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Topics on Hilbert's 16th problem

by Goo ISHIKAWA (Faculty of Sci., Hokkaido Univ.; Japan)
石川 剛郎 北大・理

Introduction. The first half of Hilbert's 16th problem is concerned with the relative position of a non-singular real algebraic curve (resp. surface) in $\mathbf{R}P^2$ (resp. $\mathbf{R}P^3$) of a fixed degree. In fact, Hilbert asked about the topology of sixth degree curves and fourth degree surfaces. Though this concrete problem was solved completely by several mathematicians in the last decade, it is still stimulating the investigation of "topology of real algebraic manifolds" or "enumerative real algebraic geometry" from the viewpoint of contemporary mathematics.

Let $H_{n,d}$ denote the $\mathbf{R}$-vector space of homogeneous polynomials $F(x_0,x_1,\ldots,x_n)$ of degree d with real coefficients. By a <u>real hypersurface</u>, we mean an element of $P(H_{n,d}) = (H_{n,d} - 0)/\mathbf{R}^{\times}$. Let $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$. For $[F] \in P(H_{n,d})$, set $\mathbf{K}Z_F = \{[x] \in \mathbf{K}P^n \mid F(x) = 0\}$. We call $[F]$ <u>non-singular</u> if $\mathbf{C}Z_F$ is non-singular.

Two non-singular real hypersurfaces $[F],[G] \in P(H_{n,d})$ are called <u>isotopic</u> if there exists a continuous family σ_t of homeomorphisms of $\mathbf{R}P^n$ such that σ_0 = identity and $\sigma_1(\mathbf{R}Z_F) = \mathbf{R}Z_G$. They are called <u>rigidly isotopic</u> if there exists a continuous family $[F]_t \in P(H_{n,d})$ of non-singular real hypersurfaces such that $[F]_0 = [F]$ and $[F]_1 = [G]$. We see that, if $[F]$ and $[G]$ are rigidly isotopic, then they are isotopic.

The following tables illustrate the known isotopy and rigid isotopy classifications for curves: (Notations will be explained)

d	Rigid isotopy classification for curves of degree ≤ 6
1	$\langle J\rangle_{I}$
2	$\langle 1\rangle_{I}$, $\langle 0\rangle_{II}$
3	$\langle J,1^{-}\rangle_{I}$, $\langle J\rangle_{II}$
4	$\langle 4\rangle_{I}$, $\langle 3\rangle_{II}$, $\langle 1\langle 1^{-}\rangle\rangle_{I}$, $\langle 2\rangle_{II}$, $\langle 1\rangle_{II}$, $\langle 0\rangle_{II}$
5	$\langle J,3^{+},3^{-}\rangle_{I}$, $\langle J,5\rangle_{II}$, $\langle J,1^{+},3^{-}\rangle_{I}$, $\langle J,4\rangle_{II}$, $\langle J,3\rangle_{II}$, $\langle J,1^{-}\langle 1^{-}\rangle\rangle_{I}$, $\langle J,2\rangle_{II}$, $\langle J,1\rangle_{II}$, $\langle J\rangle_{II}$
6	Isotopy classification
	(i) $\langle 1,1\langle 9\rangle\rangle_{I}$, $\langle 5,1\langle 5\rangle\rangle_{I}$, $\langle 9,1\langle 1\rangle\rangle_{I}$
	(ii) $\langle 1\langle 9\rangle\rangle$, $\langle 1,1\langle 8\rangle\rangle$, $\langle 4,1\langle 5\rangle\rangle$, $\langle 5,1\langle 4\rangle\rangle$, $\langle 8,1\langle 1\rangle\rangle$, $\langle 10\rangle$
	(iii) $\langle a,1\langle b\rangle\rangle$, $(0 \leq a \leq 7,\ 1 \leq b \leq 8,\ a + b \leq 8)$
	(iv) $\langle a\rangle$, $(0 \leq a \leq 9)$ (v) $\langle 1\langle 1\langle 1\rangle\rangle\rangle_{I}$:
	The eight classes $\langle 9\rangle$, $\langle 4,1\langle 4\rangle\rangle$, $\langle 1\langle 8\rangle\rangle$, $\langle 5,1\langle 1\rangle\rangle$, $\langle 3,1\langle 3\rangle\rangle$, $\langle 1,1\langle 5\rangle\rangle$, $\langle 2,1\langle 2\rangle\rangle$, and $\langle 1\langle 4\rangle\rangle$ correspond to exactly two rigid isotopy classes respectively.
	Isotopy classification for curves of degree 7
7	(i) $\langle J,a,1\langle b\rangle\rangle$, $(0 \leq a \leq 13,\ 1 \leq b \leq 13,\ a + b \leq 14)$
	(ii) $\langle J,a\rangle$, $(0 \leq a \leq 15)$ (iii) $\langle J,\langle 1\langle 1\langle 1\rangle\rangle\rangle\rangle$

