Topics on Hilbert's 16th problem

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Introduction. The first half of Hilbert's 16th problem is concerned with the relative position of a non-singular real algebraic curve (resp. surface) in $\mathbb{RP}^2$ (resp. $\mathbb{RP}^3$) of a fixed degree. In fact, Hilbert asked about the topology of sixth degree curves and fourth degree surfaces. Though this concrete problem was solved completely by several mathematicians in the last decade, it is still stimulating the investigation of "topology of real algebraic manifolds" or "enumerative real algebraic geometry" from the viewpoint of contemporary mathematics.

Let $H_{n,d}$ denote the $\mathbb{R}$-vector space of homogeneous polynomials $F(x_0,x_1,\ldots,x_n)$ of degree $d$ with real coefficients. By a real hypersurface, we mean an element of $P(H_{n,d}) = (H_{n,d} - 0)/\mathbb{R}^\times$. Let $K$ denotes $\mathbb{R}$ or $\mathbb{C}$. For $[F] \in P(H_{n,d})$, set $KZ_F = \{ [x] \in K\mathbb{P}^n \mid F(x) = 0 \}$. We call $[F]$ non-singular if $CZ_F$ is non-singular.

Two non-singular real hypersurfaces $[F],[G] \in P(H_{n,d})$ are called isotopic if there exists a continuous family $\sigma_t$ of homeomorphisms of $\mathbb{RP}^n$ such that $\sigma_0 = \text{identity}$ and $\sigma_1(RZ_F) = RZ_G$. They are called rigidly isotopic if there exists a continuous family $[F]_t \in P(H_{n,d})$ of non-singular real hypersurfaces such that $[F]_0 = [F]$ and $[F]_1 = [G]$. We see that, if $[F]$ and $[G]$ are rigidly isotopic, then they are isotopic.
The following tables illustrate the known isotopy and rigid isotopy classifications for curves: (Notations will be explained)

### Rigid isotopy classification for curves of degree \( \leq 6 \)

1. \( <J>_I \)
2. \( <1>_I, <0>_II \)
3. \( <J,1^->_I, <J>_II \)
4. \( <4>_I, <3>_II, <1<1^->_I, <2>_II, <1>_II, <0>_II \)
5. \( <J,3^+,3^->_I, <J,5>_II, <J,1^+,3^->_I, <J,4>_II, <J,3>_II, <J,1^-<1^->_I, <J,2>_II, <J,1>_II, <J>_II \)

### Isotopy classification

(i) \( <1,1<9>_I, <5,1<5>_I, <9,1<1>_I \)

(ii) \( <1<9>_>, <1,1<8>_>, <4,1<5>_>, <5,1<4>_>, <8,1<1>_>, <10> \)

(iii) \( <a,1<b>_>, (0 \leq a \leq 7, 1 \leq b \leq 8, a + b \leq 8) \)

(iv) \( <a>_>, (0 \leq a \leq 9) \)  

(v) \( <1<1<_I>_> \)

The eight classes \( <9>_>, <4,1<4>_>, <1<8>_>, <5,1<1>_>, <3,1<3>_>, <1,1<5>_>, <2,1<2>_>, \) and \( <1<4>_> \) correspond to exactly two rigid isotopy classes respectively.

### Isotopy classification for curves of degree 7

7. (i) \( <J,a,1<b>_>, (0 \leq a \leq 13, 1 \leq b \leq 13, a + b \leq 14) \)

(ii) \( <J,a>_>, (0 \leq a \leq 15) \)  

(iii) \( <J,<1<1<_I>_>>_> \)

In general, to obtain the concrete classifications, we need two types of results: (A) Restriction theorems (Estimates), (B) Existence theorems (Explicit constructions and Abstract existence theorems).

In this talk, after mentioning some restriction theorems, we
intend to review the following two important topics related to (B):

(I) Local and global bifurcations of singular real plane curves.

(II) Real structures on polarized K-3 surfaces.

