数理解析研究所講究録
690巻 1989年 37-48

PROPERTIES OF ESSENTIAL COVERINGS

by FRANCESCA ACQUISTAPACE

I want to expose some results about the possibility of triangulating smooth manifolds (and also more general objects) by the simplicial complex associated to a special kind of open covering (called essential). I want also to speak about the difficulties which arise when one tries to extend even weaker results to more singular objects.

First of all let us see how the problem arises.

The idea of working with acyclic coverings is rather old: they are used, for instance, in De Rham theorem.

The following, well known, theorem was proved in a paper of Weil (1952),[We].

<u>Theorem</u>: Let X be a normal topological space. Let $\{U_i\}_{i\in I}$ a topologically simple (locally finite) open covering. Then the nerve of this covering has the same homotopy type as X.

When X is a manifold such a covering is constructed by taking small geodetic balls (small with respect to the curvature of X).

So the problem (partially proposed already in Weil's paper) is the following:

<u>Question</u>: Do exist coverings $\mathfrak{U}$ such that the nerve $K(\mathfrak{U})$ is P.L.-homeomorphic to a smooth triangulation of the manifold V ? i.e. such that V and $K(\mathfrak{U})$ have the same P.L. type ? If so, there is a lot or a few of such coverings ?

This problem was solved by Broglia and me, up to the case of manifolds with boundary and the case of isolated singularities.

Recall the definition of nerve of a locally finite open covering $\mathfrak{U}$ of V. Suppose also that $\mathfrak{U}$ has finite order m (i.e. m + 1 open sets of $\mathfrak{U}$ do not intersect):

<u>Definition</u>: The nerve of $\mathfrak{U}$ is the simplicial complex $K(\mathfrak{U})$ defined by:

-the 0-skeleton of $K(\mathfrak{U})$ is simply I, the set of indices;

-we join two vertices i,j by a segment of the 1-skeleton if and only if $U_i \cap U_j \neq \emptyset$;

- ...

-a (r + 1)-tuple $(i_o,\dots,i_r)$ of vertices is joined by an r-simplex in the r-skeleton of $K(\mathfrak{U})$ if and only if $U_{i_0} \cap \dots \cap U_{i_r} \neq \emptyset$.

<u>Some remarks</u>.

1)If we add to $\mathfrak{U}$ an open set U contained in $U_{i_0} \cap \dots \cap U_{i_r}$, we add a new vertex to K^0 and we add to

$K(\mathfrak{U})$ some new simplexes, corresponding to the star of the new vertex. Adding and taking away such open sets acts on $K(\mathfrak{U})$ by a simple homotopy operation.

2)So we need a covering as simple as possible, i.e. without superfluous open sets. We must also impose the order of $\mathfrak{U}$ to be equal to $\dim V + 1$, since $\dim K(\mathfrak{U})$ equals the order of $\mathfrak{U}$ minus 1.

3)Another condition to be imposed is that the link of a vertex i_0 (whose vertices correspond to all the open sets U_i such that $U_i \cap U_{i_0} \neq \emptyset$) should be a P.L. sphere.

Let us give the definition of "essential covering".

Let V be a smooth manifold, $\mathfrak{U} = \left\{ U_i \right\}_{i \in I}$ be a locally finite open covering of V.

<u>Definition</u>: $\mathfrak{U}$ is essential if:

1) $\bar{U}_i \cong D^p$ = p-dimensional disk for any i.

2) $U_{i_0} \cap \dots \cap U_{i_r} \neq \emptyset \Longrightarrow \bar{U}_{i_0} \cap \dots \cap \bar{U}_{i_r} \overset{\rightarrow}{\simeq} D^p$ ($\overset{\rightarrow}{\simeq}$ means diffeomorphic after smoothing).

3) $U_{i_0} \cap \dots \cap U_{i_r} \cap \partial U_{j_1} \cap \dots \cap \partial U_{j_s} \neq \emptyset \Longrightarrow$ it is diffeomorphic after smoothing to D^{p-s}.

4) $U_{i_0} \cap \dots \cap U_{i_r} \neq \emptyset \Longrightarrow U_{i_0} \cap \dots \cap U_{i_r} \not\subset \bigcup_{j \neq i_0 \dots i_r} U_j$.