In general, to obtain the concrete classifications, we need two types of results: (A) Restriction theorems (Estimates), (B) Existence theorems (Explicit constructions and Abstruct existence theorems).

In this talk, after mentioning some restriction theorems, we intend to review the following two important topics related to (B):

(I) Local and global bifurcations of singular real plane curves.

(II) Real structures on polarized K-3 surfaces.

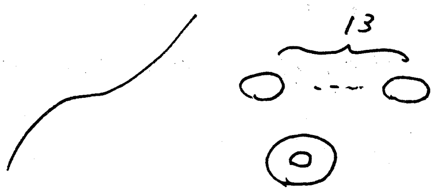
1) Let $C \subset \mathbb{R}P^2$ be a non-singular plane curve of degree d. Denote by ℓ the number of connected components of C. By Harnak(1875), $\ell \leq (1/2)(d-1)(d-2) + 1$ $(= g + 1)$. Following Prtrovskii, we call C is an <u>M-curve</u> if $\ell = g + 1$.

2)The image of an embedding $i : S^1 \longrightarrow \mathbb{R}P^2$ is called an <u>oval</u> (resp. a <u>pseudo-line</u>) if i is null-homotopic (resp. non null-homotopic). An oval divides $\mathbb{R}P^2$ to a region diffeomorphic to the open disk, which is called the <u>interior</u>, and to a region diffeomorphic to the open Möbius band, which is called the <u>exterior</u>.

3) According to Viro, we symbolize isotopy types of non-singular real plane curves as follows:

The symbol <a> means (the isotopy type) of union of a-ovals outside in each other. <J> means the pseudo-line. For a symbol <A>, <1<A>> means the union of <A> and an oval surrounding <A>. For a sequence of symbols <A_1>, <A_2>,..., <A_r>, by <A_1,A_2,...,A_r>, we mean union of them lying outside in each other.

For example, <J,13,1<1>> means the isotopy type of the curve



We see that a sympol determines the isotopy type of a non-singular plane curve.

4) An oval E of a curve C is called <u>even</u> (resp. <u>odd</u>) if E is included in the even (resp. odd) number of other ovals of C.

Set $p = \#(\text{even ovals})$ and $n = \#(\text{odd ovals})$.

Theorem (Rokhlin 1972). For an M-curve of evan degree d,

$$p - n \equiv (d/2)^2, \quad (\text{mod. } 8).$$

Example (M-curves of degree 6). Let $d = 6$ and $\ell = 11$. Then we have $p + n = 11$ and $p - n \equiv 1\ (8)$. By Bézout's theorem, the possible isotopy types are $<\alpha, 1<\beta>>$ with α, β are non-negative integers. Then we have $p = \alpha+1$ and $n = \beta$. Thus $\alpha+\beta = 10$ and $\alpha-\beta \equiv 0\ (8)$. Therefore $(\alpha,\beta) = (1,9), (5,5), (9,1)$. The possible isotopy types of M-curves of degree 6 are

$$<1,1<9>>, \ <5,1<5>> \ \text{and} \ <9,1<1>>.$$

In fact these are realized by non-singular curves of degree 6.

5) The isotopy classification of curves of degree $d \leq 5$ is classically known. The classification for $d = 6$ and $d = 7$ are established by Gudkov(1971) and by Viro(1980) respectively.

Consider on M-curves of degree 8; $d = 8$ and $\ell = 22$. Then $p+n = 22$, $p-n \equiv 0\ (8)$. Thus $(p,n) = (3,19), (7,15), (11,11), (15,7), (19,3)$. By Bézout's theorem, the possible isotopy types are $<\alpha,1<\beta>>$, $<\alpha,1<\beta>,1<\gamma>>$, $<\alpha,1<\beta>,1<\gamma>,1<\delta>>$, and $<\alpha,1<\beta,1<\gamma>>>$.

