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Kyoto University
A Difference Set Of A Cantor Set

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Abstract. An example of a regular Cantor set whose self-difference set is a Cantor set with a positive measure is given. This is a counter example of one of the questions related to the homoclinic bifurcation of surface diffeomorphisms.

§.0 Introduction.

In [2], Palis–Takens investigated the homoclinic bifurcations of surface diffeomorphisms in the following context. Let $M$ be a closed 2-dimensional manifold. We say a $C^r$-diffeomorphism $\phi : M \to M$ is persistently hyperbolic if there is a $C^r$-neighborhood $\mathcal{U}$ of $\phi$ and for every $\psi \in \mathcal{U}$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [1] for the definitions and the notations of the terminologies of dynamical systems). Let $\{\phi_{\mu}\}_{\mu \in \mathbb{R}}$ be a 1-parameter family of $C^2$-diffeomorphisms on $M$. We define $\{\phi_{\mu}\}_{\mu \in \mathbb{R}}$ has a homoclinic $\Omega$-explosion at $\mu = 0$ if:

i) For $\mu < 0$, $\phi_{\mu}$ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\tilde{\Omega}(\phi_0) = \lim_{\mu \uparrow 0} \Omega(\phi_{\mu})$ together with a homoclinic orbit of tangency $\mathcal{O}$ associated with a fixed saddle point $p$, so that $\Omega(\phi_0) = \tilde{\Omega}(\phi_0) \cup \mathcal{O}$; the product of the eigenvalues of $d\phi_0$ at $p$ is different from one in norm;

iii) The separatrices have quadratic tangency along $\mathcal{O}$ unfolding generically; $\mathcal{O}$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of $\phi_0$.

Let $\Lambda$ be a basic set of a diffeomorphism on $M$. $d^s(\Lambda)$ ($d^u(\Lambda)$) denotes the Hausdorff dimension in the transversal direction of the stable (unstable) foliation of stable (unstable) manifold of $\Lambda$ (refer [2] for the precise definition), and is called the stable (unstable) limit capacity. $B$ denotes the set of values $\mu > 0$ for which $\phi_{\mu}$ is not persistently hyperbolic.

The result of Palis–Takens is;
**Theorem** [2]. Let \( \{\phi_{\mu}; \mu \in \mathbb{R}\} \) be a family of diffeomorphisms of \( M \) with a homoclinic \( \Omega \)-explosion at \( \mu = 0 \). Suppose that \( d^s(\Lambda) + d^u(\Lambda) < 1 \), where \( \Lambda \) is the basic set of \( \phi_0 \) associated with the homoclinic tangency. Then

\[
\lim_{\delta \to 0} \frac{m(B \cap [0, \delta])}{\delta} = 0
\]

where \( m \) denotes Lebesgue measure.

This result says that if \( d^s(\Lambda) + d^u(\Lambda) < 1 \), then the measure of the parameters for which bifurcation occurs is relatively small.

Now the case of \( d^s(\Lambda) + d^u(\Lambda) > 1 \) comes into question as the next step. In the proof of the theorem above, one of the essential points is a question of how two Cantor sets in the line intersect each other when the one is slid. In [3], Palis proposed the following questions.

(Q.1) For affine Cantor sets \( X \) and \( Y \) in the line, is it true that \( X - Y \) either has measure zero or contains intervals?

(Q.2) Same for regular Cantor sets.

For two subset \( X, Y \) of \( \mathbb{R} \),

\[
X - Y = \{ x - y | x \in X, y \in Y \}.
\]

This can also be written as;

\[
X - Y = \{ \mu \in \mathbb{R} | X \cap (\mu + Y) \neq \emptyset \},
\]

namely \( X - Y \) is the set of parameters for which \( X \) and \( Y \) have an intersection point when \( Y \) is slid on the line.

Cantor set \( \Lambda \) in \( \mathbb{R} \) is called **affine**, **regular** or \( C^r \) for \( 1 \leq r \leq \infty \) if \( \Lambda \) is defined with finite number of expanding affine, \( C^2 \) or \( C^r \) maps respectively (see §2 Definition 5 for the rigorous definition).

Our result in this note is that there is a counter example of (Q.2), namely;

**Theorem.** There exists a \( C^\infty \)-Cantor set \( \Lambda \) such that

(i) \( m(\Lambda - \Lambda) > 0 \),
(ii) \( \Lambda - \Lambda \) is a Cantor set.