5) $\operatorname{ord} \mathfrak{U} = p + 1$.

<u>Remarks</u>: Conditions 1 and 2 are more or less the classical conditions for acyclicity. Condition 3 is needed for

induction: it assures that the induced covering on ∂U_i is essential. Condition 4 takes away non necessary open sets, while condition 5 implies that $\dim K(\mathfrak{U}) = \dim V$.

The definition of essential covering is easily extended to the case of a manifold with boundary, as we shall see after.

<u>Theorem</u> [Br]. If $\mathfrak{U}$ is an essential covering of V, then V and $K(\mathfrak{U})$ have the same P.L. type.

<u>Some ideas of the proof</u>.

1) The covering $\mathfrak{U}$ induces on V a "disk stratification", each stratum being the intersection of a finite number of open sets U_i with a finite number of spheres ∂U_i, minus the union of all the others open sets.

2) These strata correspond to the 1-skeleton of K' (first barycentric subdivision of $K(\mathfrak{U})$); namely:

- the p-strata are one-to-one corresponding with the vertices of K' (each vertex is a barycenter of a simplex corresponding to the not empty intersection of some open sets).
- the others strata are one-to-one corresponding with the 1-simplexes of K' by this association:

  denote by $\alpha = (i_0,\dots,i_r)$, $\beta = (j_1,\dots,j_s)$; then the stratum

$$X_{\alpha,\beta} = \left[U_{i_0} \cap \dots \cap U_{i_r} \cap \partial U_{j_1} \cap \dots \cap \partial U_{j_s}\right] - \bigcup_{j \not\in \alpha\cup\beta} U_j$$

corresponds to the 1-simplex $\tau \in (K')^1$ joining the vertex $(i_0,\ldots,i_r) = \alpha$ with the vertex $(i_0,\ldots,i_r,j_1,\ldots,j_s) = \alpha \cup \beta$.

3)Now we search for a partition of V which should correspond to a partition of (some barycentric subdivision) of $K(\mathfrak{U})$ in subcomplexes, in such a way that (any) smooth triangulation of a piece of V should be P.L. isomorphic to the corrisponding subcomplex of $K(\mathfrak{U})$.

The way how this is done is strongly inspired by the paper of Shiota-Yokoi on subanalytic hauptvermutung [Sh,Yo].

Shortly: One considers suitably truncated tubular neighborhoods of the strata. Each one is triangulated by a P.L. disk.

The corresponding objects in $K(\mathfrak{U})$ are defined as follows:

- one defines the weight of a 1-simplex in K' as being the codimension of the corresponding stratum;

- more generally the weight of an r-simplex $(\alpha_0,\ldots,\alpha_r)$ in the r-skeleton of K' (where one can suppose $\alpha_0 \subsetneq \alpha_1 \subsetneq \alpha_2 \subsetneq \ldots \subsetneq \alpha_r$) is defined as the cardinality of $\alpha_r - \alpha_0$;

Then one has the fundamental remark:

Lemma: Let τ be a 1-simplex in K' of weight r. Then the union $D(\tau)$ of all r-simplexes through τ of weight r is a P.L.-disk.

Now we are able to define a "truncated" regular neighborhood of $D(\tau)$ in K''' (yhe third barycentric subdivision) as the subcomplex corresponding to the truncated tubular neighborhood of the stratum corresponding to τ.

4)All the pieces are P.L.-disks and so what remains to be proved is that the partial P.L.-homeomorphisms glue together to give the P.L.-isomorphism we looked for. This is done by a double induction and using "Alexander trick" as in [Sh,Yo].

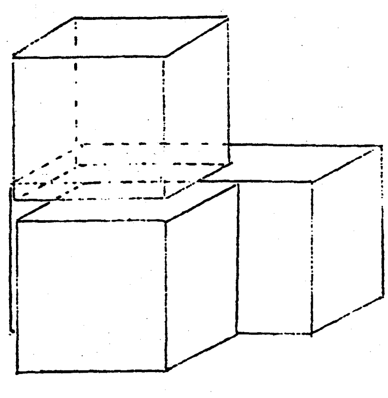
Construction of essential coverings.