1) Let \( C \subset \mathbb{R}P^2 \) be a non-singular plane curve of degree \( d \). Denote by \( l \) the number of connected components of \( C \). By Harnak (1875), \( l \leq (1/2)(d-1)(d-2) + 1 \) (\( = g + 1 \)). Following Prtovskii, we call \( C \) is an \textit{M-curve} if \( l = g + 1 \).

2) The image of an embedding \( i : S^1 \to \mathbb{R}P^2 \) is called an \textit{oval} (resp. a \textit{pseudo-line}) if \( i \) is null-homotopic (resp. non null-homotopic). An oval divides \( \mathbb{R}P^2 \) to a region diffeomorphic to the open disk, which is called the \textit{interior}, and to a region diffeomorphic to the open Möbius band, which is called the \textit{exterior}.

3) According to Viro, we symbolize isotopy types of non-singular real plane curves as follows:

The symbol \( <a> \) means (the isotopy type) of union of a-ovals outside in each other. \( <J> \) means the pseudo-line. For a symbol \( <A> \), \( <1<A>> \) means the union of \( <A> \) and an oval surrounding \( <A> \). For a sequence of symbols \( <A_1>, <A_2>, \ldots, <A_r> \), by \( <A_1,A_2,\ldots,A_r> \), we mean union of them lying outside in each other.

For example, \( <J,13,1<> \) means the isotopy type of the curve

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{curve.png}
\end{array}
\]
We see that a symbol determines the isotopy type of a non-singular plane curve.

4) An oval $E$ of a curve $C$ is called even (resp. odd) if $E$ is included in the even (resp. odd) number of other ovals of $C$. Set $p = \#(\text{even ovals})$ and $n = \#(\text{odd ovals})$.

Theorem (Rokhlin 1972). For an $M$-curve of even degree $d$, $p - n \equiv (d/2)^2$, (mod. 8).

Example ($M$-curves of degree 6). Let $d = 6$ and $\ell = 11$. Then we have $p + n = 11$ and $p - n \equiv 1 \ (8)$. By Bézout's theorem, the possible isotopy types are $<\alpha,1<\beta>$ with $\alpha,\beta$ are non-negative integers. Then we have $p = \alpha + 1$ and $n = \beta$. Thus $\alpha + \beta = 10$ and $\alpha - \beta \equiv 0 \ (8)$. Therefore $(\alpha,\beta) = (1,9), (5,5), (9,1)$. The possible isotopy types of $M$-curves of degree 6 are

$<1,1<9>>$, $<5,1<5>>$ and $<9,1<1>>$.

In fact these are realized by non-singular curves of degree 6.

5) The isotopy classification of curves of degree $d \leq 5$ is classically known. The classification for $d = 6$ and $d = 7$ are established by Gudkov(1971) and by Viro(1980) respectively.

Consider on $M$-curves of degree 8; $d = 8$ and $\ell = 22$. Then $p + n = 22$, $p - n \equiv 0 \ (8)$. Thus $(p,n) = (3,19), (7,15), (11,11), (15,7), (19,3)$. By Bézout's theorem, the possible isotopy types
are \( <\alpha, 1<\beta>, <\alpha, 1<\beta>, 1<\gamma>, <\alpha, 1<\beta>, 1<\gamma>, 1<\delta>, \) and \( <\alpha, 1<\beta, 1<\gamma>>. \)

**Theorem (Viro 1980).** For an M-curve of degree 8 of type \( <\alpha, 1<\beta>, 1<\gamma>, 1<\delta>, \) if \( \beta\gamma\delta \neq 0 \), then \( \beta, \gamma \) and \( \delta \) are all odd.

Nowaday, we have the following table:

<table>
<thead>
<tr>
<th>d</th>
<th>g+1</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>22</td>
<td>58 ( \leq ? \leq 104 )</td>
</tr>
</tbody>
</table>

Here, \( \nu = \#(\text{isotopy classes of M-curves of degree d}). \)

**Conjecture (Viro).** For an M-curves of degree 8, if \( \gamma \neq 0 \), then \( \delta \) and \( \gamma \) are odd.

If Conjecture is true, then we have \( \nu \leq 79 \) for \( d = 8 \).