In the succeeding sections, we give an outline of the proof of this theorem. The complete proof will appear elsewhere.

\[ \] 

\[ \] 

\section{1 \textbf{Definition of the Cantor sets } \Lambda(s), \Gamma(s).} 

First of all, we define two cantor set depending on a sequence of real numbers.

\textbf{Definition 1.} Let \( I = [x_1, x_2] \) be a closed interval and \( \lambda \) a real number with \( 0 < \lambda < \frac{1}{2} \).

We define,

\[
I_0(\lambda; I) = [x_1, x_1 + \lambda(z_2 - x_1)] \quad I_1(\lambda; I) = [x_2 - \lambda(z_2 - x_1), z_2].
\]

\textbf{Definition 2 (Cantor set } \Lambda(s) ). Let \( I^0 = [0, 1] \) and \( s = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) be a one sided sequence of real numbers with \( 0 < \lambda_i < \frac{1}{2} \) for all \( i \geq 1 \).

We define the Cantor set \( \Lambda(s) \) as follows.

Let \( I_0^1 = I_0(\lambda_1; I^0), I_1^1 = I_1(\lambda_1; I^0) \) and \( I^1 = I_0^1 \cup I_1^1 \). \( \Delta_n \) denotes the set of all sequences of 0 and 1 of length \( n \). When \( I_{\beta}^{n-1} \)'s are defined for all \( \beta \in \Delta_n-1 \), we define;

\[
I_{\beta 0}^n = I_0(\lambda_n; I_{\beta}^{n-1}) \quad I_{\beta 1}^n = I_1(\lambda_n; I_{\beta}^{n-1}).
\]

Inductively, we can define \( I_{\alpha}^n \) for all \( \alpha \in \Delta_n \) and for all \( n \geq 0 \). Define

\[
I^n = \bigcup_{\alpha \in \Delta_n} I_{\alpha}^n
\]

and

\[
\Lambda(s) = \bigcap_{n \geq 0} I^n.
\]

This is clearly a Cantor set by the definition.

Next, we define another Cantor set \( \Gamma(s) \).
DEFINITION 3. Let $J = [x_1, x_2]$ and $0 < \lambda < \frac{1}{3}$. We define,

$$J_0(\lambda; J) = [x_1, x_1 + \lambda(x_2 - x_1)]$$
$$J_1(\lambda; J) = \left[\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)\right]$$
$$J_2(\lambda; J) = [x_2 - \lambda(x_2 - x_1), x_2].$$

DEFINITION 4. Let $J^0 = [-1, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Let

$$J_0^1 = J_0(\lambda_1; J^0)$$
$$J_1^1 = J_1(\lambda_1; J^0)$$
$$J_2^1 = J_2(\lambda_1; J^0)$$

and $\Pi_n$ denote the set of all sequences of $0, 1, 2$ of length $n$. When $J_{\delta}^{n-1}$'s are defined for all $\delta \in \Pi_{n-1}$, we define;

$$J_0^n = J_0(\lambda_n; J_{\delta}^{n-1})$$
$$J_1^n = J_1(\lambda_n; J_{\delta}^{n-1})$$
$$J_2^n = J_2(\lambda_n; J_{\delta}^{n-1}).$$

Inductively, we can define $J_\gamma^n$ for all $\gamma \in \Pi_n$ and for all $n \geq 0$. Define

$$J^n = \bigcup_{\gamma \in \Pi_n} J_\gamma^n$$

and

$$\Gamma(s) = \bigcap_{n \geq 0} J^n.$$

These cantor sets have the following relation.

THEOREM 1. Let $s = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Then,

$$\Lambda(s) - \Lambda(s) = \Gamma(s).$$
§.2 Outline of the proof.

**Definition 5.** Let \( \Lambda \) be a Cantor set on a closed interval \( I \). \( \Lambda \) is called affine, regular or \( C^r \)-Cantor set for \( 1 \leq r \leq \infty \) if there are closed disjoint intervals \( I_1, \ldots, I_k \) on \( I \) and onto affine, \( C^2 \) or \( C^r \)-maps \( f_i : I_i \to I \) for all \( 1 \leq i \leq k \) such that:

(i) \( |f'_i(x)| > 1 \quad \forall x \in I_i \)

(ii) \( \Lambda = \bigcap_{n=0}^{\infty} \{ \bigcup_{\sigma \in \Sigma_n^k} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} \cdots f_{\sigma(n)}^{-1}(I) \} \),

where \( \Sigma_n^k = \{ \sigma : \{1, \ldots, n\} \to \{1, \ldots, k\} \} \).

