A Difference Set Of A Cantor Set

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Abstract. An example of a regular Cantor set whose self-difference set is a Cantor set with a positive measure is given. This is a counter example of one of the questions related to the homoclinic bifurcation of surface diffeomorphisms.

§.0 Introduction.

In [2], Palis–Takens investigated the homoclinic bifurcations of surface diffeomorphisms in the following context. Let $M$ be a closed 2-dimensional manifold. We say a $C^r$-diffeomorphism $\phi : M \to M$ is persistently hyperbolic if there is a $C^r$-neighborhood $U$ of $\phi$ and for every $\psi \in U$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [1] for the definitions and the notations of the terminologies of dynamical systems). Let $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ be a 1-parameter family of $C^2$-diffeomorphisms on $M$. We define $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ has a homoclinic $\Omega$-explosion at $\mu = 0$ if:

i) For $\mu < 0$, $\phi_\mu$ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\tilde{\Omega}(\phi_0) = \lim_{\mu \uparrow 0} \Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $O$ associated with a fixed saddle point $p$, so that $\Omega(\phi_0) = \tilde{\Omega}(\phi_0) \cup O$; the product of the eigenvalues of $d\phi_0$ at $p$ is different from one in norm;

iii) The separatrices have quadratic tangency along $O$ unfolding generically; $O$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of $\phi_0$.

Let $\Lambda$ be a basic set of a diffeomorphism on $M$. $d^s(\Lambda)$ ($d^u(\Lambda)$) denotes the Hausdorff dimension in the transversal direction of the stable (unstable) foliation of stable (unstable) manifold of $\Lambda$ (refer [2] for the precise definition), and is called the stable (unstable) limit capacity. $B$ denotes the set of values $\mu > 0$ for which $\phi_\mu$ is not persistently hyperbolic.

The result of Palis–Takens is;
THEOREM [2]. Let $\{\phi_{\mu}; \mu \in \mathbb{R}\}$ be a family of diffeomorphisms of $M$ with a homoclinic $\Omega$-explosion at $\mu = 0$. Suppose that $d^s(\Lambda) + d^u(\Lambda) < 1$, where $\Lambda$ is the basic set of $\phi_0$ associated with the homoclinic tangency. Then

$$\lim_{\delta \to 0} \frac{m(B \cap [0, \delta])}{\delta} = 0$$

where $m$ denotes Lebesgue measure.

This result says that if $d^s(\Lambda) + d^u(\Lambda) < 1$, then the measure of the parameters for which bifurcation occurs is relatively small.

Now the case of $d^s(\Lambda) + d^u(\Lambda) > 1$ comes into question as the next step. In the proof of the theorem above, one of the essential points is a question of how two Cantor sets in the line intersect each other when the one is slid. In [3], Palis proposed the following questions.

(Q.1) For affine Cantor sets $X$ and $Y$ in the line, is it true that $X - Y$ either has measure zero or contains intervals?

(Q.2) Same for regular Cantor sets.

For two subset $X,Y$ of $\mathbb{R}$,

$$X - Y = \{ z - y \mid z \in X, y \in Y \}.$$  

This can also be written as;

$$X - Y = \{ \mu \in \mathbb{R} \mid X \cap (\mu + Y) \neq \emptyset \},$$

namely $X - Y$ is the set of parameters for which $X$ and $Y$ have a intersection point when $Y$ is slid on the line.

Cantor set $\Lambda$ in $\mathbb{R}$ is called affine, regular or $C^r$ for $1 \leq r \leq \infty$ if $\Lambda$ is defined with finite number of expanding affine, $C^2$ or $C^r$ maps respectively (see §2 Definition 5 for the rigorous definition).

Our result in this note is that there is a counter example of (Q.2), namely;

THEOREM. There exists a $C^\infty$-Cantor set $\Lambda$ such that

(i) $m(\Lambda - \Lambda) > 0,$
(ii) $A - A$ is a Cantor set.

In the succeeding sections, we give an outline of the proof of this theorem. The complete proof will appear elsewhere.

§1 Definition of the Cantor sets $\Lambda(s)$, $\Gamma(s)$.

First of all, we define two cantor set depending on a sequence of real numbers.

