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Kyoto University
STRATIFICATION OF THE DISCRIMINANT VARIETIES OF TYPE $A_\ell$ and $B_\ell$

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§1. Introduction Let $R$ be a reduced irreducible root system in $\mathbb{R}^t$. Let $\mathcal{H} = \{H_\alpha\} (\alpha \in \Lambda)$ be the corresponding arrangement of the hyperplanes. The Weyl group $W$ is the group generated by the reflections along $\{H_\alpha ; \alpha \in \Lambda\}$. It acts on $\mathbb{C}^t$ so that the quotient space $\mathbb{C}^t/W$ is isomorphic to the affine space $\mathbb{C}^t$ whose affine coordinate ring is the ring of the invariant polynomial $C[\xi_1, \ldots, \xi_t]^W$ (Chapter 6, [1]). Let $|\mathcal{H}| = \bigcup_{\alpha \in \Lambda} H_\alpha$. The action on the complement $\mathbb{C}^t - |\mathcal{H}|$ is free and $|\mathcal{H}|$ is $W$-invariant. We call the quotient space $|\mathcal{H}|/W$ the discriminant variety of the root system and we denote it by $D$. The discriminant variety is a hypersurface in the quotient space $\mathbb{C}^t/W$. There are many interesting results by many authors about the topology of the arrangement $|\mathcal{H}|$ or $\mathbb{C}^{t+1} - |\mathcal{H}|$. See Orlik [6] and its references. The complement $\mathbb{C}^t - D$ is known to be a $K(\pi,1)$-space by [2] and [3]. Let $\mathcal{S}$ be a stratification of $|\mathcal{H}|$ which is compatible with the $W$–action. For instance, we can take the minimal stratification $S_{\min} = \{H_\Xi ; \Xi \subseteq \Lambda\}$ where $H_\Xi = \bigcap_{\alpha \in \Xi} H_\alpha - \bigcup_{\alpha \not\in \Xi} H_\alpha$. For a given $\mathcal{S}$, $D$ inherits a canonical stratification $\overline{\mathcal{S}}$ which is defined by the images of the strata of $\mathcal{S}$. The purpose of this paper is to show that the discriminant variety for the arrangements of type $A_\ell$ and $B_\ell$ has canonical regular stratifications which are constructed in the above way. Here the regularity means the b-regularity in the sense of Whitney [7]. It is known that the b-regularity implies the a-regularity ([5]). For $A_{\ell+1}$ and $B_{\ell+1}$, we can simply take $\mathcal{S} = S_{\min}$.

Let $T$ be an analytic stratification of an analytic variety $V$ in an open set $U$ of $\mathbb{C}^n$. Let $(M, N)$ be a pair of strata of $T$ with $M \supset N$ and let $q \in N$. Let $p(u) (0 \leq u < 1)$ be a real analytic curve such that $p(0) = q$ and $p(u) \in M$ for $u > 0$. Let $T = \lim_{u \to 0} T_{p(u)}M$. We say that the pair $(M, N)$ has a unique tangential limit at $q$ if this limit $T$ depends only on $q$ and $M$. If $T$ enjoys this property at any point $q$ of $N$ for any pair $(M, N)$, we say that $T$ has the unique tangential limits property. Of course, the existence of a stratification with the unique tangential limits property poses a strong geometric restriction on $V$.

We will show that the stratifications $\overline{\mathcal{S}}$ for $A_{\ell+1}$ and $B_{\ell+1}$-discriminants have the unique tangential limits property.

§2. $A_\ell$-arrangement. We first consider the $A_\ell$-arrangement. As a root system, $A_\ell$ is the restriction of $B_{\ell+1}$ to the following hyperplane

$$L : \xi_1 + \cdots + \xi_{\ell+1} = 0.$$
The corresponding arrangement \( \mathcal{H} \) consists of \( \binom{\ell+1}{2} \) hyperplanes \( \{\xi_i - \xi_j = 0 \} \) \((i < j)\) and the Weyl group \( W \) is the symmetric group \( S_{\ell+1} \). The invariant ring is generated by

\[
s_i = \sum_{\tau \in S_{\ell+1}} \xi_{\tau(1)} \cdots \xi_{\tau(i)} \quad (i = 1, \ldots, \ell + 1).
\]

We refer to Chapter 6 of [1] for the basic results about the irreducible root systems. We use the following symmetric polynomials for the calculation’s sake.

\[
\tau_i = \xi_{1}^{i} + \cdots + \xi_{\ell+1}^{i} \quad (i = 1, \ldots, \ell + 1).
\]

Note that \( \\{\tau_1, \ldots, \tau_{\ell+1}\} \) is also a basis of the ring of invariant polynomials and that \( s_1 = \tau_1 = 0 \) on \( L \). We define the mapping \( \Phi : C^{\ell+1} \rightarrow C^{\ell+1} \) by \( \Phi(\xi_1, \ldots, \xi_{\ell+1}) = (\tau_1, \ldots, \tau_{\ell+1}) \). Let \( \bar{L} \) be the hyperplane in the quotient space defined by \( \tau_1 = 0 \). Let \( \phi_L : L \rightarrow \bar{L} \) and \( \phi : |\mathcal{H}| \rightarrow D \) be the respective restriction of \( \Phi \) to \( L \) and \(|\mathcal{H}| \). We have the following commutative diagrams.