**Problem.** Classify M-cueves of degree 8.

For M-curves of degree 10, there are results of Chislenko and Shustin. For curves in \( \mathbb{RP}^1 \times \mathbb{RP}^1 \), the are results of Hilbert, Gudkov and Matsuoka.

6) Rigid isotopy. Let \( PH_{n,d} \) be as in introduction. Denote
by $\Sigma$ the set of singular hypersurfaces in $PH_n,d$. Then we have
the natural bijection between the set of rigid isotopy classes of
non-singular hypersurfaces of degree $d$ in $\mathbb{R}P^n$ and $\pi_0(PH_n,d - \Sigma)$.

7) Plane curve ($n = 2$). Let $[F] \in PH_2,d$. Consider the
involution on the Riemann surface $CZ_F$ defined by the complex
conjugation. The fixed point set is the real locus $RZ_F$.

According to Klein, $[F]$ is called of type I (resp. II) if
$CZ_F - RZ_F$ is not connected (resp. connected).

If $[F]$ is of type I, the orientation of $CZ_F$ induces an
orientation, so called the complex orientation, of $RZ_F$ (up to
simultaneous reversing).

The type and the complex orientation in the case of type I are
rigid isotopy invariants.

For example, $<J,1^+,3^->_I$ means a rigid isotopy class of type I
with the following complex orientation:

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\[\begin{array}{c}
\circlearrowleft \\
\circlearrowright \\
\end{array}\]
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There exist the rigid isotopy classifications for $d \leq 6$.

8) Construction. One of methods to construct non-singular
curves in prescribed isotopy classes is perturbation of singular
curves: $\text{Singular curves} \xrightarrow{\text{perturb}} \text{Non-singular curves}$

In the classical method of Harnack, Brusotti and so on, only
curves with $A_1$-type singularities are considered:

\[
\begin{array}{c}
\times \\
\rightarrow \\
\end{array}
\]

Viro treats the perturbation of $J_{10}$-singularity defined by locally $f(x,y) = (y - \alpha_1 x^2) (y - \alpha_2 x^2) (y - \alpha_3 x^2) = 0$.

**Theorem** (Viro 1982). There exist exactly 31-classes of perturbations of $J_{10}$. (See [1986, Viro]).

By this and the gluing method of Viro, we can construct an M-curve of degree 6 of type $<5,1<5>$:

\[
\begin{array}{c}
\circ \\
\rightarrow \\
\end{array}
\]

Problem. Let $F : \mathbb{R}^2 \times R^{10},0 \rightarrow \mathbb{R},0$ be the versal deformation of $J_{10}$-singularity and $\Sigma \subset R^{10}$ be the bifurcation locus. Then $\#(\text{local connected components of } R^{10} - \Sigma \text{ at } 0) = 31$.

9) Space surface ($n = 3$). The rigid isotopy classifications of space surfaces of degree $\leq 2$ is classically well-known and that of space surfaces of degree 3 is done by Klein. For example, the number of rigid isotopy classes is equals to 1, 3, 5 in the case $d =$
16

1, 2, 3 respectively.

Now consider the rigid isotopy classification problem of space surfaces of degree 4.

Set $S = CZ_F \subset CP^3$, for $[F] \in PH_{3,4} - \Sigma$. Then the first betti number $b_1(S) = 0$ and $\Lambda^2 T^* S$ is trivial, that is, $S$ is a K-3 surface. As is well-known, $L = H^2(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. With respect to the cup-product, the signature of $L$ is $(3, 19)$, and $L \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi^3 \oplus (-E_8)^{\Phi^2}$. Denote by $\tau : S \rightarrow S$ the complex conjugation and set $\varphi = \tau^* : L \rightarrow L$. Let $h \in L$ be the Poincaré dual of the hyperplane section. Then we have $\varphi(h) = -h$.

Consider the triple $(L, \varphi, h)$. Set $L^\varphi = \{x \in L \mid \varphi(x) = x\}$ and denote by $(t_+, t_-)$ the signature of $L^\varphi$. Then we have $t_+ = 1$ and the Euler characteristic $\chi(RS) = 2t_- - 18$.

