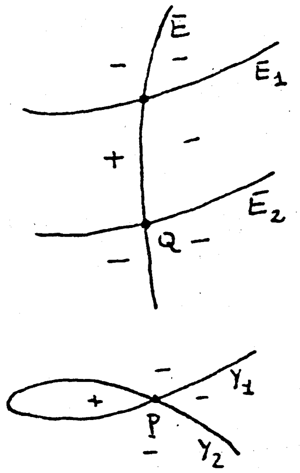


Figure 4

By previous remarks we have that ψ is flat at Q.

Remark that $d\pi_1 \wedge \dots \wedge d\pi_n$ is not identically zero on M, since π is an isomorphism outside $\pi^{-1}(Z)$

At this point we can conclude by applying the following lemma:

<u>Lemma</u> :Let $f^*:\mathbb{R}[[u_1,\dots,u_n]] \longrightarrow \mathbb{R}[[x_1,\dots,x_n]]$ be the homomorphism defined by

$$x_i = f_i(u_1,\dots,u_n)$$

with $f_i(u_1,\dots,u_n) \in \mathbb{R}[[u_1,\dots,u_n]]$ and $f_i(0) = 0$
Then if f^* is not injective $df_1\wedge\dots\wedge df_n \equiv 0$.

<u>Proof</u>:Suppose $\operatorname{Ker} f^* \neq \left\{ 0 \right\}$ and choose $F \in \operatorname{Ker} f^*$ such that $F \neq 0$ and F is of minimal order. Then

$$F(f_1,\dots,f_n) = 0$$

Remark that not all the derivatives $\frac{\partial F}{\partial x_i}$ can be zero because if so F is constant and hence $F = 0$.
Since the derivatives have order less than F, by differentiating one finds a linear relation among $df_1,\dots,df_n$, namely

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\Big|_{x_i=f_i(u)} df_i = 0$$

where not all the coefficients are equal to zero.
This is enough to conclude that the vectors df_i are lineary dependent on the quotient field of $\mathbb{R}[[u_1,\dots,u_n]]$ and so $df_1\wedge\dots\wedge df_n \equiv 0$. ■

REFERENCES

[B.T]

F.Broglia - A.Tognoli: *Approximation of C^{∞}-functions without changing their zero-set.* To appear

F.Broglia - A.Tognoli: *Approximation des fonctions C^{∞} et points de platitude.* To appear in C.R.A.S.

[H]

H.Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic O .* Ann. of Math. 79 pp 109.324 (1964)

[Hi]

M.W.Hirsc: *Differential topology.* Springer 1976

[T]

A.Tognoli: *Algebraic approximation of manifolds and spaces.* Sem.Bourbaki 548 (1979.80)