\[
\begin{array}{ccc}
C^{\ell+1} & \xrightarrow{\Phi} & L \\
\downarrow \phi & & \downarrow \phi_L \\
C^{\ell+1} & \xrightarrow{\Phi} & \bar{L} \\
\end{array}
\]

Here the horizontal maps are the respective inclusion maps. It is well-known that \( D \) is defined by \( \prod_{1 \leq i < j} (\xi_i - \xi_j)^2 = 0 \) which can be written in a weighted homogeneous polynomial of \( s_1, \ldots, s_{\ell+1} \) or equivalently of \( \tau_1, \ldots, \tau_{\ell+1} \). This is equal to the discriminant polynomial of \( x^{\ell+1} - s_1 x^{\ell} + \cdots + (-1)^{\ell+1} s_{\ell+1} = 0 \) in the usual sense ([4]).

Now we consider the stratification \( \mathcal{S} = \mathcal{S}_{\min} \) of \(|\mathcal{H}| \). Let \( C_1 \) be the set of the non-maximal subdivisions of the set \( \{1, \ldots, \ell+1\} \). Namely an element \( \mathcal{F} \) of \( C_1 \) can be written as \( \{I_1, \ldots, I_k\} \) where \( I_i \cap I_j = 0 \) for \( i \neq j \) and \( \bigcup_{j=1}^{k} I_j = \{1, \ldots, \ell+1\} \). The maximal element \( \mathcal{M} = \{\{1\}, \ldots, \{\ell+1\}\} \) is excluded as \( M(\mathcal{M}) = C^{\ell+1} \). Note that the Weyl group \( W \) acts canonically on \( C_1 \). Let \( C_2 \) be the set of the non-maximal partitions of the integer \( \ell + 1 \). An element \( \mathcal{K} \) of \( C_2 \) is written as \( \{m_1, \ldots, m_k\} \) such that \( \sum_{j=1}^{k} m_j = \ell + 1 \) with \( m_j > 0 \). For a subset \( I \) of \( \{1, \ldots, \ell+1\} \), we denote its cardinality by \(|I|\). Then there is a canonical surjection from \( C_1 \) to \( C_2 \) by \( \mathcal{F} \mapsto |\mathcal{F}| \) where \( |\mathcal{F}| = \{|I_1|, \ldots, |I_k|\} \). For each \( \mathcal{F} = \{I_1, \ldots, I_k\} \), we define

\[
M(\mathcal{F}) = \{ \xi = (\xi_i) \in C^{\ell+1} \mid \xi_i = \xi_j \Leftrightarrow \exists a \in \mathbb{Z}; \{i, j\} \subset I_a \}.
\]

It is clear that \( \{M(\mathcal{F})\}_{\mathcal{F} \in C_1} \) is equal to \( \mathcal{S} = \mathcal{S}_{\min} \) which is a regular stratification of \(|\mathcal{H}| \). Let \( \mathcal{F} = \{I_1, \ldots, I_k\} \) and \( \mathcal{G} = \{J_1, \ldots, J_m\} \) be elements of \( C_1 \). \( \mathcal{F} \) is called a subdivision of \( \mathcal{G} \) if for each \( i \), there exists a \( j \) such that \( I_i \subset J_j \). We define a partial ordering in \( C_1 \) (respectively in \( C_2 \)) by \( \mathcal{F} \succeq \mathcal{G} \) if and only if \( \mathcal{F} \) is a subdivision of \( \mathcal{G} \). (Respectively \(|\mathcal{F}| \succeq |\mathcal{G}| \) \( \Leftrightarrow |\mathcal{F}| \) is a subpartition of \(|\mathcal{G}| \).) The canonical map \( \mathcal{F} \mapsto |\mathcal{F}| \) is obviously order-preserving.
PROPOSITION (2.5). Let $\mathcal{F}, \mathcal{F}' \in C_{1}$. The following conditions are equivalent.

(i) $\overline{M(\mathcal{F})} \supseteq M(\mathcal{F}')$. (ii) $\overline{M(\mathcal{F})} \cap M(\mathcal{F}') \neq \emptyset$. (iii) $\mathcal{F} \supseteq \mathcal{F}'$.

PROPOSITION (2.6). Let $\mathcal{F}, \mathcal{F}' \in C_{1}$. (I) The following conditions are equivalent.

(i) $\phi(M(\mathcal{F})) = \phi(M(\mathcal{F}'))$. (ii) $\phi(M(\mathcal{F})) \cap \phi(M(\mathcal{F}')) \neq \emptyset$.

(iii) There exists an element $g \in W$ such that $g(M(\mathcal{F})) = M(\mathcal{F}')$. (iv) $|\mathcal{F}| = |\mathcal{F}'|$ in $C_{2}$.