1)For $\mathbb{R}^n$.

We begin with a "net" of equal segments in $\mathbb{R}$, say with vertices in $\mathbb{Z}$.; then consider $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as devided in stripes

$$\Sigma_h = \mathbb{R} \times [h,h+1] , h \in \mathbb{Z}$$

The net on $\mathbb{R}$ induces a net of squares in Σ_0: repeat this net in the even stripes, and traslate it by a small vector (parallel to $\mathbb{R}$) in the odd stripes.



Do the same for $\mathbb{R}^3$.

To define the net for $\mathbb{R}^n$ you need only the matrix of translations

$$A = (a_i^j), \quad i,j = 1,\dots,n-1,$$

where $0 < a_i^{n-1} < a_i^{n-2} \dots,$ $0 < a_{n-1}^{n-1} < 1$ and $v_j = (a_i^j,\dots,,a_i^j,0,\dots,0)$ defines the translation that extends the net to $\mathbb{R}^{j+1}$.

The cubes of the net have the following properties:

- they have disjoint interiors;
- if $Q_1 \cap \dots \cap Q_k \neq \emptyset$, then it is a parallelepiped in a n-k+1 plane;
- n+2 cubes do not intersect.

Take small thickenings of the cubes and smooth the boundaries without changing the incidence properties (do it carefully!). What you get is an essential covering of $\mathbb{R}^n$, generated by the net.

2) For a generic transversal h-plane the induced covering is essential.

(It is a little delicate to verify this point).

3) For a compact manifold V.

Choose a suitable embedding $V \subset \mathbb{R}^n$, with n sufficiently large with respect to the dimension p of V, in such a way that the coordinate functions are Morse functions on V. Now an essential covering for $\mathbb{R}^n$, generated by a net of sufficiently small cubes, induces an essential covering of V: eventually one needs only to move V a little, if it meets some $n-p-1$ face of the net.

4) For a non compact manifold V.

One has to consider more general nets of cubes, such that their size decreases as one approaches infinity. The proof is more or less the same.

There are a lot of such coverings because they refine any given locally finite open covering.

<u>Extensions of the results</u>

1) Manifolds with boundaries.

There is a natural way of defining essential coverings for such a V: namely they must induce essential coverings both on ∂V and on $2V$. In this case $K(\mathfrak{U})$ is a P.L. manifold with boundary and one has $\partial K(\mathfrak{U}) = K(\mathfrak{U}_{|\partial V})$.

2) Isolated singularities.

If $X \subset \mathbb{R}^n$ is a closed stratified set such that the complement of the union of the zero strata is a manifold, one can reason as follows.

For each singular point $x_i \in X$, take a small ball $B(x_i, r_i)$ with center x_i. Then $X - \cup B(x_i, r_i)$ is a smooth manifold with boundary. More over $\bar{B}_i = \overline{B(x_i, r_i)} \cap X$ is homeomorphic to the cone on the smooth manifold ∂B_i. So, for small ε_i, consider an essential covering $\mathfrak{U}$ of

$$V = X - \cup B(x_i, r_i - \varepsilon_i)\hbar$$

and complete it by the B_i's. The nerve $K(\mathfrak{U}, B_i)$ is obtained by adding to $K(\mathfrak{U})$ the cones on the pieces of $\partial(K(\mathfrak{U}))$

corresponding to the ∂B_i.

The case of a general stratified set.

The first step is to construct a topologically simple open covering $\mathfrak{U}$ of X. By Weil's theorem, for such a covering, $K(\mathfrak{U})$ would have the same homotopy type as X.

Let us recall the definitions.

Definition: A locally finite open covering $\mathfrak{U}$ is topologically simple if any non empty intersection of open sets in $\mathfrak{U}$ has the extension property.

This means the following:

Definition: A topological space Z has the extension property if for each closed set F in a normal space A, and for each continous map $f:F \longrightarrow Z$, f extends to a continous map $\bar{f}:A \longrightarrow Z$; if f extends only in a neighborhood of F, one says that Z has the local extension property.

