A COUNTER EXAMPLE OF STRONG BAUM-CONNES CONJECTURES

FOR FOLIATED MANIFOLDS

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§1 Introduction Related to a new index theory, Baum and Connes conjectured in [1] that the analytic and topological $K$-theory for foliations and dynamical systems are isomorphic each other under the $K$-theoretic index map. Since then, there appeared several papers supporting the conjecture (cf:[1]~[9] except [4]). However, Skandalis recently showed in [4] that there exists a counter example of the strong Connes-Kasparov conjecture for $K$-theory of $C^*$-crossed products. In his proof, a central tool is the property T due to Kazhdan in semisimple Lie groups of real rank one.

Modifying his idea, we shall show in this note that there exists a counter example of the strong Baum-Connes conjecture for foliated manifolds.

§2 Preliminaries Let $(M,F)$ be a foliated manifold and $G$ its holonomy groupoid. Taking the source and range maps $s,r$ from $G$ to $M$ respectively, one can define the foliation $\tilde{F}$ of $G$ coming from the tensor product of the pull backs $s^*(F)$ and $r^*(F)$ of $F$ by $s$ and $r$
respectively. Let $\Omega^{1/2}$ be the half density bundle over $G$ tangential to $\mathfrak{F}$ and denote by $C_c(\Omega^{1/2})$ the $*$-algebra of all continuous sections of $\Omega^{1/2}$ with compact support. The $*$-algebraic operation is defined as follows:

$$(f \cdot g)(\tau) = \int_{\tau = \tau_1 \tau_2} f(\tau_1)g(\tau_2)$$

$$f^*(\tau) = \overline{f}(\tau^{-1})$$

for all $f, g \in C_c(\Omega^{1/2})$ where $\overline{f}(\tau) = \overline{f(\tau)}$. Given any $x \in M$, let $H_x$ be the Hilbert space consisting of all $L^2$-sections of $\Omega^{1/2}$ over $G$ and $\pi_x$ the $*$-representation of $C_c(\Omega^{1/2})$ on $H_x$ defined by

$$(\pi_x(f)\xi)(\tau) = \int_{\tau = \tau_1 \tau_2} f(\tau_1)\xi(\tau_2)$$

for all $f \in C_c(\Omega^{1/2})$ and $\xi \in H_x$. The completion $C^*_r(M,F)$ of $C_c(\Omega^{1/2})$ with respect to $\|f\| = \sup_{x \in M} \|\pi_x(f)\|$ is called a foliation $C^*$-algebra associated to $(M,F)$.

We now consider the $K$-theory $K_a(M,F)$ of $C^*_r(M,F)$ which is called the analytic $K$-theory of $(M,F)$. We also have another $K$-theory of $(M,F)$ using a pure topological way. It is called the topological $K$-theory of $(M,F)$ which is denoted by $K_t(M,F)$. By Baum-Connes[1], if $G$ is torsion free, then $K_t(M,F)$ is nothing more than the twisted $K$-theory $K^\nu(BG)$ of the classifying space $BG$ of $G$ by the transverse bundle $\nu$ of $F$. More precisely, the latter is the $K$-theory $K(B\tilde{\nu}/S\tilde{\nu})$ of the Thom space $B\tilde{\nu}/S\tilde{\nu}$ of the $\nu$-bundle $\tilde{\nu}$ over $BG$. We then state the Baum-Connes conjecture for foliated manifolds as follows:

**Baum-Connes Conjecture:** Given a smooth foliated manifold $(M,F)$, the $K$-index map is an isomorphism from $K_t(M,F)$ to $K_a(M,F)$. 

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There exists several papers supporting the above conjecture (cf:[1]-[9] except [4]).

Suppose G is torsion free and ν has a spin$^C$ structure, then we know that $K^0(BG)$ is equal to $K(BG)$ via Thom isomorphism. Related to the conjecture, we may formulate the following conjecture which is a strong version of the Baum-Connes' one:

**Strong Baum-Connes Conjecture:** Given any smooth foliated manifold $(M,F)$ where G is torsion free and ν is spin$^C$, then the K-index map is an KK-isomorphism from $K^*_c(M,F)$ to $K_a(M,F)$ in the sense of Kasparov.

§3 Construction

Let G be a connected semisimple linear Lie group which is locally isomorphic to $Sp(n,1)$ or $F_4$ $(n \geq 2)$. Then it is of real rank one without Kazhdan's property T. Thanks to Skandalis[4], it is non K-nuclear, which means that the K-theory of the reduced group C*-algebra is no longer KK-isomorphic to that of nuclear C*-algebras. Using this fact, he found a counter example of the strong Connes-Kasparov conjecture for C*-crossed products. We now modify his idea in order to find a counter example of the strong Baum-Connes conjecture for foliated manifolds.