(II) $\overline{\phi(M(\mathcal{F}))} \supseteq \phi(M(\mathcal{F}'))$ if and only if $|\mathcal{F}| \succeq |\mathcal{F}'|$.

PROOF: Proposition (2.5) is immediate from the definition of $M(\mathcal{F})$. We prove Proposition (2.6). The equivalence (iii) $\Leftrightarrow$ (iv) is obvious. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are also trivial. Assume that $\phi(\xi) = \phi(\xi')$ for some $\xi \in M(\mathcal{F})$ and $\xi' \in M(\mathcal{F}')$. This implies that there exists a $g \in W$ such that $g(\xi) = \xi'$. As $\mathcal{H}$ is invariant by the action of $W$, we can write $g(M(\mathcal{F})) = M(\mathcal{G})$ for some $\mathcal{G} \in C_{1}$. As $\{M(\mathcal{F})\}_{\mathcal{F} \in C_{1}}$ are disjoint, this implies $\mathcal{F}' = \mathcal{G}$. Thus (ii) $\Rightarrow$ (iii). As $\overline{\phi(M(\mathcal{F}))} = \phi(M(\mathcal{F}'))$, the assertion (II) is an immediate consequence of (I) and Proposition (2.5).

DEFINITION (2.7). For $\mathcal{K} \in C_{2}$, we define $V(\mathcal{K}) = \phi(M(\mathcal{F}))$ where $|\mathcal{F}| = \mathcal{K}$.

We define an important vector-valued function $X(z)$ by

$$(2.8) \quad X(z) = (x, x^2, \ldots, x^{\ell+1}).$$

Let $X'(z) = (1, 2x, \ldots, (\ell + 1)x^{\ell})$ be the derivative of $X(z)$. Then $\Phi(\xi) = \sum_{i=1}^{\ell+1} X(\xi_i)$ and the tangential map $d\Phi_{\xi} : T_{\xi} C^{\ell+1} \to T_{\Phi(\xi)} C^{\ell+1}$ satisfies $d\Phi_{\xi}(\frac{\partial}{\partial \xi_i}) = \sum_{j=1}^{\ell+1} j \xi_i^{j-1} \frac{\partial}{\partial \tau_j}$. We identify the tangent space $T_{\Phi(\xi)} C^{\ell+1}$ with $C^{\ell+1}$ in a canonical way. Then the above equality says

$$(2.9) \quad d\Phi_{\xi}(\frac{\partial}{\partial \xi_i}) = X'(\xi_i), \quad i = 1, \ldots, \ell + 1.$$ 

For any subset $I$ of $\{1, \ldots, \ell + 1\}$, we define

$$(2.10) \quad \frac{\partial}{\partial \xi_I} = \frac{1}{|I|} \sum_{i \in I} \frac{\partial}{\partial \xi_i}, \quad \xi_I = \frac{1}{|I|} \sum_{i \in I} \xi_i.$$ 

Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and let $\xi \in M(\mathcal{F})$. As $\xi_j$ does not depend on $j \in I_i$ for $i$ being fixed, we have $\xi_j = \xi_{I_i}$ for any $j \in I_i$.

PROPOSITION (2.11). Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and let $\xi \in M(\mathcal{F})$.

(i) $T_{\xi} M(\mathcal{F})$ is the $(k-1)$-dimensional vector space which is equal to

$$T_{\xi} M(\mathcal{F}) = \left\{ \sum_{t=1}^{k} \lambda_t \frac{\partial}{\partial \xi_{I_t}} ; \sum_{t=1}^{k} \lambda_t = 0 \right\}.$$
The restriction $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ is a finite covering.

(iii) $V(|\mathcal{F}|)$ is non-singular and

$$T_{\phi(\xi)}V(|\mathcal{F}|) = \left\{ \sum_{t=1}^{k} \lambda_{t}X'(\xi_{I_{t}}) : \sum_{t=1}^{k} \lambda_{t} = 0 \right\}.$$ 

PROOF: (i) is obvious by the definition of $M(\mathcal{F})$. Thus

$$d\Phi_{\xi}(T_{\xi}M(\mathcal{F})) = \left\{ \sum_{t=1}^{k} \lambda_{t}X'(\xi_{I_{2}}) : \sum_{t=1}^{k} \lambda_{t} = 0 \right\}.$$ 

By the Vandermonde determinant formula, this image has dimension $(k - 1)$. Thus the restriction $\phi|M(\mathcal{F})$ is a submersion and the local image by $\phi$ is smooth. Thus the local images near $\xi$ and $\eta$ by $\phi$ coincide. This proves that $V(|\mathcal{F}|)$ is smooth and the assertions (ii) and (iii) follow immediately.

