<table>
<thead>
<tr>
<th>Title</th>
<th>A Difference Set Of A Cantor Set (The Study of Dynamical Systems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author</td>
<td>SANNAMI, ATSURO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1989), 696: 38-46</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1989-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/101430">http://hdl.handle.net/2433/101430</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A Difference Set Of A Cantor Set

ATSURO SANNAMI

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060 Japan

Abstract. An example of a regular Cantor set whose self-difference set is a Cantor set with a positive measure is given. This is a counter example of one of the questions related to the homoclinic bifurcation of surface diffeomorphisms.

§0 Introduction.

In [2], Palis–Takens investigated the homoclinic bifurcations of surface diffeomorphisms in the following context. Let $M$ be a closed 2-dimensional manifold. We say a $C^r$-diffeomorphism $\phi : M \rightarrow M$ is persistently hyperbolic if there is a $C^r$-neighborhood $U$ of $\phi$ and for every $\psi \in U$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set ( refer [1] for the definitions and the notations of the terminologies of dynamical systems ). Let $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ be a 1-parameter family of $C^2$-diffeomorphisms on $M$. We define $\{\phi_\mu\}_{\mu \in \mathbb{R}}$ has a homoclinic $\Omega$-explosion at $\mu = 0$ if:

i) For $\mu < 0$, $\phi_\mu$ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\tilde{\Omega}(\phi_0) = \lim_{\mu \uparrow 0} \Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $O$ associated with a fixed saddle point $p$, so that $\Omega(\phi_0) = \tilde{\Omega}(\phi_0) \cup O$; the product of the eigenvalues of $d\phi_0$ at $p$ is different from one in norm;

iii) The separatrices have quadratic tangency along $O$ unfolding generically; $O$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of $\phi_0$.

Let $\Lambda$ be a basic set of a diffeomorphism on $M$. $d^*(\Lambda)$ ( $d^u(\Lambda)$ ) denotes the Hausdorff dimension in the transversal direction of the stable ( unstable ) foliation of stable ( unstable ) manifold of $\Lambda$ ( refer [2] for the precise definition ), and is called the stable ( unstable ) limit capacity. $B$ denotes the set of values $\mu > 0$ for which $\phi_\mu$ is not persistently hyperbolic.

The result of Palis–Takens is;
**Theorem [2].** Let \( \{\phi_{\mu}; \mu \in \mathbb{R}\} \) be a family of diffeomorphisms of \( M \) with a homoclinic \( \Omega \)-explosion at \( \mu = 0 \). Suppose that \( d^{s}(\Lambda) + d^{u}(\Lambda) < 1 \), where \( \Lambda \) is the basic set of \( \phi_{0} \) associated with the homoclinic tangency. Then

\[
\lim_{\delta \to 0} \frac{m(B \cap [0, \delta])}{\delta} = 0
\]

where \( m \) denotes Lebesgue measure.

This result says that if \( d^{s}(\Lambda) + d^{u}(\Lambda) < 1 \), then the measure of the parameters for which bifurcation occurs is relatively small.

Now the case of \( d^{s}(\Lambda) + d^{u}(\Lambda) > 1 \) comes into question as the next step. In the proof of the theorem above, one of the essential points is a question of how two Cantor sets in the line intersect each other when the one is slid. In [3], Palis proposed the following questions.

(Q.1) For affine Cantor sets \( X \) and \( Y \) in the line, is it true that \( X - Y \) either has measure zero or contains intervals?

(Q.2) Same for regular Cantor sets.

For two subset \( X, Y \) of \( \mathbb{R} \),

\[
X - Y = \{ x - y \mid x \in X, y \in Y \}.
\]

This can also be written as:

\[
X - Y = \{ \mu \in \mathbb{R} \mid X \cap (\mu + Y) \neq \emptyset \},
\]

namely \( X - Y \) is the set of parameters for which \( X \) and \( Y \) have an intersection point when \( Y \) is slid on the line.

