A Difference Set Of A Cantor Set

ATSURO SANNAMI

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060 Japan

Abstract. An example of a regular Cantor set whose self-difference set is a Cantor set with a positive measure is given. This is a counter example of one of the questions related to the homoclinic bifurcation of surface diffeomorphisms.

§.0 Introduction.

In [2], Palis–Takens investigated the homoclinic bifurcations of surface diffeomorphisms in the following context. Let $M$ be a closed 2-dimensional manifold. We say a $C^r$-diffeomorphism $\phi : M \to M$ is persistently hyperbolic if there is a $C^r$-neighborhood $\mathcal{U}$ of $\phi$ and for every $\psi \in \mathcal{U}$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [1] for the definitions and the notations of the terminologies of dynamical systems). Let $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ be a 1-parameter family of $C^2$-diffeomorphisms on $M$. We define $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ has a homoclinic $\Omega$-explosion at $\mu = 0$ if:

i) For $\mu < 0$, $\phi_\mu$ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\tilde{\Omega}(\phi_0) = \lim_{\mu \uparrow 0} \Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $\mathcal{O}$ associated with a fixed saddle point $p$, so that $\Omega(\phi_0) = \tilde{\Omega}(\phi_0) \cup \mathcal{O}$; the product of the eigenvalues of $d\phi_0$ at $p$ is different from one in norm;

iii) The separatrices have quadratic tangency along $\mathcal{O}$ unfolding generically; $\mathcal{O}$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of $\phi_0$.

Let $\Lambda$ be a basic set of a diffeomorphism on $M$. $d^s(\Lambda)$ ( $d^u(\Lambda)$ ) denotes the Hausdorff dimension in the transversal direction of the stable ( unstable ) foliation of stable ( unstable ) manifold of $\Lambda$ (refer [2] for the precise definition), and is called the stable ( unstable ) limit capacity. $B$ denotes the set of values $\mu > 0$ for which $\phi_\mu$ is not persistently hyperbolic.

The result of Palis–Takens is;
THEOREM [2]. Let \( \{\phi_{\mu}; \mu \in \mathbb{R}\} \) be a family of diffeomorphisms of \( M \) with a homoclinic \( \Omega \)-explosion at \( \mu = 0 \). Suppose that \( d^u(\Lambda) + d^u(\Lambda) < 1 \), where \( \Lambda \) is the basic set of \( \phi_0 \) associated with the homoclinic tangency. Then

\[
\lim_{\delta \to 0} \frac{m(B \cap (0, \delta))}{\delta} = 0
\]

where \( m \) denotes Lebesgue measure.

This result says that if \( d^u(\Lambda) + d^u(\Lambda) < 1 \), then the measure of the parameters for which bifurcation occurs is relatively small.

Now the case of \( d^u(\Lambda) + d^u(\Lambda) > 1 \) comes into question as the next step. In the proof of the theorem above, one of the essential points is a question of how two Cantor sets in the line intersect each other when the one is slid. In [3], Palis proposed the following questions.

(Q.1) For affine Cantor sets \( X \) and \( Y \) in the line, is it true that \( X - Y \) either has measure zero or contains intervals?

(Q.2) Same for regular Cantor sets.

For two subset \( X, Y \) of \( \mathbb{R} \),

\[
X - Y = \{ x - y \mid x \in X, y \in Y \}.
\]

This can also be written as;

\[
X - Y = \{ \mu \in \mathbb{R} \mid X \cap (\mu + Y) \neq \emptyset \},
\]

namely \( X - Y \) is the set of parameters for which \( X \) and \( Y \) have a intersection point when \( Y \) is slid on the line.

Cantor set \( \Lambda \) in \( \mathbb{R} \) is called affine, regular or \( C^r \) for \( 1 \leq r \leq \infty \) if \( \Lambda \) is defined with finite number of expanding affine, \( C^2 \) or \( C^r \) maps respectively (see §2 Definition 5 for the rigorous definition).

Our result in this note is that there is a counter example of (Q.2), namely;

THEOREM. There exists a \( C^\infty \)-Cantor set \( \Lambda \) such that

(i) \( m(\Lambda - \Lambda) > 0 \),

2
(ii) $\Lambda - \Lambda$ is a Cantor set.