Let us examine the order of the covering $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ more explicitly. Let $\{\alpha_{1}, \ldots, \alpha_{m}\} = \{n : \exists i, n = |I_{i}| \}$. Clearly we have $m \leq k$ and $\{\alpha_{i}\}$ are mutually distinct. Let $\rho_{i}$ be the number of $j$'s such that $|I_{j}| = \alpha_{i}$ ($i = 1, \ldots, k$). We consider the subgroups

$$W(\mathcal{F}) = \{ g \in W : g(M(\mathcal{F})) = M(\mathcal{F}) \}, \quad I(\mathcal{F}) = \{ g \in W : g|M(\mathcal{F}) = id \}.$$ 

Then $I(\mathcal{F})$ is a normal subgroup of $W(\mathcal{F})$ and the quotient group $W(\mathcal{F})/I(\mathcal{F})$ acts freely on $M(\mathcal{F})$ with the quotient space $V(|\mathcal{F}|)$. More precisely let $\bar{g} \in W(\mathcal{F})/I(\mathcal{F})$. Then for each $s = 1, \ldots, m$, $\bar{g}$ induces a permutation of $\{\xi_{I_{j}} : |I_{j}| = \alpha_{s}\}$. Thus we have

PROPOSITION (2.12). There is a canonical isomorphism $W(\mathcal{F})/I(\mathcal{F}) \cong S_{\rho_{1}} \times \cdots \times S_{\rho_{m}}$. Thus the order of the above covering is $\rho_{1}! \cdots \rho_{m}!$.

Let $f(x)$ be a vector valued rational function of one variable. We define the rational functions $f_{k}(x_{1}, \ldots, x_{k})$ ($k = 1, \ldots, \ell + 1$) inductively by $f_{1}(x_{1}) = f(x_{1})$ and

$$f_{k}(x_{1}, \ldots, x_{k}) = \frac{f_{k-1}(x_{1}, \ldots, x_{k-1}) - f_{k-1}(x_{1}, \ldots, x_{k-2}, x_{k})}{(x_{k-1} - x_{k})}.$$ 

We call $f_{k}(x_{1}, \ldots, x_{k})$ the k-fold derived function of $f(x)$. 

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**PROPOSITION (2.14).** We have the following formulae.

(i) \[ f(x_k) = f(x_1) + \sum_{j=2}^{k} \left( \prod_{h=1}^{j-1} (x_k - x_h) \right) f_j(x_1, \ldots, x_j) \]

(ii) \[ f_{s+1}(x_1, \ldots, x_s, x_{s+k}) = f_{s+1}(x_1, \ldots, x_{s+1}) + \sum_{j=2}^{k} \left( \prod_{h=1}^{j-1} (x_{s+k} - x_{s+h}) \right) f_{s+j}(x_1, \ldots, x_{s+j}) \]

**PROOF:** As (i) is a special case of (ii), we prove (ii) by the induction on \( k \). The assertion on \( k = 1 \) is trivial. We assume the assertion for \( k - 1 \). By the definition of the derived function, we have

\[
\begin{align*}
f_{s+1}(x_1, \ldots, x_s, x_{s+k}) - f_{s+1}(x_1, \ldots, x_s, x_{s+1}) &= (x_{s+k} - x_{s+1})f_{s+2}(x_1, \ldots, x_{s+1}, x_{s+k}) \\
&= \left( \prod_{h=1}^{j-1} (x_{s+k} - x_{s+h}) \right) f_{s+j}(x_1, \ldots, x_{s+j})
\end{align*}
\]

This completes the proof.

Now we consider the derived functions \( X_k(x_1, \ldots, x_k) \) and \( X_k'(x_1, \ldots, x_k) \) of \( X(x) \) and \( X'(x) \) respectively. The following Lemma plays an important role throughout this paper.

**LEMMA (2.15).** Let \( a_{k,j} \) and \( b_{k,j} \) be the \( j \)-th coordinate of \( X_k(x_1, \ldots, x_k) \) and \( X_k'(x_1, \ldots, x_k) \) respectively. Then \( a_{k,j}, b_{k,j} \) are symmetric polynomials of \( x_1, \ldots, x_k \) defined by

(i) \[ a_{k,k+j} = \sum_{\nu_1 + \cdots + \nu_k = j+1} x_1^{\nu_1} \cdots x_k^{\nu_k}, \quad b_{k,k+j} = (k+j) \sum_{\nu_1 + \cdots + \nu_k = j} x_1^{\nu_1} \cdots x_k^{\nu_k} \]

(ii) \( X_k(x, \ldots, x) = X^{(k-1)}(x)/(k-1)! \), \( X_k'(x, \ldots, x) = X^{(k)}(x)/(k-1)! \)

where \( X^{(j)}(x) = \left( \frac{d}{dx} \right)^j X(x) \).

**PROOF:** (i) is immediate from the inductive calculation and the equality: \( (x^a - y^a)/(x - y) = x^{a-1} + x^{a-2}y + \cdots + y^{a-1} \). The assertion (ii) follows immediately from (i).