Theorem (Viro 1980). For an M-curve of degree 8 of type $<\alpha,1<\beta>,1<\gamma>,1<\delta>>$, if $\beta\gamma\delta \neq 0$, then β, γ and δ are all odd.

Nowaday, we have the following table:

d	g+1	ν
2	1	1
4	4	1
6	11	3
8	22	$58 \leq ? \leq 104$

Here, ν = #(isotopy classes of M-curves of degree d).

Conjecture (Viro). For an M-curves of degree 8, if $\gamma \neq 0$, then β and γ are odd.

If Conjecture is true, then we have $\nu \leq 79$ for d = 8.

Problem. Classify M-cueves of degree 8.

For M-curves of degree 10, there are results of Chislenko and Shustin. For curves in $\mathbb{R}P^1\times \mathbb{R}P^1$, the are results of Hilbert, Gudkov and Matsuoka.

6) Rigid isotopy. Let $PH_{n,d}$ be as in introduction. Denote

by Σ the set of singular hypersurfaces in $PH_{n,d}$. Then we have the natural bijection between the set of rigid isotopy classes of non-singular hypersurfaces of degree d in $\mathbb{R}P^n$ and $\pi_0(PH_{n,d} - \Sigma)$.

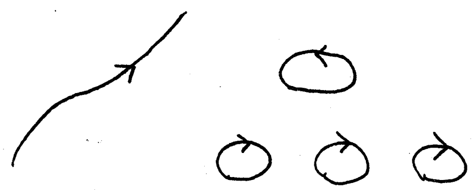
7) Plane curve ($n = 2$). Let $[F] \in PH_{2,d}$. Consider the involution on the Riemann surface $\mathbb{C}Z_F$ defined by the complex conjugation. The fixed point set is the real locus $\mathbb{R}Z_F$.

According to Klein, $[F]$ is called of <u>type</u> I (resp. II) if $\mathbb{C}Z_F - \mathbb{R}Z_F$ is not connected (resp. connected).

If $[F]$ is of type I, the orientation of $\mathbb{C}Z_F$ induces an orientation, so called the <u>complex orientation</u>, of $\mathbb{R}Z_F$ (up to simulteneous reversing).

The type and the complex orientation in the case of type I are rigid isotopy invariants.

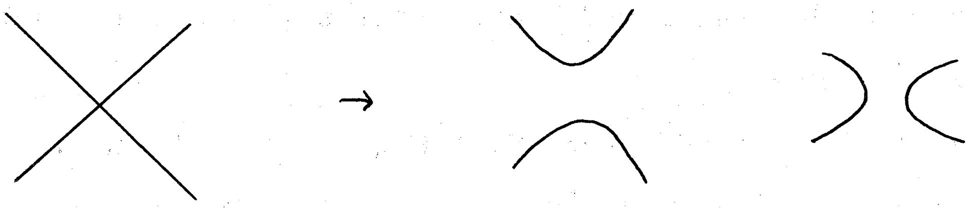
For example, $\langle J, 1^+, 3^- \rangle_I$ means a rigid isotopy class of type I with the following complex orientation:



There exist the rigid isotopy classifications for $d \leq 6$.

8) Construction. One of methods to construct non-singular curves in prescribed isotopy classes is perturbation of singular curves: $\boxed{\text{Singular curves} \xrightarrow[\text{perturb}]{} \text{Non-singular curves}}$

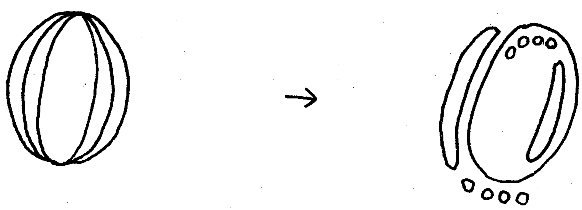
In the classical method of Harnack, Brusotti and so on, only curves with A_1-type singularities are considered:



Viro treats the perturbation of J_{10}-singularity defined by locally $f(x,y) = (y - \alpha_1 x^2)(y - \alpha_2 x^2)(y - \alpha_3 x^2) = 0$.

Theorem (Viro 1982). There exist exactly 31-classes of perturbatoin of J_{10}. (See [**1986**, Viro]).