Cantor set \( \Lambda \) in \( \mathbb{R} \) is called affine, regular or \( C^r \) for \( 1 \leq r \leq \infty \) if \( \Lambda \) is defined with finite number of expanding affine, \( C^{2} \) or \( C^{r} \) maps respectively (see §2 Definition 5 for the rigorous definition).

Our result in this note is that there is a counter example of (Q.2), namely:

**Theorem.** There exists a \( C^\infty \)-Cantor set \( \Lambda \) such that

(i) \( m(\Lambda - \Lambda) > 0 \),
(ii) \( \Lambda - \Lambda \) is a Cantor set.

In the succeeding sections, we give an outline of the proof of this theorem. The complete proof will appear elsewhere.

§.1 Definition of the Cantor sets \( \Lambda(s) \), \( \Gamma(s) \).

First of all, we define two cantor set depending on a sequence of real numbers.

**Definition 1.** Let \( I = [x_1, x_2] \) be a closed interval and \( \lambda \) a real number with \( 0 < \lambda < \frac{1}{2} \). We define,

\[
I_0(\lambda; I) = [x_1, x_1 + \lambda(x_2 - x_1)] \\
I_1(\lambda; I) = [x_2 - \lambda(x_2 - x_1), x_2].
\]

**Definition 2 (Cantor set \( \Lambda(s) \)).** Let \( I^0 = [0, 1] \) and \( s = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) be a one sided sequence of real numbers with \( 0 < \lambda_i < \frac{1}{2} \) for all \( i \geq 1 \). We define the Cantor set \( \Lambda(s) \) as follows.

Let \( I_0^1 = I_0(\lambda_1; I^0), \quad I_1^1 = I_1(\lambda_1; I^0) \) and \( I^1 = I_0^1 \cup I_1^1 \). \( \Delta_n \) denotes the set of all sequences of 0 and 1 of length \( n \). When \( I^{n-1}_{\beta} \)'s are defined for all \( \beta \in \Delta_{n-1} \), we define;

\[
I_{\beta 0}^n = I_0(\lambda_n; I^{n-1}_{\beta}) \\
I_{\beta 1}^n = I_1(\lambda_n; I^{n-1}_{\beta}).
\]

Inductively, we can define \( I^m_\alpha \) for all \( \alpha \in \Delta_n \) and for all \( n \geq 0 \). Define

\[
I^n = \bigcup_{\alpha \in \Delta_n} I^n_\alpha
\]

and

\[
\Lambda(s) = \bigcap_{n \geq 0} I^n.
\]

This is clearly a Cantor set by the definition.

Next, we define another Cantor set \( \Gamma(s) \).
DEFINITION 3. Let \( J = [x_1, x_2] \) and \( 0 < \lambda < \frac{1}{3} \). We define,

\[
\begin{align*}
J_0(\lambda; J) &= [x_1, x_1 + \lambda(x_2 - x_1)] \\
J_1(\lambda; J) &= \left[\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)\right] \\
J_2(\lambda; J) &= [x_2 - \lambda(x_2 - x_1), x_2] .
\end{align*}
\]

DEFINITION 4. Let \( J^0 = [-1, 1] \) and \( s = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) be a one sided sequence of real numbers with \( 0 < \lambda_i < \frac{1}{3} \) for all \( i \geq 1 \). Let

\[
\begin{align*}
J_0^1 &= J_0(\lambda_1; J^0) \\
J_1^1 &= J_1(\lambda_1; J^0) \\
J_2^1 &= J_2(\lambda_1; J^0)
\end{align*}
\]

and \( \Pi_n \) denote the set of all sequences of 0, 1, 2 of length \( n \). When \( J_{\delta}^{n-1} \)'s are defined for all \( \delta \in \Pi_{n-1} \), we define;

\[
\begin{align*}
J_0^{n} &= J_0(\lambda_n; J_{\delta}^{n-1}) \\
J_1^{n} &= J_1(\lambda_n; J_{\delta}^{n-1}) \\
J_2^{n} &= J_2(\lambda_n; J_{\delta}^{n-1}).
\end{align*}
\]

Inductively, we can define \( J_{\gamma}^{n} \) for all \( \gamma \in \Pi_n \) and for all \( n \geq 0 \). Define

\[
J^n = \bigcup_{\gamma \in \Pi_n} J_{\gamma}^{n}
\]

and

\[
\Gamma(s) = \bigcap_{n \geq 0} J^n.
\]

These cantor sets have the following relation.