For the natural action of $PGL(4)$ on $PH_{3,4} - \Sigma$, we have

Theorem (Nikulin 1979). The construction in above induces a bijection

$$
\pi_0((PH_{3,4} - \Sigma)/PGL(4)) \longleftrightarrow \chi(L, \varphi, h) \begin{cases} 
L \text{ is free of sign } (3,19), \\
\varphi \text{ is an involution of } L, \\
h^2 = 4, \ h \text{ is primitive,} \\
\varphi(h) = -h \text{ and } t_+ = 1 \\
\text{isom.}
\end{cases}
$$

The right hand side has 134 elements. 72 of them correspond to surfaces with null-homotopic real part $RS \subset RP^3$.

Since $\pi_0(PGL(4)) = 2$, the natural mapping

$$
\pi : \pi_0(PH_{3,4} - \Sigma) \rightarrow \pi_0((PH_{3,4} - \Sigma)/PGL(4))
$$

is 2:1 or 1:1 over each point of the target.
Let \( r : \mathbb{RP}^3 \to \mathbb{RP}^3 \) be the reflection along a plane in \( \mathbb{RP}^3 \), for instance, \( r[x_0, x_1, x_2, x_3] = [-x_0, x_1, x_2, x_3] \).

We call a rigid isotopy class \([F]\) is \textit{amphicheiral} if \([F \circ r]\) is rigidly isotopic to \([F]\).

Then we see that \( \pi \) is one to one over \( \pi[F] \) if and only if \([F]\) is amphicheiral.

The next result is the criterion of amphicheirality described by only on real loci \( RS \).

\textbf{Theorem} (Kharlamov 1984). Denote by \( p \) the number of spherical components of \( RS \), and by \( q \) the half of dimension of \( H_1(RS,R) \). Consider the following four cases:

(a) \( RS \) is null-homotopic and \( p \geq 3, q \geq 3 \),
(b) \( RS \) is null-homotopic and \( p < 3 \) or \( q < 3 \),
(c) \( RS \) is non null-homotopic and \( p \geq 4 \),
(d) \( RS \) is non null homotopic and \( p < 4 \).

If (a) or (c), then \([F]\) is not amphicheiral. If (b) or (d), then \([F]\) is amphicheiral.

As a result, there exist exactly 170 rigid isotopy classes of non-singular surfaces in \( \mathbb{RP}^3 \) of degree 4:

\[ \#(\pi_0(\mathbb{PH}_{3,4} - \Sigma)) = 170. \]

\textbf{Bibliography on classical theory of real algebraic varieties}
This is a collection of papers concerning to the first half of
the Hilbert's 16th problem, complex manifolds with conjugations, singularities defined over the real and so on.

Of course, there may be many important papers which do not appear in this bibliography.

1875

1891

1906

1913

1931


1932
1933

1938
I.G. Petrovskii, On the topology of real plane algebraic curves, Ann. of Math., (2) 39, 189-209.

1939

1944

1949


1950

1951
O.A. Oleinik, On algebraic curves on an algebraic surface, Usp.

On the topology of real algebraic curves on an algebraic surface, Mat. Sb., 29, 133-156.

Estimates of the Betti numbers of real algebraic hypersurfaces, Mat. Sb., 28, 635-640.


1964


1965


1966

D.A. Gudkov, G.A. Utkin, M.A. Tai, A complete classification of irreducible curves of the fourth order, Mat. Sb., 69, 222-256.

1968


1969


1971

N.L. Alling, N. Greenleaf, Fundation of the theory of Klein

V.I. Arnol'd, The arrangement of the ovals of real plane
algebraic curves, involutions of four-dimensional smooth manifolds,

1972

V.M. Kharlamov, Maximal number of components of the fourth
Appl., 6, 345-346.


V.A. Rokhlin, Congruences modulo 16 in Hilbert's sixteenth
6, 301-306.