In the succeeding sections, we give an outline of the proof of this theorem. The complete proof will appear elsewhere.

§.1 Definition of the Cantor sets $\Lambda(s)$, $\Gamma(s)$.

First of all, we define two cantor set depending on a sequence of real numbers.

**Definition 1.** Let $I = [x_1, x_2]$ be a closed interval and $\lambda$ a real number with $0 < \lambda < \frac{1}{2}$. We define,

\[
I_0(\lambda; I) = [x_1, x_1 + \lambda(2 - x_1)] \\
I_1(\lambda; I) = [x_2 - \lambda(2 - x_1), x_2].
\]

**Definition 2 (Cantor set $\Lambda(s)$).** Let $I^0 = [0, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$ for all $i \geq 1$. We define the Cantor set $\Lambda(s)$ as follows.

Let $I_0^1 = I_0(\lambda_1; I^0)$, $I_1^1 = I_1(\lambda_1; I^0)$ and $I^1 = I_0^1 \cup I_1^1$. $\Delta_n$ denotes the set of all sequences of 0 and 1 of length $n$. When $\Gamma^{n-1}_\beta$'s are defined for all $\beta \in \Delta_{n-1}$, we define;

\[
\Gamma_{\beta 0}^n = I_0(\lambda_n; \Gamma_{\beta}^{n-1}) \\
\Gamma_{\beta 1}^n = I_1(\lambda_n; \Gamma_{\beta}^{n-1}).
\]

Inductively, we can define $\Gamma^n$ for all $\alpha \in \Delta_n$ and for all $n \geq 0$. Define

\[
\Gamma^n = \bigcup_{\alpha \in \Delta_n} \Gamma^n_{\alpha}
\]

and

\[
\Lambda(s) = \bigcap_{n \geq 0} \Gamma^n.
\]

This is clearly a Cantor set by the definition.

Next, we define another Cantor set $\Gamma(s)$. 
**Definition 3.** Let $J = [x_1, x_2]$ and $0 < \lambda < \frac{1}{3}$. We define,

$$J_0(\lambda; J) = [x_1, x_1 + \lambda(x_2 - x_1)]$$

$$J_1(\lambda; J) = \left[\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)\right]$$

$$J_2(\lambda; J) = [x_2 - \lambda(x_2 - x_1), x_2].$$

**Definition 4.** Let $J^0 = [-1, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Let

$$J_0^1 = J_0(\lambda_1; J^0)$$

$$J_1^1 = J_1(\lambda_1; J^0)$$

$$J_2^1 = J_2(\lambda_1; J^0)$$

and $\Pi_n$ denote the set of all sequences of 0, 1, 2 of length $n$. When $J_{\delta}^{n-1}$'s are defined for all $\delta \in \Pi_{n-1}$, we define;

$$J_0^n = J_0(\lambda_n; J_{\delta}^{n-1})$$

$$J_1^n = J_1(\lambda_n; J_{\delta}^{n-1})$$

$$J_2^n = J_2(\lambda_n; J_{\delta}^{n-1}).$$

Inductively, we can define $J_\gamma^n$ for all $\gamma \in \Pi_n$ and for all $n \geq 0$. Define

$$J^n = \bigcup_{\gamma \in \Pi_n} J_\gamma^n$$

and

$$\Gamma(s) = \bigcap_{n \geq 0} J^n.$$

These cantor sets have the following relation.

**Theorem 1.** Let $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Then,

$$\Lambda(s) - \Lambda(s) = \Gamma(s).$$
§.2 Outline of the proof.

**Definition 5.** Let $\Lambda$ be a Cantor set on a closed interval $I$. $\Lambda$ is called affine, regular or $C^r$--Cantor set for $1 \leq r \leq \infty$ if there are closed disjoint intervals $I_1, \cdots, I_k$ on $I$ and onto affine, $C^2$ or $C^r$--maps $f_i : I \rightarrow I$ for all $1 \leq i \leq k$ such that;

(i) $|f'_i(x)| > 1 \quad \forall x \in I_i$

(ii) $\Lambda = \bigcap_{n=0}^{\infty} \left\{ \bigcup_{\sigma \in \Sigma_n^k} f^{-1}_{\sigma(1)} f^{-1}_{\sigma(2)} \cdots f^{-1}_{\sigma(n)}(I) \right\}$, where $\Sigma_n^k = \{ \sigma : \{1, \cdots, n\} \rightarrow \{1, \cdots, k\} \}$. 