By this and the gluing method of Viro, we can construct an M-curve of degree 6 of type $\langle 5,1\langle 5\rangle\rangle$:



Problem. Let $F : \mathbf{R}^2 \times \mathbf{R}^{10}, 0 \longrightarrow \mathbf{R}, 0$ be the versal deformation of J_{10}-singularity and $\Sigma \subset \mathbf{R}^{10}$ be the bifurcation locus. Then #(local connected components of $\mathbf{R}^{10} - \Sigma$ at 0) = 31?.

9) Space surface (n = 3). The rigid isotopy classificatins of space surfaces of degree ≤ 2 is classically well-known and that of space surfaces of degree 3 is done by Klein. For example, the number of rigid isotopy classes is equals to 1, 3, 5 in the case d =

1, 2, 3 respectively.

Now consider the rigid isotopy classification problem of space surfaces of degree 4.

Set $S = \mathbb{C}Z_F \subset \mathbb{C}P^3$, for $[F] \in PH_{3,4} - \Sigma$. Then the first betti number $b_1(S) = 0$ and Λ^2T^*S is trivial, that is, S is a K-3 surface. As is well-known, $L = H^2(S,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. With respect to the cup-product, the signature of L is $(3,19)$, and $L \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. Denote by $\tau : S \longrightarrow S$ the complex conjugation and set $\varphi = \tau^* : L \longrightarrow L$. Let $h \in L$ be the Poincaré dual of the hyperplane section. Then we have $\varphi(h) = -h$.

Consider the triple (L,φ,h). Set $L^\varphi = \{x \in L \mid \varphi(x) = x\}$ and denote by (t_+,t_-) the signature of L^φ. Then we have $t_+ = 1$ and the Euler characteristic $\chi(\mathbb{R}S) = 2t_- - 18$.

For the natural action of $PGL(4)$ on $PH_{3,4} - \Sigma$, we have

Theorem (Nikulin 1979). The construction in above induces a bijection

$$\pi_0((PH_{3,4}-\Sigma)/PGL(4)) \longleftrightarrow \left\{(L,\varphi,h) \;\middle|\; \begin{array}{l} L \text{ is free of sign } (3,19), \\ \varphi \text{ is an involution of } L, \\ h^2 = 4,\ h \text{ is primitive}, \\ \varphi(h) = -h \text{ and } t_+ = 1 \end{array}\right\}/\text{isom.}.$$

The right hand side has 134 elements. 72 of them correspond to surfaces with null-homotopic real part $\mathbb{R}S \subset \mathbb{R}P^3$.

Since $\pi_0(PGL(4)) = 2$, the natural mapping

$$\pi : \pi_0(PH_{3,4} - \Sigma) \longrightarrow \pi_0((PH_{3,4} - \Sigma)/PGL(4))$$

is 2:1 or 1:1 over each point of the target.

Let $r : \mathbb{R}P^3 \longrightarrow \mathbb{R}P^3$ be the reflextion along a plane in $\mathbb{R}P^3$, for instance, $r[x_0, x_1, x_2, x_3] = [-x_0, x_1, x_2, x_3]$.

We call a rigid isotopy class $[F]$ is <u>amphicheiral</u> if $[F \circ r]$ is rigidly isotopic to $[F]$.

Then we see that π is one to one over $\pi[F]$ if and only if $[F]$ is amphicheiral.

The next result is the criterion of amphicheirality described by only on real loci $\mathbb{R}S$.

Theorem (Kharlamov 1984). Denote by p the number of spherical components of $\mathbb{R}S$, and by q the half of dimension of $H_1(\mathbb{R}S, \mathbb{R})$. Consider the following four cases:

(a) $\mathbb{R}S$ is null-homotopic and $p \geq 3$, $q \geq 3$,

(b) $\mathbb{R}S$ is null-homotopic and $p < 3$ or $q < 3$,

(c) $\mathbb{R}S$ is non null-homotopic and $p \geq 4$,

(d) $\mathbb{R}S$ is non null homotopic and $p < 4$.

If (a) or (c), then $[F]$ is not amphicheiral. If (b) or (d), then $[F]$ is amphicheiral.

As a result, there exist exactly 170 rigid isotopy classes of non-singular surfaces in $\mathbb{R}P^3$ of degree 4;

$$\#(\pi_0(PH_{3,4} - \Sigma)) = 170.$$

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This is a collection of papers concerning to the first half of the Hilbert's 16th problem, complex manifolds with conjugations, singularities defined over the real and so on.

Of course, there may be many important papers which do not appear in this bibliography.

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