DEFINITION 1. Let $I = [x_1, x_2]$ be a closed interval and $\lambda$ a real number with $0 < \lambda < \frac{1}{2}$. We define,

$\displaystyle I_0(\lambda; I) = [x_1, x_1 + \lambda(x_2 - x_1)]$
$\displaystyle I_1(\lambda; I) = [x_2 - \lambda(x_2 - x_1), x_2]$.

DEFINITION 2 (CANTOR SET $\Lambda(s)$). Let $I^0 = [0, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$ for all $i \geq 1$. We define the Cantor set $\Lambda(s)$ as follows.

Let $I^1_0 = I_0(\lambda_1; I^0)$, $I^1_1 = I_1(\lambda_1; I^0)$ and $I^1 = I^1_0 \cup I^1_1$. $\Delta_n$ denotes the set of all sequences of 0 and 1 of length $n$. When $I^\beta_{n-1}$'s are defined for all $\beta \in \Delta_{n-1}$, we define;

$\displaystyle I^\beta_0 = I_0(\lambda_n; I^\beta_{n-1})$
$\displaystyle I^\beta_1 = I_1(\lambda_n; I^\beta_{n-1})$.

Inductively, we can define $I^{\alpha}_n$ for all $\alpha \in \Delta_n$ and for all $n \geq 0$. Define

$\displaystyle I^n = \bigcup_{\alpha \in \Delta_n} I^{\alpha}_n$

and

$\Lambda(s) = \bigcap_{n \geq 0} I^n$.

This is clearly a Cantor set by the definition.

Next, we define another Cantor set $\Gamma(s)$. 

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DEFINITION 3. Let $J = [x_1, x_2]$ and $0 < \lambda < \frac{1}{3}$. We define,

\begin{align*}
J_0(\lambda; J) &= [x_1, x_1 + \lambda(x_2 - x_1)] \\
J_1(\lambda; J) &= \left[\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)\right] \\
J_2(\lambda; J) &= [x_2 - \lambda(x_2 - x_1), x_2].
\end{align*}

DEFINITION 4. Let $J^0 = [-1, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Let

\begin{align*}
J_0^1 &= J_0(\lambda_1; J^0) \\
J_1^1 &= J_1(\lambda_1; J^0) \\
J_2^1 &= J_2(\lambda_1; J^0)
\end{align*}

and $\Pi_n$ denote the set of all sequences of $0, 1, 2$ of length $n$. When $J_\delta^{n-1}$'s are defined for all $\delta \in \Pi_{n-1}$, we define;

\begin{align*}
J_{\delta 0}^n &= J_0(\lambda_n; J^{n-1}_\delta) \\
J_{\delta 1}^n &= J_1(\lambda_n; J^{n-1}_\delta) \\
J_{\delta 2}^n &= J_2(\lambda_n; J^{n-1}_\delta).
\end{align*}

Inductively, we can define $J^n_\gamma$ for all $\gamma \in \Pi_n$ and for all $n \geq 0$. Define

$$J^n = \bigcup_{\gamma \in \Pi_n} J^n_\gamma$$

and

$$\Gamma(s) = \bigcap_{n \geq 0} J^n.$$

These cantor sets have the following relation.

THEOREM 1. Let $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Then,

$$\Lambda(s) - \Lambda(s) = \Gamma(s).$$
§.2 Outline of the proof.

Definition 5. Let $\Lambda$ be a Cantor set on a closed interval $I$. $\Lambda$ is called affine, regular or $C^r$–Cantor set for $1 \leq r \leq \infty$ if there are closed disjoint intervals $I_1, \cdots, I_k$ on $I$ and onto affine, $C^2$ or $C^r$–maps $f_i : I_i \to I$ for all $1 \leq i \leq k$ such that:

(i) $|f'_i(x)| > 1 \quad \forall x \in I_i$

(ii) $\Lambda = \bigcap_{n=0}^{\infty} \left\{ \bigcup_{\sigma \in \Sigma_n^k} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} \cdots f_{\sigma(n)}^{-1}(I) \right\}$,

where $\Sigma_n^k = \{ \sigma : \{1, \cdots, n\} \to \{1, \cdots, k\} \}$.

Our main result is restated as follows.