**LEMMA (2.16).** Let \( \xi \in M(\mathcal{F}) \) and let \( \mathcal{F} = \{ I_1, \ldots, I_k \} \). Then

\[ X_t'(\xi_{I_{\sigma(1)}}, \ldots, \xi_{I_{\sigma(t)}}) \in T_{\phi(\xi)} V(|\mathcal{F}|) \quad \text{for any } t = 2, \ldots, k \text{ and } \sigma \in S_t \]
PROOF: By Proposition (2.11), we have that
\[ X'(\xi_{I}) - X'(\xi_{I_{j}}) = (\xi_{I} - \xi_{I_{j}})X'_{I}(\xi_{I_{i}}, \xi_{I_{j}}) \in T_{\phi(\xi)}V(|\mathcal{F}|) \quad (i \neq j). \]
This implies that \( X'_{I}(\xi_{I_{i}}, \xi_{I_{j}}) \in T_{\phi(\xi)}V(|\mathcal{F}|) \) for \( i \neq j \). Now the assertion follows by an easy inductive argument.

The following is a generalization of the Vandermonde determinant formula and it plays a key role to show the linear independence of certain vectors in the later arguments.

**LEMMA (2.17).** (Generalized Vandermonde formula) Let \( \lambda_{1}, \ldots, \lambda_{k} \) be mutually distinct complex numbers and let \( \mathcal{N} = \{ \nu_{1}, \ldots, \nu_{k} \} \) be an element of \( \mathcal{C}_{2} \). Then we have the formula:
\[
\det\left( \begin{array}{cccc}
^{t}X'_{\nu_{1}}(\lambda_{1}), & \ldots, & ^{t}X^{(\nu_{1})}(\lambda_{1}), & \ldots, & ^{t}X'_{\nu_{k}}(\lambda_{k}), & \ldots, & ^{t}X^{(\nu_{k})}(\lambda_{k})
\end{array} \right) = (\ell+1)! \prod_{j>i}(\lambda_{j} - \lambda_{i})^{\nu_{j} \nu_{i}}.
\]
In particular, \( \{ X^{(j)}(\lambda_{i}) \} \) \( (j = 1, \ldots, \nu_{i}, \ i = 1, \ldots, k) \) are linearly independent.

**PROOF:** Let \( \Psi(x_{1}, \ldots, x_{\ell+1}) = \det(^{t}X'(x_{1}), \ldots, ^{t}X'(x_{\ell+1})) \). Then it is easy to see that
\[
(2.18) \quad \Psi(x_{1}, \ldots, x_{\ell+1}) = (\ell+1)! \prod_{j>i}(x_{j} - x_{i})
\]
by the Vandermonde determinant formula. We consider the differential operators:
\[
D_{i} = \left( \frac{\partial}{\partial x_{\nu_{1} + \cdots + \nu_{i-1} + 2}} \right)^{1} \cdots \left( \frac{\partial}{\partial x_{\nu_{1} + \cdots + \nu_{i}}} \right)^{\nu_{i}-1}
\quad \text{and} \quad D = D_{1} \cdots D_{k}.
\]
Let \( E = \{ (j, h) \ ; \ \nu_{1} + \cdots + \nu_{i-1} + 1 \leq h < j \leq \nu_{1} + \cdots + \nu_{i}, \ i = 1, \ldots, k \} \) and let \( \mathcal{E} \) be the ideal generated by \( \{ x_{j} - x_{h} \ ; \ (j, h) \in E \} \). As \( \sum_{j=1}^{\nu_{i}-1} j = \binom{\nu_{i}}{2} \), it is easy to see that
\[
(2.19) \quad D\Psi \equiv (\ell+1)! \prod_{(j, h) \notin E} (x_{j} - x_{h}) \mod \mathcal{E}.
\]
Thus the assertion follows immediately from
\[
\det(^{t}X'(\lambda_{1}), \ldots, ^{t}X^{(\nu_{1})}(\lambda_{1}), \ldots, ^{t}X'(\lambda_{k}), \ldots, ^{t}X^{(\nu_{k})}(\lambda_{k})) = (D\Psi)(\lambda_{1}, \ldots, \lambda_{1}, \ldots, \lambda_{k}, \ldots, \lambda_{k}) = (\ell+1)! \prod_{j>i}(\lambda_{j} - \lambda_{i})^{\nu_{j} \nu_{i}}.
\]
Here the last equality is due to (2.19).

§3. Regularity and the limit of the tangent space. Now we are ready to show the regularity of the stratification \( \overline{S} \) of the discriminant variety of \( A_{\ell+1} \)-arrangement and the unique
tangential limits property. Let $M(\mathcal{F})$ and $M(\mathcal{G})$ be stratum of $S$ such that $\overline{M(\mathcal{F})} \supset M(\mathcal{G})$. Let $q$ be an arbitrary point of the stratum $V(|\mathcal{G}|)$ and let $p(u)$ and $q(u)$ be real analytic curves defined on the interval $[0,1]$ such that (i) $p(0) = q(0) = q$ and $q(u) \in V(|\mathcal{G}|)$ for any $u \in [0,1]$. (ii) $p(u) \in V(|\mathcal{F}|)$ for $u > 0$. We also assume that

$$
\lim_{u \rightarrow 0} T_{p(u)} V(|\mathcal{F}|) = T, \quad \lim_{u \rightarrow 0} [\overline{p}(u), \overline{q}(u)] = \gamma.
$$