THEOREM 1. Let \( s = (\lambda_1, \lambda_2, \lambda_3, \cdots) \) be a sequence of real numbers with \( 0 < \lambda_i < \frac{1}{3} \) for all \( i \geq 1 \). Then,

\[
\Lambda(s) - \Lambda(s) = \Gamma(s).
\]
§.2 Outline of the proof.

**Definition 5.** Let $\Lambda$ be a Cantor set on a closed interval $I$. $\Lambda$ is called affine, regular or $C^r$-Cantor set for $1 \leq r \leq \infty$ if there are closed disjoint intervals $I_1, \cdots, I_k$ on $I$ and onto affine, $C^2$ or $C^r$-maps $f_i : I_i \to I$ for all $1 \leq i \leq k$ such that;

(i) $|f'_i(x)| > 1 \ \forall x \in I_i$

(ii) $\Lambda = \bigcap_{n=0}^{\infty} \{ \bigcup_{\sigma \in \Sigma_n^k} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} \cdots f_{\sigma(n)}^{-1}(I) \}$

where $\Sigma_n^k = \{ \sigma : \{1, \cdots, n\} \to \{1, \cdots, k\} \}$.

Our main result is restated as follows.

**Theorem 2.** There exists a sequence of real numbers $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$ such that;

(i) $\Lambda(s)$ is a $C^\infty$-Cantor set,

(ii) $m(\Lambda(s) - \Lambda(s)) > 0$,

where $m(\ )$ denotes the Lebesgue measure.

From now on, we shall give the outline of the proof of this Theorem 2.

Let $\{r_n\}_{n \geq 0}$ be a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} r_n < 1.$$  

(1)

We define $\{\lambda_n\}_{n \geq 1}$ using this $\{r_n\}_{n \geq 0}$ as follows.

$$(2) \quad \left\{ \begin{array}{l}
\lambda_1 = \frac{1}{3}(1 - r_0) \\
\lambda_{n+1} = \frac{1}{3} \left( \frac{1 - \sum_{i=0}^{n} r_i}{1 - \sum_{i=0}^{n-1} r_i} \right)
\end{array} \right.$$  

It is easily seen that

$$0 < \lambda_n < \frac{1}{3} \quad \forall n \geq 1.$$  

(3)  

These numbers has the following relations.
Lemma 3.
\[ \sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n} \lambda_j \quad \forall n \geq 0. \]

Lemma 4.
\[ r_n = 3^n (1 - 3\lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \quad \forall n \geq 0. \]

where, we assume \( \prod_{j=1}^{0} \lambda_j = 1 \) for the simplicity of notation.

Using these lemmas, we can show (ii) of Theorem 2. In fact, the following lemma holds.

Lemma 5. Let \( \{r_n\}_{n \geq 0} \) be a sequence of positive real numbers such that \( \sum_{n=0}^{\infty} r_n < 1 \), and \( \{\lambda_n\}_{n \geq 1} \) be the sequence defined by (2). Then, \( m(\Gamma(s)) > 0 \).

§.3 The regularity of \( \Lambda(s) \).

We define a sequence \( \{r_n\}_{n \geq 0} \) (and so \( \{\lambda_n\}_{n \geq 1} \)), and prove that \( \Lambda(s) \) is \( C^\infty \). First of all, we fix a \( C^\infty \)-function \( h(t) \) on \([0,1]\) with the following properties.