Theorem 2. There exists a sequence of real numbers $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with $0 < \lambda_i < \frac{1}{8}$ for all $i \geq 1$ such that:

(i) $\Lambda(s)$ is a $C^\infty$–Cantor set,

(ii) $m(\Lambda(s) - \Lambda(s)) > 0$,

where $m(\ )$ denotes the Lebesgue measure.

From now on, we shall give the outline of the proof of this Theorem 2.

Let $\{r_n\}_{n \geq 0}$ be a sequence of positive real numbers such that

(1) $\sum_{n=0}^{\infty} r_n < 1$.

We define $\{\lambda_n\}_{n \geq 1}$ using this $\{r_n\}_{n \geq 0}$ as follows.

(2) \[
\begin{cases}
\lambda_1 = \frac{1}{3} (1 - r_0) \\
\lambda_{n+1} = \frac{1}{3} \left( \frac{1 - \sum_{i=0}^{n} r_i}{1 - \sum_{i=0}^{n-1} r_i} \right)
\end{cases}
\]

It is easily seen that

(3) $0 < \lambda_n < \frac{1}{3} \quad \forall n \geq 1$.

These numbers has the following relations.
Lemma 3.

\[ \sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \quad \forall n \geq 0. \]

Lemma 4.

\[ r_n = 3^n (1 - 3 \lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \quad \forall n \geq 0. \]

where, we assume \( \prod_{j=1}^{0} \lambda_j = 1 \) for the simplicity of notation.

Using these lemmas, we can show (ii) of Theorem 2. In fact, the following lemma holds.

Lemma 5. Let \( \{r_n\}_{n \geq 0} \) be a sequence of positive real numbers such that \( \sum_{n=0}^{\infty} r_n < 1 \), and \( \{\lambda_n\}_{n \geq 1} \) be the sequence defined by (2). Then, \( m(\Gamma(s)) > 0 \).

§.3 The regularity of \( \Lambda(s) \).

We define a sequence \( \{r_n\}_{n \geq 0} \) (and so \( \{\lambda_n\}_{n \geq 1} \)), and prove that \( \Lambda(s) \) is \( C^\infty \). First of all, we fix a \( C^\infty \)-function \( h(t) \) on \([0,1]\) with the following properties.

(i) \( h(t) \geq 0 \),
(ii) \( \int_{0}^{1} h(t) dt = 1 \),
(iii) for all \( n \geq 0 \),

\[ \begin{cases} \lim_{t \downarrow 0} h^{(n)}(t) = 0, \\ \lim_{t \uparrow 1} h^{(n)}(t) = 0. \end{cases} \]

To define \( \{r_n\}_{n \geq 0} \), we define the following sequences. For each integers \( n \geq 0 \), let

\[ q_n = \max\{q_0, q_1, \ldots, q_{n-1}, 1, \sup_{t \in [0,1]} |h^{(n)}(t)|\}. \]

For \( n \geq 0 \), we define,

\[ r_n = \frac{4^{-(n^2+2)}}{q_n}. \]
Since \( r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)} \), we have,

\[
\sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12} .
\]

Therefore, \( \{r_n\}_{n \geq 0} \) satisfy (1). We define another sequence of positive real numbers:

\[
m_n = \frac{3(3r_{n-1} - r_n)}{2^{n-1}(1 - \sum_{i=0}^{n-1} r_i)} \quad \forall n \geq 1 .
\]

Since \( \{r_n\}_{n \geq 0} \) is monotonically decreasing and by (4), \( m_n > 0 \) for all \( n \geq 1 \).

\( U^0 \) denotes the open interval between \( I_0^1 \) and \( I_1^1 \), namely;

\[
U^0 = I^0 \setminus (I_0^1 \cup I_1^1) .
\]

In general, \( U_{\alpha}^{n-1} \) (\( \alpha \in \Delta_{n-1} \)) denotes the open interval between \( I_{\alpha 0}^n \) and \( I_{\alpha 1}^n \) in \( I_{\alpha}^{n-1} \), namely;

\[
U_{\alpha}^{n-1} = I_{\alpha}^{n-1} \setminus (I_{\alpha 0}^n \cup I_{\alpha 1}^n) .
\]

Let \( \ell_n = \ell(I_{\alpha}^n) \). Then, by the definition,

\[
\ell_n = \lambda_n \ell_{n-1} .
\]