Here $[\overline{p}(u), \overline{q}(u)]$ is the line spanned by $\overline{p}(u) - \overline{q}(u)$. Changing the parameter $u$ by $u^{1/m}$ for some integer $m$ if necessary, we may assume that there are lifting real analytic curves $p(u)$ and $q(u)$ in $\overline{M(\mathcal{F})}$ and $M(\mathcal{G})$ respectively so that $p(u) = \phi(p(u))$ and $q(u) = \phi(q(u))$ respectively. We may assume that $p(0) = q(0)$ and let $\eta = p(0) \in M(\mathcal{G})$. Let $\mathcal{G} = \{J_{1}, \ldots, J_{m}\}$. By Proposition (2.5), we can write $\mathcal{F} = \{J_{i,j} ; i=1, \ldots , m, i=l, \ldots,v_{i}\}$ where $J_{1,j} \subset J_{i}$ for $j=1,$ $\ldots,$ $\nu_{i}$.

**Theorem (3.2).** $\overline{S}$ is a regular stratification with the unique tangential limits property. Namely

(i) $T$ is generated by

$$
\left\{ \sum_{i=1}^{m} \lambda_{i}X'(\eta_{J_{i}}) ; \sum_{i=1}^{m} \lambda_{i} = 0 \right\} \cup \left\{ X^{(j)}(\eta_{J_{i}}) ; 1 \leq i \leq m, 2 \leq j \leq \nu_{i} \right\}.
$$

(ii) (Regularity) $\gamma \in T$.

**Proof:** By Proposition (2.11), the vectors $X'(\eta_{J_{i}}) = \sum_{i=1}^{m} \lambda_{i}X'(p(u)_{J_{i,1}})+\cdots+\lambda_{m}X'(p(u)_{J_{m,1}})$ with $\sum_{i=1}^{m} \lambda_{i} = 0$ are contained in $T_{p(u)} V(|\mathcal{F}|)$. Thus by taking the limit as $u \rightarrow 0$, we see that $\sum_{i=1}^{m} \lambda_{i}X'(\eta_{J_{i}}) \in T$. This gives only a subspace of $T$ of dimension $m - 1$. We still need $\nu_{1} + \cdots + \nu_{m} - m$ independent vectors to generate $T$. For this purpose, we apply Lemma (2.15). We know that $X_{k}^{'(p(u)_{J_{i,1}},\ldots,p(u)_{J_{i,k}})} \in T_{p(u)} V(|\mathcal{F}|)$ (2 $\leq k \leq \nu_{i},$ $1 \leq i \leq m$). We take the limits of these vectors as $u \rightarrow 0$ and we apply Lemma (2.15) to obtain that $X^{(j)}(\eta_{J_{i}}) \in T$ (2 $\leq j \leq \nu_{i},$ $1 \leq i \leq m$). Now we apply Lemma (2.17) to see that the vectors $\{X^{(j)}(\eta_{J_{i}}) ; 1 \leq i \leq m, 1 \leq j \leq \nu_{i}\}$ are linearly independent. This completes the proof of (i).

Now we consider the regularity (ii). Using the equality $\sum_{j=1}^{\nu_{i}} |J_{i,j}| = |J_{i}|$, we have

$$
p(u) - q(u) = \sum_{i=1}^{m} \sum_{j=1}^{\nu_{i}} |J_{i,j}| (X(p(u)_{J_{i,j}}) - X(q(u)_{J_{i}})).
$$

Using Proposition (2.14), we can write

$$
X(p(u)_{J_{i,j}}) - X(q(u)_{J_{i}}) = \sum_{h=1}^{j} \alpha_{i,j,h}(u)X_{h+1}(q(u)_{J_{i,1}},p(u)_{J_{1,1}},\ldots,p(u)_{J_{1,h}}).
$$
where $\alpha_{i,j,h}(u)$ is defined by

$$
\alpha_{i,j,h}(u) = (p(u)_{J,j} - q(u)_{J,j}) \prod_{k=1}^{h-1} (p(u)_{J,j} - p(u)_{J,k}), \quad h = 1, \ldots, \nu_i.
$$

Substituting (3.4) in (3.3), we obtain

$$
p(u) - q(u) = \sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h}(u) X_{h+1}(q(u)_{J,i} - p(u)_{J,i}).
$$

where $\alpha_{i,h}(u) = \sum_{j=h}^{\nu_i} |J_{i,j}| \alpha_{i,j,h}(u)$. In particular, we have

$$
\alpha_{i,1}(u) = \sum_{j=1}^{\nu_i} |J_{i,j}| (p(u)_{J,j} - q(u)_{J,j}).
$$

We define a non-negative integer $\beta$ by

$$
\beta = \min \{ \text{order } (\alpha_{i,h}(u)) : i = 1, \ldots, m, \ h = 1, \ldots, \nu_i \}
$$

and let $\alpha_{i,h}(u) = \alpha_{i,h} u^\beta + \text{(higher terms)}$. Then (3.6) and Lemma (2.15) imply that

$$
p(u) - q(u) = \left( \sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J,i})/h! \right) u^\beta + \text{(higher terms)}.
$$