Our main result is restated as follows.

**Theorem 2.** There exists a sequence of real numbers $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with $0 < \lambda_i < \frac{1}{8}$ for all $i \geq 1$ such that;

(i) $\Lambda(s)$ is a $C^\infty$--Cantor set,

(ii) $m(\Lambda(s) - \Lambda(s)) > 0$,

where $m()$ denotes the Lebesgue measure.

From now on, we shall give the outline of the proof of this Theorem 2.

Let $\{\tau_n\}_{n \geq 0}$ be a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} \tau_n < 1.$$  

We define $\{\lambda_n\}_{n \geq 1}$ using this $\{\tau_n\}_{n \geq 0}$ as follows.

$$\begin{cases} 
\lambda_1 = \frac{1}{3}(1 - \tau_0) \\
\lambda_{n+1} = \frac{1}{3} \left( \frac{1 - \sum_{i=0}^{n} \tau_i}{1 - \sum_{i=0}^{n-1} \tau_i} \right)
\end{cases}$$

It is easily seen that

$$0 < \lambda_n < \frac{1}{3} \quad \forall n \geq 1.$$

These numbers has the following relations.
**Lemma 3.**

\[
\sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \quad \forall n \geq 0 .
\]

**Lemma 4.**

\[
r_n = 3^n (1 - 3 \lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \quad \forall n \geq 0 .
\]

where, we assume \( \prod_{j=1}^{0} \lambda_j = 1 \) for the simplicity of notation.

Using these lemmas, we can show (ii) of Theorem 2. In fact, the following lemma holds.

**Lemma 5.** Let \( \{r_n\}_{n \geq 0} \) be a sequence of positive real numbers such that \( \sum_{n=0}^{\infty} r_n < 1 \), and \( \{\lambda_n\}_{n \geq 1} \) be the sequence defined by (2). Then, \( m(\Gamma(s)) > 0 \).

§.3 The regularity of \( \Lambda(s) \).

We define a sequence \( \{r_n\}_{n \geq 0} \) (and so \( \{\lambda_n\}_{n \geq 1} \)), and prove that \( \Lambda(s) \) is \( C^\infty \). First of all, we fix a \( C^\infty \)-function \( h(t) \) on \([0,1]\) with the following properties.

(i) \( h(t) \geq 0 \),

(ii) \( \int_{0}^{1} h(t) dt = 1 \),

(iii) for all \( n \geq 0 \),

\[
\begin{cases}
\lim_{t \downarrow 0} h^{(n)}(t) = 0, \\
\lim_{t \uparrow 1} h^{(n)}(t) = 0 .
\end{cases}
\]

To define \( \{r_n\}_{n \geq 0} \), we define the following sequences. For each integers \( n \geq 0 \), let

\[
q_n = \max\{q_0, q_1, \ldots, q_n-1, 1, \sup_{t \in [0,1]} |h^{(n)}(t)|\} .
\]

For \( n \geq 0 \), we define,

\[
r_n = \frac{4^{-(n^2+2)}}{q_n} .
\]
Since $r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)}$, we have,

\[\sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12}.\]

Therefore, $\{r_n\}_{n \geq 0}$ satisfy (1). We define another sequence of positive real numbers;

\[m_n = \frac{3(3r_{n-1} - r_n)}{2^{n-1}(1 - \sum_{i=0}^{n-1} r_i)} \quad \forall n \geq 1.\]

Since $\{r_n\}_{n \geq 0}$ is monotonically decreasing and by (4), $m_n > 0$ for all $n \geq 1$.