Let \( u_n = \ell(U_{\alpha}^n) \), and \( U_{\alpha}^n = [x_\alpha, y_\alpha] \). Then,

\[
u_n = \ell_n - 2\ell_{n+1} ,
\]

and

\[
u_n = y_\alpha - x_\alpha .
\]

We prove the smoothness of \( A(s) \) as follows. We define a non-negative \( C^\infty \)-function \( f(t) \) on \([0, \lambda_1]\) and define;

\[
g(t) = \int_0^t (f(s) + 3) ds .
\]

We put:

\[
\begin{cases}
g_0(t) = g(t) & \text{on } [0, \lambda_1] \\
g_1(t) = g(t - 1 + \lambda_1) & \text{on } [1 - \lambda_1, 1] .
\end{cases}
\]
and prove that these $g_0$ and $g_1$ define $\Lambda(s)$.

**DEFINITION OF $f(t)$**. Recall that we have already defined a $C^\infty$-function $h(t)$ on $[0, 1]$. We define $f(t)$ using this $h(t)$ as follows. Let $[x'_\alpha, y'_\alpha]$ be the interval of length $\frac{\ell_n}{3}$ in the middle of $U^n_\alpha$ such that

$$[x'_\alpha, y'_\alpha] = [x_\alpha + \frac{1}{2}(u_n - \frac{\ell_n}{3}), y_\alpha - \frac{1}{2}(u_n - \frac{\ell_n}{3})].$$

We define $f(t)$ as follows.

(i) On $U^n_\alpha$ ($n \neq 0$),

$$\begin{cases} 
  f(t) = m_n h(\frac{t - x'_\alpha}{\frac{\ell_n}{3}}) & t \in [x'_\alpha, y'_\alpha] \\
  f(t) = 0 & \text{otherwise}.
\end{cases}$$

(ii) On $\Lambda(s)$, $f(t) = 0$.

What we have to show are;

(1) $f(t)$ is a $C^\infty$-function on $[0, \lambda_1]$.

(2) $g_0$ and $g_1$ define $\Lambda(s)$.

To show the smoothness of $f(t)$, we define a function $f_n(t)$ for any $n \geq 0$ as follows. Since $f(t)$ is $C^\infty$ on $U = \bigcup_{n \geq 1, \alpha \in \Delta_n} U^n_\alpha$ ($= [0, \lambda_1] \backslash \Lambda(s)$), $f^{(n)}(t)$ exists for all $n \geq 0$ on $U$. We define,

$$\begin{cases} 
  f_n(t) = f^{(n)}(t) & \text{for } t \in U \\
  f_n(t) = 0 & \text{otherwise (i.e. } t \in \Lambda(s) \text{).}
\end{cases}$$

The smoothness is shown by proving that;

**LEMMA 6.** For any $n \geq 0$, $f_n$ is differentiable at any $t \in [0, \lambda_1]$ and $f'_n(t) = f_{n+1}(t)$.

For the proof of (II), we need some lemmas. Let $I^n_\alpha = [r^n_\alpha, s^n_\alpha]$.

**LEMMA 7.** For all $\alpha, \alpha' \in \Delta_n$,

$$\int_{I^n_\alpha} f(t) dt = \int_{I^n_{\alpha'}} f(t) dt.$$
**Lemma 8.** For all $n \geq 1$,

\[ \int_0^{t_n} f(t)dt = \frac{1}{3} m_n \ell_n + 2 \int_0^{t_{n+1}} f(t)dt. \]

**Lemma 9.** For all $n \geq 1$,

\[ \ell_{n-1} = g_0(\ell_n). \]

We have to prove that,

\[ \Lambda(s) = \bigcap_{n \geq 0} \left\{ \bigcup_{\sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0) \right\}. \]

Recall that $\Sigma_n^2 = \{0,1\}^{\{1,\cdots,n\}}$ and $I^0 = [0,1]$. This is obtained by showing the following lemma.

**Lemma 10.** For all $n \geq 0$ and $\alpha \in \Delta_n$,

\[ g_0(I_0^{n+1}) = I_\alpha^n, \quad g_1(I_1^{n+1}) = I_\alpha^n. \]

**References**