By the Generalized Vandermonde formula (Lemma (2.17)), we can see easily that

$$
\sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J,i})/h! \neq 0 \quad \text{and} \quad \gamma = \left[ \sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J,i})/h! \right].
$$

Here $[v]$ denotes the line generated by the vector $v$. Thus the assertion (ii) of Theorem (3.2) follows immediately from (i) and (3.10) and the following.

**Assertion (3.11):** $\sum_{i=1}^{m} \alpha_{i,1} = 0$.

**Proof:** By (3.7) we have

$$
\sum_{i=1}^{m} \alpha_{i,1}(u) = \sum_{i=1}^{m} \alpha_{i,1} t^\beta + \text{(higher terms)} = \sum_{i=1}^{m} \sum_{j=1}^{\nu_i} |J_{i,j}| (p(u)_{J,i} - q(u)_{J,i}) = 0.
$$

The last equality is derived from the fact that $p(u)$ and $q(u)$ are in the hyperplane $L$. Now the assertion is immediate from the above equality.

§4. $B_{\ell+1}$-arrangement. Let $R$ be the root system of type $B_{\ell+1}$ in $\mathbb{R}^{\ell+1}$. The corresponding arrangement $\mathcal{H}$ consists of $2\left(\begin{array}{c}
\ell+1 \\
2
\end{array}\right) + \ell + 1$ hyperplanes: $\{\xi_i \pm \xi_j = 0\}$ and $\{\xi_i = 0\}$. The Weyl group
$W$ is isomorphic to a semi-direct product of the symmetric group $S_{\ell+1}$ and the abelian group $(\mathbb{Z}/2\mathbb{Z})^{\ell+1}$ (Chapter 6, [1]). The invariant polynomial ring is generated by

$$t_i = \sum_{\tau \in S_{\ell+1}} \xi_{\tau(1)}^2 \cdots \xi_{\tau(i)}^2, \quad i = 1, \ldots, \ell + 1. \tag{4.1}$$

We will use the following generators.

$$\zeta_i = \xi_1^{2i} + \cdots + \xi_{\ell+1}^{2i}, \quad i = 1, \ldots, \ell + 1. \tag{4.2}$$

Let $\Phi : C^{\ell+1} \to C^{\ell+1}/W \cong C^{\ell+1}$ be the map defined by $\xi \mapsto (\zeta_1(\xi), \ldots, \zeta_{\ell+1}(\xi))$. We take $S = S_{\text{min}}$. The stratification $S$ can be described as follows. Let $E_1$ be the set of the subdivisions of the non-empty subsets of $\{1, \ldots, \ell + 1\}$. Namely an element $F \in E_1$ can be written as $F = \{I_1, \ldots, I_k\}$ where each $I_i$ is non-empty and $I_i \cap I_j = \emptyset$ for $i \neq j$. Let $S(F) = \bigcup_{i=1}^{k} I_i$ and $F^c = \{1, \ldots, \ell + 1\} - S(F)$. Let $E_2$ be the set of the partitions of the integer $m$ for $m = 1, \ldots, \ell + 1$. There is a canonical surjective mapping from $E_1$ to $E_2$ by $F \mapsto |F| = \{|I_1|, \ldots, |I_k|\}$. Let

$$M(F) = \{ \xi \in C^{\ell+1} ; (i) \, \xi_i = 0 \iff i \in F^c, (ii) \, \xi_i^2 = \xi_j^2 \iff \{i,j\} \subseteq \exists \mathcal{I}\}$$

We omit $M = \{\{1\}, \ldots, \{\ell+1\}\}$ and $|M|$ from $E_1$ and $E_2$ respectively as $M(M)$ and $V(|M|)$ are nothing but the complement $C^{\ell+1} - H$ and $C^{\ell+1} - D$. Let $\alpha = \sum_{i=1}^{k} |I_i| - k$. Then $M(F)$ is a disjoint union of $2^\alpha$ connected components corresponding to the sign of $\xi_i = \pm \xi_j$ in the definition of $M(F)$. But they are in the same $W$-orbit. (Recall that the reflection along $\{\xi_i = 0\}$ is the multiplication by $-1$ in the $i$-th coordinate.) Thus each connected component is mapped by $\phi$ onto the same stratum of $\bar{S}$. We define partial orderings in $E_1$ and $E_2$ as follows. Let $F = \{I_1, \ldots, I_k\}$ and $G = \{J_1, \ldots, J_n\}$. $F \succeq G$ if and only if (i) $F^c \subseteq G^c$, (ii) $\tilde{F} \succeq \tilde{G}$ in $C_1$. Here $\tilde{F}$ is defined by $\{F^c, I_1, \ldots, I_k\} \in C_1$. Similarly we define $|F| \succeq |G|$ if and only if (i) $|F^c| \leq |G^c|$, (ii) $|\tilde{F}| \succeq |\tilde{G}|$ in $C_2$.

