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TOPOLOGY OF SPACES OF RATIONAL MAPPS AND NONLINEAR INTEGRABLE SYSTEMS OF LAX TYPE

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ABSTRACT. In this paper we discuss some ideas and results related to a characterization of spaces of rational mapps of degree k in terms of nonlinear integrable systems of Lax type.

First we address the problem of parametrizing the space of rational mapps: $CP^1 \to CP^1$ of degree k taking ∞ to 0 in terms of solutions of certain nonlinear integrable systems. The origin of this problem is in the study of moduli space $\mathcal{M}_k$ of SU(2) Yang-Mills-Higgs k-monopoles by S.K. Donaldson [1]. Here the k-monopoles is a finite ($4\pi k$) potential energy static solution of the Yang-Mills-Higgs equations on Min^4 having 4k-1 real parameters. He proved

THEOREM (Donaldson) *There is a one-to-one correspondence between the extended moduli space* $\tilde{\mathcal{M}}_k$ *of* SU(2) k-*monopoles and the space* $\mathcal{R}at^C(k)$ *of rational mapps*: $CP^1 \to CP^1$ *of degree* k *which takes* ∞ *to* 0.

The space $\mathcal{M}_k$ has 4k real parameters and is a circle bundle of the moduli space $\mathcal{M}_k$ of k-monopoles. Each map of $\mathcal{R}at^{\mathbb{C}}(k)$ can be regarded as a scattering date for k-monopole. Being based upon the Donaldson's theorem and the result by G. Segal [2] on the topology of rational functions, recently the author and T.E. Duncan [3] consider the topology of the moduli space of k-monopoles. Let $\mathcal{L}_k$ be the loop space of mapps: $S^2 \to S^2$ of degree k taking ∞ to 0. It is proved

PROPOSITION([3]) *The inclusion* $\mathcal{M}_k \subset \mathcal{L}_k$ *induces a homotopy equivalence up to dimension* k, *namely,* $\pi_j(\mathcal{M}_k) \simeq \pi_j(\mathcal{L}_k)$ *for* $1 \leq j \leq k-1$.

This helps us to determine explicitely the homotopy groups of the moduli space $\mathcal{M}_k$. We note $\pi_0(\mathcal{M}_k) \simeq \{0\}$. Some topological features of $\mathcal{M}_k$ are also discussed in [3]. For example, the integer k is equal to the first Chern class of certain holomorphic vector bundle on S^1 and the Euler characteristic of $\mathcal{M}_k$ is zero.

We can recall that many classes of rational functions have arisen as rational solutions of various nonlinear integrable systems such as the KP equation and the stational Einstein equations. Here let us consider, conversely, what nonlinear integrable system completely parametrizes a given space of rational mapps (or functions)? Let $\mathcal{R}at^{\mathbb{C}}_{gen}(k)$ be the space of generic rational functions of the form $f(z)=\Sigma_{j=1}^{k} a_j/(z-z_j)$, where $a_j \neq 0$ and $z_i \neq z_j$. It is shown in [3] that the (finite nonperiodic complex) Toda equation defines a flow on the space $\mathcal{R}at^{\mathbb{C}}_{gen}(k)$. In the subsequent discussions we consider a subspace of $\mathcal{R}at^{\mathbb{C}}(k)$, denoted by $\mathcal{R}at^{\mathbb{R}}_{p}(k)$, of *real* coefficient rational functions of degree k and the *fixed* monic denominator p(z). See [4] for details.

The motivation for studying $\mathcal{R}at_p^R(k)$ is that, surprisingly, the deformations on complex z-plane which leave invariant the singularities of solutions are closely related to the (complete) integrability of nonlinear dynamical systems and nonlinear PDEs under consideration. Indeed, there is an interesting conjecture [5] that a dynamical system described by a differential system or a PDE is *integrable* whenever the solutions have the Painlevé property, namely, that their only movable singularities are poles. It is to be noted that the celebrated Kovalevskaja's top [6] was found by supposing that general solution of equation of motion should be a single-valued analytic function having no singularities except for poles.

For a given set of real parameters $p=(p_0,\cdots,p_{k-1})$, we see that each function of $\mathcal{R}at_p^R(k)$ takes the form

$$f(z) = \frac{q_{k-1}z^{k-1}+\cdots+q_0}{z^k+p_{k-1}z^{k-1}+\cdots+p_0} = \frac{q(z)}{p(z)} \tag{1}$$

where q_j are real parameters, $p(z)$ and $q(z)$ do not have any common factor. It is worth noting that $f(z)$ can be written uniquely as

$$f(z) = C_0^T(zI-A_0)^{-1}B_0 \tag{2}$$

where $C_0^T=(0\cdots01)$, $B_0^T=(q_0\cdots q_{k-1})$, the superscript T denotes the transporsed and A_0 is a $k\times k$ companion matrix such that

$$\det(zI-A_0) = p(z), \quad \mathrm{rank}\left[B_0\ A_0B_0\cdots A_0^{k-1}B_0\right] = k. \tag{3}$$

The set (A_0, B_0, C_0) is called a *cyclic triplet*. The expression (2)

of f(z) is called the observable canonical form in linear systems theory. We can associate f(z) of $\mathcal{R}at_p^R(k)$ with a controllable and observable linear dynamical system

$$\frac{dx(\tau)}{d\tau} = A_0x(\tau)+B_0u(\tau), \quad y(\tau) = C_0^Tx(\tau), \tag{4}$$

where $x(\tau)$ is a k-vector, $u(\tau)$ and $y(\tau)$ are scalar variables. Conversely, f(z) is called the transfer function of the system (4). See [3] about an application of linear systems theory to the topology of rational functions.