$U^0$ denotes the open interval between $I_0^1$ and $I_1^1$, namely;

\[U^0 = I^0 \setminus (I_0^1 \cup I_1^1).\]

In general, $U_{\alpha}^{n-1}$ ($\alpha \in \Delta_{n-1}$) denotes the open interval between $I_{\alpha 0}^n$ and $I_{\alpha 1}^n$ in $I_{\alpha}^{n-1}$, namely;

\[U_{\alpha}^{n-1} = I_{\alpha}^{n-1} \setminus (I_{\alpha 0}^n \cup I_{\alpha 1}^n).\]

Let $\ell_n = \ell(I_{\alpha}^n)$. Then, by the definition,

\[\ell_n = \lambda_n \ell_{n-1}.\]

Let $u_n = \ell(U_{\alpha}^n)$, and $U_{\alpha}^n = [x_{\alpha}, y_{\alpha}]$. Then,\n
\[u_n = \ell_n - 2\ell_{n+1},\]

and\n
\[u_n = y_{\alpha} - x_{\alpha}.\]

We prove the smoothness of $A(s)$ as follows. We define a non-negative $C^\infty$-function $f(t)$ on $[0, \lambda_1]$ and define;

\[g(t) = \int_{0}^{t} (f(s) + 3)ds.\]

We put;

\[\begin{cases} 
  g_0(t) = g(t) & \text{on } [0, \lambda_1] \\
  g_1(t) = g(t - 1 + \lambda_1) & \text{on } [1 - \lambda_1, 1].
\end{cases}\]
and prove that these $g_0$ and $g_1$ define $\Lambda(s)$.

**DEFINITION OF $f(t)$.** Recall that we have already defined a $C^\infty$-function $h(t)$ on $[0, 1]$. We define $f(t)$ using this $h(t)$ as follows. Let $[x'_\alpha, y'_\alpha]$ be the interval of length $\frac{\ell_n}{3}$ in the middle of $U^n_\alpha$ such that

$$[x'_\alpha, y'_\alpha] = [x_\alpha + \frac{1}{2}(u_n - \frac{\ell_n}{3}), y_\alpha - \frac{1}{2}(u_n - \frac{\ell_n}{3})].$$

We define $f(t)$ as follows.

(i) On $U^n_\alpha$ ($n \neq 0$),

$$f(t) = \begin{cases} m_n h\left(\frac{t - x'_\alpha}{\frac{\ell_n}{3}}\right) & t \in [x'_\alpha, y'_\alpha] \\ f(t) = 0 & \text{otherwise} \end{cases}$$

(ii) On $\Lambda(s)$, $f(t) = 0$.

What we have to show are:

(I) $f(t)$ is a $C^\infty$-function on $[0, \lambda_1]$.

(II) $g_0$ and $g_1$ define $\Lambda(s)$.

To show the smoothness of $f(t)$, we define a function $f_n(t)$ for any $n \geq 0$ as follows. Since $f(t)$ is $C^\infty$ on $U = \bigcup_{n \geq 1, \alpha \in \Delta_n} U^n_\alpha$ ($= [0, \lambda_1] \setminus \Lambda(s)$), $f^{(n)}(t)$ exists for all $n \geq 0$ on $U$. We define,

$$f_n(t) = \begin{cases} f^{(n)}(t) & \text{for } t \in U \\ f_n(t) = 0 & \text{otherwise (i.e. } t \in \Lambda(s) \text{).} \end{cases}$$

The smoothness is shown by proving that;

**LEMMA 6.** For any $n \geq 0$, $f_n$ is differentiable at any $t \in [0, \lambda_1]$ and $f_n'(t) = f_{n+1}(t)$.

For the proof of (II), we need some lemmas. Let $I^n_\alpha = [r^n_\alpha, s^n_\alpha]$.

**LEMMA 7.** For all $\alpha, \alpha' \in \Delta_n$,

$$\int_{I^n_\alpha} f(t) dt = \int_{I^n_{\alpha'}} f(t) dt.$$
Lemma 8. For all $n \geq 1$,

$$\int_0^{\ell_n} f(t)dt = \frac{1}{3} m_n \ell_n + 2 \int_0^{\ell_{n+1}} f(t)dt .$$

Lemma 9. For all $n \geq 1$,

$$\ell_{n-1} = g_0(\ell_n) .$$

We have to prove that,

$$\Lambda(s) = \bigcap_{n \geq 0} \{ \bigcup_{\sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0) \} .$$

Recall that $\Sigma_n^2 = \{0,1\}^{\{1,\cdots,n\}}$ and $I^0 = [0,1]$. This is obtained by showing the following lemma.

Lemma 10. For all $n \geq 0$ and $\alpha \in \Delta_n$,

$$g_0(I_0^{n+1}) = I_\alpha^n, \quad g_1(I_1^{n+1}) = I_\alpha^n .$$

References