Now the following propositions are completely parallel to Proposition (2.5) and Proposition (2.6).

**Proposition (4.3).** Let $F, G \in E_1$. The following conditions are equivalent.

(i) $\overline{M(F)} \supseteq M(G)$. (ii) $\overline{M(F)} \cap M(G) \neq \emptyset$. (iii) $F \succeq G$.

**Proposition (4.4).** Let $F, G \in E_1$. The following conditions are equivalent.

(i) $\phi(M(F)) = \phi(M(G))$. (ii) There exists a $g \in W$ such that $g(M(F)) = M(G)$. (iii) $|F| = |G|$.

Thus for a $K \in E_2$ we can define $V(K) = \phi(M(F))$ for any $F \in E_1$ such that $|F| = K$. Now we study the tangential map. Note that

$$d\Phi_\xi(\frac{\partial}{\partial \xi_i}) = 2\xi_i X'(\xi_i^2). \tag{4.5}$$
For each $I \subset \{1, \ldots, \ell + 1\}$, we define $m(I) = \min \{i \mid i \in I\}$. Let $\mathcal{F} = \{I_1, \ldots, I_k\} \in \mathcal{E}_1$ and let $\xi \in \mathcal{F}$. We define $\bar{\xi} \in M(\mathcal{F})$ by

\[\bar{\xi}_j = \begin{cases} \xi_{m(I_i)} & \text{if } j \in I_i \\ 0 & \text{if } j \in \mathcal{F}^c. \end{cases}\]

It is easy to see that $\bar{\xi}$ is in the W-orbit of $\xi$. We also define

\[\frac{\partial}{\partial \xi_{I_i}} = \frac{1}{|I_i|} \sum_{j \in I_i} (\xi_j/\xi_{m(I_i)}) \frac{\partial}{\partial \xi_j}.\]

Note that $\xi_j/\xi_{m(I_i)} = \pm 1$ and $\xi^2_j = \xi^2_{m(I_i)} = \bar{\xi}_I^2$ for each $j \in I_i$. It is easy to see that $\frac{\partial}{\partial \xi_{I_i}} \in T_\xi M(\mathcal{F})$ and $d\Phi_\xi(\frac{\partial}{\partial \xi_{I_i}}) = 2\bar{\xi}_{I_i}X'(\bar{\xi}_I^2)$. Now Proposition (2.11) and Lemma (2.15) can be translated into the following form.

**Proposition (4.7).** Let $\mathcal{F} = \{I_1, \ldots, I_k\} \in \mathcal{E}_1$. Then

(i) The dimension of $T_\xi M(\mathcal{F})$ is $k$ and it is generated by $\left\{ \frac{\partial}{\partial \xi_{I_i}} \mid i = 1, \ldots, k \right\}$.

(ii) The restriction $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ is a finite covering.

(iii) $V(|\mathcal{F}|)$ is non-singular and $T_\phi(\xi) V(|\mathcal{F}|)$ is generated by $\left\{ X'(\bar{\xi}_I^2) \mid i = 1, \ldots, k \right\}$.

**Lemma (4.8).** Let $\mathcal{F}$ be as in Proposition (4.7). Then

\[X'_s(\bar{\xi}_I^2, \ldots, \bar{\xi}_I^2) \in T_\phi(\xi) V(|\mathcal{F}|) \quad \text{for } s = 1, \ldots, k.\]

Let $\mathcal{F} \succeq \mathcal{G}$ and let $\mathcal{G} = \{J_1, \ldots, J_m\}$. We can write $\mathcal{F} = \{J_{i,j} \mid i = 0, \ldots, m, j = 1, \ldots, \nu_i\}$ so that $J_{i,j} \subset J_i$ where $J_0 = \mathcal{G}^c$ by definition. Let $\bar{p}(u)$, $\bar{q}(u)$, $\bar{p}(u)$, $\bar{q}(u)$, $\eta$, $T$, and $\gamma$ be as §3. We consider the equality $\bar{p}(u) - \bar{q}(u) = \sum_{i=0}^{m} \sum_{j=1}^{\nu_i} |J_{i,j}| (X(\bar{p}(u)_{I_i}) - X(\bar{q}(u)_{I_i}))$. Then using Lemma (4.8), we do the same argument as for the $A_{\ell+1}$-discriminant to obtain

**Theorem (4.9).** $\mathcal{S}$ is a regular stratification with the unique tangential limits property. Namely

(i) $T$ is generated by $\left\{ X^{(j)}(\eta_{I_i}^2) \mid i = 0, \ldots, m, j = 1, \ldots, \nu_i \right\}$. (ii) (Regularity) $\gamma \in T$.

For the stratification of discriminant variety of $D_\ell$, see [8].

**References**


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