In the case of a closed Whitney object $X \subset \mathbb{R}^n$, each relatively compact contractible open set $B \subset X$ has the extension property. This can be proved as follows.

- An open set U in $\mathbb{R}^n$ has the local extension property (take an extension $\bar{f}:A \longrightarrow \mathbb{R}^n$ by Tietze theorem, then take $\bar{f}^{-1}(U) \supset F$).

- If B is a deformation retract of some open neighborhood $N \supset B$, then B has the local extension property (it is sufficient to compose the extension $\bar{f}:U \longrightarrow N$ with the retraction).

- If B is contractible and has the local extension property, then it has also the global one: if $f: F \longrightarrow B$ extends to $\bar{f}:U \longrightarrow B$, take $F' = A - U$ and choose a continous $t:A \longrightarrow \mathbb{R}$ such that $t^{-1}(0) = F'$ and $t^{-1}(1) = F \subset U$. Now f extends to g defined by:

$$g(x) = \begin{cases} r(t(x),\bar{f}(x)), & \text{if } x \in U \\ y_0 & \text{if } x \in F' \end{cases}$$

where $r(s,x)$ is the retraction of B to the point y_0.

- If $B \subset X$ is contractible and relatively compact, then B is a deformation retract of some open neighborhood in $\mathbb{R}^n$. To prove this consider a closed ball C in $\mathbb{R}^n$ with $B \subset C$ and extend to C the stratification of $C \cap X$. Then, [Th], $C-X$ is diffeomorphic to $M - \partial M$, where M is a compact manifold, and $\partial C \cup (X \cap C)$ is a quotient of ∂M. So $(X \cap C) \cup \partial C$ has a collar (T,p) in C. The open set which retracts to B is $p^{-1}(B) \subset T|_B$.

So it is sufficient to construct a simple covering of X, i.e. a locally finite relatively compact open covering in which any non empty intersection of open sets is contractible.

We thaught of essentially two kinds of open sets.

The first one are small balls. One can nicely describe inductively such a covering: then what one has to prove is the following.

- Let $B(x_0,r)$ be a ball with x_0 belonging to the stratum S: suppose r so small that B is transversal to any stratum adherent to S and it is inside a tubular neighborhood of S. Then B retracts to any $y \in S \cap B(x_0,r)$.

The proof is very simple if $y = x_0$; for $y \neq x_0$ one can consider the vector field defined by the projection of $x-y$ on the tangent space to the stratum T of x. But this vector field (which does not vanish if r is sufficently small) fails to point inside at any point of ∂B , and so does not retract B to y.

So we are no more able to prove that $B(x_0,r_0) \cap B(x_1,r_1)$ is contractible when x_0 and x_1 are in the same stratum S, and the construction does not work.

The second kind of open sets is like this.

For a point $x_0 \in S$ take a small ball $B(x_0,r)$ and then define $Bx_0 = B(x_0,r) \cap S$; finally take $\pi^{-1}(Bx_0)$ in a tubular neighborhood of S belonging to a system of controlled tubes.

Take a locally finite transversal covering with open sets of this kind. Then, in principle, it seems easier to prove that each finite intersection retracts on its intersection with the minimal stratum. Then one has a simple covering of a manifold. But there are still some difficulties in doing induction for an intersection involving many different strata.

In any case we hope to conclude in a short time.

REFERENCES.

[Ac,Br]: F.Acquistapace, F.Broglia: *Essential Coverings for Stratified Sets with Only Isolated Singularities.* Boll. U.M.I. (7) 1-A (1987).

[Br]: F.Broglia: *Essential Coverings and Triangulations.* Annali di Mat. Pura e Applic. (IV), Vol.CXLVI (1987)

[Sh,Yo]: M.Shiota, M.Yokoi: *Triangulations of subanalytic sets and locally subanalytic manifolds.* Transactions of the Am. Math. Soc. Vol.286 (1984)

[Th]: R.Thom: *Ensembles et morphismes stratifiés.* Bull. Am. Math Soc. (1969).

[We]: A.Weil: *Sur les théorème de De Rham.* Comm. Math. Helv., 26 (1952).