Let V be a vector group on which G acts faithfully. By Borel's result [12], there exist torsion free uniform lattices $\Gamma$, $\Delta$ of G, V respectively such that $\Delta$ is $\Gamma$-invariant. By Wang's result [10], the semidirect product $V \times_s G$ of V by G has the property T which is no longer of real rank one. Since $\Delta \times_s \Gamma$ is a uniform lattice of $V \times_s G$, it also has the property T. By the similar way as in the proof of Skandalis [4], we have the following lemma which is of
Lemma 1. \( C^*_\Gamma(\Delta \times \Gamma) \) is non KK-nuclear.

**Remark** Dr. Matsumoto generalized the above lemma in more general setting.

Since \( \Delta \) is a torsion free uniform lattice of \( \Gamma \), the character group \( \Delta \) of \( \Delta \) is a torus which may be chosen as of even dimension.

Let \( H \) be a maximal compact subgroup of \( G \) and \( M = (G/H) \times \Delta \) the orbit space of \( (G/H) \times \Delta \) by the diagonal \( \Gamma \)-action. As \( \Gamma \) is torsion free, \( M \) is a smooth manifold. Let \( F = (G/H) \times \Gamma(t) : t \in \Delta \). Then we have the following lemma:

**Lemma 2.** \((M,F)\) is a foliated \( \Delta \)-bundle over \( \Gamma \backslash G/H \).

Let us consider the reduced crossed product \( (C(\Delta) \times \Gamma) \overset{\text{r}}{\otimes} \) of \( C(\Delta) \) by the holonomy action \( \alpha \) of \( \Gamma \) on \( \Delta \) and \( BC(\cdot) \) means the \( C^* \)-algebra of all compact operators on a Hilbert space \( \cdot \). By the joint work with Natsume (cf:[11]), we have by Lemma 1 the following Lemma:

**Lemma 3.** \( C^*_\Gamma(M,F) \) is isomorphic to \( (C(\Delta) \times \Gamma) \overset{\alpha}{\otimes} BC(L^2(\Gamma \backslash G/H)) \).

It then easily follows from Lemma 2 that

**Corollary 4.** \( K_a(M,F) = K_a(\Delta,\Gamma) = K(C(\Delta) \times \Gamma) \).

In what follows, we shall determine \( K_t(M,F) \). Let us consider the leaf \( \ell_\chi \) in \( F \) passing through \( \chi \in \Delta \). Then its fundamental group \( \pi_1(\ell_\chi) \) is the stabilizer \( \Gamma_\chi \) of \( \Gamma \) at \( \chi \). Since \( \Delta \) is the torus of even dimension, the holonomy map \( h_\chi \) of \( \ell_\chi \) is a homomorphism from \( \pi_1(\ell_\chi) \) into \( \pi_1(\Delta) \). Therefore, \( h_\chi(\Gamma_\chi) \) is torsion free for all \( \chi \in \Delta \), which means that \( G \) is also torsion free. By Baum-Connes [1], we then have
the following lemma:

**Lemma 5.** \( K_t(M,F) = K^\nu(BG) \) where \( \nu = T(M)/F \).

In our case, as \( T(M)/F = T(\Delta) \), \( \nu \) has a complex structure. By Thom isomorphism, we deduce the following lemma:

**Lemma 6.** \( K^\nu(BG) = K(BG) \).

Combining all the lemmas cited above, we obtain the following main theorem:

**Theorem 7.** Let \( G \) be a connected semisimple Lie group which is locally isomorphic to \( Sp(n,1) \) or \( F_4(n \geq 2) \) acting on a vector group \( V \) faithfully. Let \( \Gamma, \Delta \) be torsion free uniform lattices of \( G, V \) respectively such that \( \Gamma \) is \( \Delta \)-invariant. If \( (M,F) \) is a foliated \( \Delta \)-bundle over \( \Gamma \backslash G/H \), then it is a counter example for the strong Baum-Connes conjecture where \( \Delta \) is the character group of \( \Delta \) and \( H \) is a maximal compact subgroup of \( G \).

**Proof.** By definition, the holonomy groupoid \( G \) of \( F \) has no torsion and \( \nu = T(M)/F \) has a complex structure. Suppose the \( K \)-index map is a \( KK \)-isomorphism from \( K_t(M,F) \) to \( K_a(M,F) \). This means by Lemma 3,5 and 6 that there exists an invertible element of \( KK((C(\Delta)\times_\alpha \Gamma)_r, C_0(BG)) \) which implement the \( K \)-index map where \( C_0(BG) \) is the \( C^* \)-algebra of all continuous sections of \( E\Gamma \times_{\Gamma} C(\Delta) \) vanishing at infinity. Since \( C_0(BG) \) is nuclear and \( (C(\Delta)\times_\alpha \Gamma)_r = C_\Gamma(\Delta \times_\Gamma \Gamma) \) up to isomorphism, this contradicts to Lemma 1. Q.E.D.

**Remark.** In spite of the above theorem, the original Baum-Connes conjecture still remains open even for the above example.
References