Our main result is restated as follows.

**Theorem 2.** There exists a sequence of real numbers \( s = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) with \( 0 < \lambda_i < \frac{1}{8} \) for all \( i \geq 1 \) such that:

(i) \( \Lambda(s) \) is a \( C^\infty \)-Cantor set,

(ii) \( m(\Lambda(s) - \Lambda(s)) > 0 \),

where \( m() \) denotes the Lebesgue measure.

From now on, we shall give the outline of the proof of this Theorem 2.

Let \( \{r_n\}_{n \geq 0} \) be a sequence of positive real numbers such that

\[
\sum_{n=0}^{\infty} r_n < 1 .
\]

We define \( \{\lambda_n\}_{n \geq 1} \) using this \( \{r_n\}_{n \geq 0} \) as follows.

\[
\begin{aligned}
\lambda_1 &= \frac{1}{3} (1 - r_0) \\
\lambda_{n+1} &= \frac{1}{3} \left( \frac{1 - \sum_{i=0}^{n} r_i}{1 - \sum_{i=0}^{n-1} r_i} \right)
\end{aligned}
\]

It is easily seen that

\[
0 < \lambda_n < \frac{1}{3} \quad \forall n \geq 1 .
\]

These numbers has the following relations.
**Lemma 3.**

\[ \sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \quad \forall n \geq 0. \]

**Lemma 4.**

\[ r_n = 3^n (1 - 3\lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \quad \forall n \geq 0. \]

where, we assume \( \prod_{j=1}^{0} \lambda_j = 1 \) for the simplicity of notation.

Using these lemmas, we can show (ii) of Theorem 2. In fact, the following lemma holds.

**Lemma 5.** Let \( \{r_n\}_{n \geq 0} \) be a sequence of positive real numbers such that \( \sum_{n=0}^{\infty} r_n < 1 \), and \( \{\lambda_n\}_{n \geq 1} \) be the sequence defined by (2). Then, \( m(\Gamma(s)) > 0 \).

### §3 The regularity of \( \Lambda(s) \).

We define a sequence \( \{r_n\}_{n \geq 0} \) (and so \( \{\lambda_n\}_{n \geq 1} \), and prove that \( \Lambda(s) \) is \( C^\infty \). First of all, we fix a \( C^\infty \)-function \( h(t) \) on \( [0,1] \) with the following properties.

(i) \( h(t) \geq 0 \),

(ii) \( \int_0^1 h(t) dt = 1 \),

(iii) for all \( n \geq 0 \),

\[ \begin{cases} \lim_{t \downarrow 0} h^{(n)}(t) = 0, \\ \lim_{t \uparrow 1} h^{(n)}(t) = 0. \end{cases} \]

To define \( \{r_n\}_{n \geq 0} \), we define the following sequences. For each integers \( n \geq 0 \), let

\[ q_n = \max\{q_0, q_1, \ldots, q_{n-1}, 1, \sup_{t \in [0,1]} |h^{(n)}(t)| \}. \]

For \( n \geq 0 \), we define,

\[ r_n = \frac{4^{-n^2+2}}{q_n} \]
Since $r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)}$, we have,

$$
\sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12}.
$$

Therefore, $\{r_n\}_{n \geq 0}$ satisfy (1). We define another sequence of positive real numbers:

$$m_n = \frac{3(3r_{n-1} - r_n)}{2^{n-1}(1 - \sum_{i=0}^{n-1} r_i)} \quad \forall n \geq 1.$$

Since $\{r_n\}_{n \geq 0}$ is monotonically decreasing and by (4), $m_n > 0$ for all $n \geq 1$.

$U^0$ denotes the open interval between $I_0^1$ and $I_1^1$, namely;

$$U^0 = I^0 \setminus (I_0^1 \cup I_1^1).$$

In general, $U_{\alpha}^{n-1}$ ($\alpha \in \Delta_{n-1}$) denotes the open interval between $I_{\alpha 0}^n$ and $I_{\alpha 1}^n$ in $I_{\alpha}^{n-1}$, namely;

$$U_{\alpha}^{n-1} = I_{\alpha}^{n-1} \setminus (I_{\alpha 0}^n \cup I_{\alpha 1}^n).$$