1973

V.I. Arnol'd, The topology of real algebraic curves (the works

D.A. Gudkov, Construction of a curve of the sixth order of type

D.A. Gudkov, A.D. Krakhnov, Periodicity of the Euler
characteristic of real algebraic (M-1)-varieties, Funkt. Anal.

V.M. Kharlamov, New congruences for the Euler characteristic of

W. Massey, The quotient space of the complex projective plane
under conjugation is a 4 sphere, Geom. Dedicata 2., 371-374.

1974


1975


S.M. Natanzon, Moduli of real algebraic curves, Usp. Mat. Nauk 30-1, 251-252.


1976


1977


1978


A.B. Korchagin, New possibility on the method of Brusotti for the construction of $M$-curves of an order $\geq 8$, in [LA1978], pp. 149-159.


G.M. Polotovskii, (M-1) and (M-2)-ramified curves of sixth order, in [LA1978], pp. 130-148.

R. Silhol, Geometrie algebrique sur un corps non algebriquement clos, Comm. Algebra, 6-11, 1131-1155.


1979


1980

B. Chevalier, Sur une manièere de construire des courbes réelles dans $\mathbb{P}^2(\mathbb{R})$, in [R,1980], pp. 27-50.


H. Jaffee, Real algebraic curves, Topology, 19, 81-87.

A. Marin, Quelques remarques sur les courbes algébriques planes réelles, in [R,1980], pp. 51-68.


S. Ochanine, Sur l'utilisation de la théorie des groupes formals dans le 16ème problème de Hilbert, in [R,1980], pp. 81-100.


O. Ya. Viro, Curves of degree 7, curves of degree 8 and the


1981


V.M. Karpushkin, The number of components of the complement to the level hypersurface of a harmonic polynomial, Vestnik Moscow Univ. Mat., 34-6, 3-4.


Real algebraic surfaces, Amer. Math. Soc.


1982


M. Seppälä, Real structure of Teichmüller space, Manuscripta


1983


R. Silhol, Diviseurs sur les varietes algebriques reelles, Boll. U. M. I..


, Progress over the last five years in the topology of real algebraic varieties, Proc. International Conference of Mathematicians, Warsaw, 525-611.


1984


M. Letizia, Quotients by complex conjugation of nonsingular quadrics and cubics in $\mathbb{P}^\mathbb{C}_2$ defined over $\mathbb{R}$, Pacific J. Math., 110-2, 307-314.

M.G. Marinari, M. Reimondo, On complete intersection real curves, Rocky Mt. J. Math., 14, 919-920.

J.-J. Risler, Complexité et géométrie réelle (d'après A.Khovansky), Séminaire Bourbaki 37e année 1984-85, n° 637.


________, Real algebraic surfaces with rational or elliptic fiberings, Math. Z., 186, 465-499.

1985


B. Chevallier, Sur les courbes maximales de Harnack, 300, (I)4 109-114.


J.J. Risler, Type topologique des surfaces algébriques réelles de degré 4 dans $\mathbb{R}P^3$, Astérisque, 126, 153-168.


_________ , Estimates for the number of singular points of algebraic curves, Differential and integral equation, Gos. Univ. Gorki., pp. 81-83.


1986


_________ , Courbes maximales de Harnack et discriminant,
in [R,1986], pp. 41-65.


V.M. Kharlamov (Harlamov), Surfaces non amphichérales de degré 4 dans \( \mathbb{R}P^3 \), in [R,1986], pp. 67-72.


F. Loeser, Un analogue local de l'inégalité de Petrowsky-Oleinik, in [R,1986], pp. 73-83.


M. Seppälä, On moduli of real curves, in [R,1986], 85-95.

R. Silhol, Bounds for the connected components and the first Betti number mod two of a real algebraic surface, Compositio Math., 60, 53-63.


1987


W. Kucharz, Real algebraic curves as complete intersections, Math. Z., 194, 259-266.


_________, A new M-curves of degree eight, Mat. Zametki, 42-2, 180-186.


_________, in [ASV, 1987], pp. 149-163.


1988