Next, let us consider the system ($0 \leqq j \leqq k-1$) of nonlinear PDEs of Lax type

$$\frac{\partial A(t)}{\partial t_j} = [A^j(t)_L{}^T - A^j(t)_L, A(t)], \quad A(0) = A_0, \tag{5}$$

where t is a finite set of time variable, $t=(t_0,\cdots,t_{k-1})$, A(t) is a symmetric $n\times n$ matrix function of t, and subscript L indicates the strictly lower-triangular part. We see that the initial value problem (5) can be solved uniquely by the QR factorization of

$$\exp(\Sigma_{j=0}^{k-1} t_j A_0{}^j). \tag{6}$$

There always exists the factors of this QR factorization. Let us introduce a time evolution of the vectors B_0 and C_0. Consider the system ($0 \leqq j \leqq k-1$) of linear PDEs

$$\frac{\partial B(t)}{\partial t_j} = (A^j(t)+A^j(t)_L{}^T - A^j(t)_L)B(t), \quad B(0) = B_0,$$

$$\frac{\partial C(t)}{\partial t_j} = (A^j(t)_L{}^T - A^j(t)_L)C(t), \quad C(0) = C_0. \tag{7}$$

The following lemma is the key connection between the system (5), (7) and the space $\mathcal{R}at_p^R(k)$.

LEMMA([4]) *If* (A_0, B_0, C_0) *is a cyclic triplet, then* $(A(t), B(t), C(t))$ *is also a cyclic triplet for every* t *of* R^k.

Thus any set of solution of (5) and (7) gives a rational function $f(z; t)$ of $\mathcal{R}at_p^R(k)$ via the formula

$$f(z; t) = C(t)^T(zI-A(t))^{-1}B(t). \tag{8}$$

It is also proved in [4] that $\det(zI-A(t))=p(z)$, and consequently, the flow $f(z)\longrightarrow f(z; t)$ do not have any movable singularities. We note that any member of (5) and (7) is compatible to each other. After the celebrated KP hierarchy and the Toda lattice hierarchy, We call the system (5) with the supplementary system (7) the *cyclic-Toda hierarchy*. The C^∞-solution space for *any* equivalence class $[(A_0, B_0, C_0)]$ of initial cyclic triplets satisfying (3) is called the ***moduli space*** of the cyclic-Toda hierarchy. The main result in [4] is

THEOREM *The space* $\mathcal{R}at_p^R(k)$ *of rational functions of degree* n *and fixed denominator* $p(z)$ *is homeomorphic to the moduli space of the cyclic-Toda hierarchy.*

Clearly the cyclic-Toda flow leaves invariant the poles of $f(z)$. We see that the cyclic-Toda hierarchy completely parametrizes the space $\mathcal{R}at_p^R(k)$ of rational functions (which have no singularities except for poles). Recall the Kovalevskaya's criterion [6]. The space $\mathcal{R}at_p^R(k)$ has 2^r for $0\leq r\leq k-1$, or 2^r-1 for $r=k$ connected components, where r is

a number of real distinct roots of $p(z)$. It is also proved [4] that the flow of cyclic-Toda hierarchy for the fixed initial value (A_0, B_0, C_0) is identified with one of the connected components of $\mathcal{R}at_p^R(k)$. Note also that poles of $f(z)$ (roots of $p(z)$) move about complex z-plane according to the choice of the initial value A_0 through $\det(zI-A_0)=p(z)$. Furthermore, it is easy to see that the cyclic-Toda hierarchy has the Painlevé property defined in [5].

Finally we shall take a case where $\tilde{p}(z)$ admits k real distinct roots. The initial value A_0 can be transformed into a Jacobi matrix,

$$A_0 = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{k-1} \\ & & a_{k-1} & b_k \end{bmatrix}, \tag{9}$$

where $a_j \neq 0$. From (5) with (7) we obtain so-called the Jacobi-Toda hierarchy whose moduli space is homeomorphic to $\mathcal{R}at_{\tilde{p}}^R(k)$.
By setting $B_0^T=C_0^T=(0\cdots0\ 1)$, we obtain a subsystem whose flow is identified with the connected components $\mathcal{R}at_{\tilde{p}}^R(k,0)$ of $\mathcal{R}at_{\tilde{p}}^R(k)$. Here $\mathcal{R}at_{\tilde{p}}^R(k,0)$ has Cauchy index k and is diffeomorphic to R^k. The original (finite nonperiodic) Toda equation is a special member of the subsystem parametrized by t_1. The relationship between the original Toda equation and the rational function of $\mathcal{R}at_{\tilde{p}}^R(k,0)$ was first pointed out by J. Moser [7].

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