Let $\ell_n = \ell(I_{\alpha}^n)$. Then, by the definition,

$$\ell_n = \lambda_n \ell_{n-1}.$$

Let $u_n = \ell(U_{\alpha}^n)$, and $U_{\alpha}^n = [x_{\alpha}, y_{\alpha}]$. Then,

$$u_n = \ell_n - 2\ell_{n+1},$$

and

$$u_n = y_{\alpha} - x_{\alpha}.$$

We prove the smoothness of $A(s)$ as follows. We define a non-negative $C^\infty$-function $f(t)$ on $[0, \lambda_1]$ and define;

$$g(t) = \int_0^t (f(s) + 3) \, ds.$$

We put;

$$
\begin{cases}
g_0(t) = g(t) & \text{on } [0, \lambda_1] \\
g_1(t) = g(t - 1 + \lambda_1) & \text{on } [1 - \lambda_1, 1].
\end{cases}
$$
and prove that these \(g_0\) and \(g_1\) define \(\Lambda(s)\).

**Definition of \(f(t)\).** Recall that we have already defined a \(C^\infty\)-function \(h(t)\) on \([0, 1]\). We define \(f(t)\) using this \(h(t)\) as follows. Let \([x'_{\alpha}, y'_{\alpha}]\) be the interval of length \(\frac{l_n}{3}\) in the middle of \(U^n_{\alpha}\) such that

\[
[x'_{\alpha}, y'_{\alpha}] = [x_{\alpha} + \frac{1}{2}(u_n - \frac{l_n}{3}), y_{\alpha} - \frac{1}{2}(u_n - \frac{l_n}{3})].
\]

We define \(f(t)\) as follows.

(i) On \(U^n_{\alpha} (n \neq 0)\),

\[
\begin{align*}
  f(t) &= m_n h\left(\frac{t - x'_{\alpha}}{\frac{l_n}{3}}\right) \quad t \in [x'_{\alpha}, y'_{\alpha}] \\
  f(t) &= 0 \quad \text{otherwise}.
\end{align*}
\]

(ii) On \(\Lambda(s)\), \(f(t) = 0\).

What we have to show are;

(I) \(f(t)\) is a \(C^\infty\)-function on \([0, \lambda_1]\).

(II) \(g_0\) and \(g_1\) define \(\Lambda(s)\).

To show the smoothness of \(f(t)\), we define a function \(f_n(t)\) for any \(n \geq 0\) as follows. Since \(f(t)\) is \(C^\infty\) on \(U = \cup_{n \geq 1, \alpha \in \Delta_n} U^n_{\alpha} ( = [0, \lambda_1] \setminus \Lambda(s) )\), \(f^{(n)}(t)\) exists for all \(n \geq 0\) on \(U\). We define,

\[
\begin{align*}
  f_n(t) &= f^{(n)}(t) \quad \text{for} \quad t \in U \\
  f_n(t) &= 0 \quad \text{otherwise (i.e.} \quad t \in \Lambda(s) \). 
\end{align*}
\]

The smoothness is shown by proving that;

**Lemma 6.** For any \(n \geq 0\), \(f_n\) is differentiable at any \(t \in [0, \lambda_1]\) and \(f'_n(t) = f_{n+1}(t)\).

For the proof of (II), we need some lemmas. Let \(I^n_{\alpha} = [x^n_{\alpha}, s^n_{\alpha}]\).

**Lemma 7.** For all \(\alpha, \alpha' \in \Delta_n\),

\[
\int_{I^n_{\alpha}} f(t) dt = \int_{I^n_{\alpha'}} f(t) dt.
\]
LEMMA 8. For all \( n \geq 1 \),

\[
\int_{0}^{t_n} f(t) dt = \frac{1}{3} m_n t_n + 2 \int_{0}^{t_{n+1}} f(t) dt.
\]

LEMMA 9. For all \( n \geq 1 \),

\[
l_{n-1} = g_0(l_n).
\]

We have to prove that,

\[
\Lambda(s) = \bigcap_{n \geq 0} \{ \bigcup_{\sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0) \}.
\]

Recall that \( \Sigma_n^2 = \{0,1\}^{\{1,\cdots,n\}} \) and \( I^0 = [0,1] \). This is obtained by showing the following lemma.

LEMMA 10. For all \( n \geq 0 \) and \( \alpha \in \Delta_n \),

\[
g_0(I_{0\alpha}^{n+1}) = I^n_{\alpha}, \quad g_1(I_{1\alpha}^{n+1}) = I^n_{\alpha}.
\]

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