(i) \( h(t) \geq 0 \),
(ii) \( \int_{0}^{1} h(t) dt = 1 \),
(iii) for all \( n \geq 0 \),
\[ \left\{ \begin{array}{l} \lim_{t \downarrow 0} h^{(n)}(t) = 0, \\ \lim_{t \uparrow 1} h^{(n)}(t) = 0. \end{array} \right. \]

To define \( \{r_n\}_{n \geq 0} \), we define the following sequences. For each integers \( n \geq 0 \), let
\[ q_n = \max\{q_0, q_1, \ldots, q_{n-1}, 1, \sup_{t \in [0,1]} |h^{(n)}(t)| \} . \]

For \( n \geq 0 \), we define,
\[ r_n = \frac{4^{-(n^2+2)}}{q_n} . \]
Since $r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)}$, we have,

\[(4) \quad \sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12} .\]

Therefore, $\{r_n\}_{n \geq 0}$ satisfy (1). We define another sequence of positive real numbers:

$$m_n = \frac{3(3r_{n-1} - r_n)}{2^{n-1}(1 - \sum_{i=0}^{n-1} r_i)} \quad \forall n \geq 1 .$$

Since $\{r_n\}_{n \geq 0}$ is monotonically decreasing and by (4), $m_n > 0$ for all $n \geq 1$.

$U^0$ denotes the open interval between $I^1_0$ and $I^1_1$, namely;

$$U^0 = I^0 \setminus (I^1_0 \cup I^1_1) .$$

In general, $U^{n-1}_\alpha$ ($\alpha \in \Delta_{n-1}$) denotes the open interval between $I_{\alpha 0}^n$ and $I_{\alpha 1}^n$ in $\Gamma^{-1}_\alpha$, namely;

$$U^{n-1}_\alpha = I^{n-1}_\alpha \setminus (I_{\alpha 0}^n \cup I_{\alpha 1}^n) .$$

Let $\ell_n = \ell(I^n_\alpha)$. Then, by the definition,

$$\ell_n = \lambda_n \ell_{n-1} .$$

Let $u_n = \ell(U^n_\alpha)$, and $U^n_\alpha = [x_\alpha, y_\alpha]$. Then,

$$u_n = \ell_n - 2\ell_{n+1} ,$$

and

$$u_n = y_\alpha - x_\alpha .$$

We prove the smoothness of $\Lambda(s)$ as follows. We define a non-negative $C^\infty$-function $f(t)$ on $[0, \lambda_1]$ and define;

$$g(t) = \int_0^t (f(s) + 3)ds .$$

We put;

$$\left\{ \begin{array}{l} g_0(t) = g(t) \quad \text{on} \ [0, \lambda_1] \\ g_1(t) = g(t - 1 + \lambda_1) \quad \text{on} \ [1 - \lambda_1, 1] . \end{array} \right.$$
and prove that these $g_0$ and $g_1$ define $\Lambda(s)$.

**Definition of $f(t)$**. Recall that we have already defined a $C^\infty$-function $h(t)$ on $[0, 1]$. We define $f(t)$ using this $h(t)$ as follows. Let $[x'_\alpha, y'_\alpha]$ be the interval of length $\frac{t_n}{3}$ in the middle of $U^n_\alpha$ such that

$$[x'_\alpha, y'_\alpha] = [x_\alpha + \frac{1}{2}(u_n - t_n), y_\alpha - \frac{1}{2}(u_n - t_n)].$$

We define $f(t)$ as follows.

(i) On $U^n_\alpha$ (n $\neq$ 0),

$$
\begin{cases}
  f(t) = m_n h\left(\frac{t-x'_\alpha}{\ell_\alpha}\right) & t \in [x'_\alpha, y'_\alpha] \\
  f(t) = 0 & \text{otherwise}.
\end{cases}
$$

(ii) On $\Lambda(s)$, $f(t) = 0$.

What we have to show are:

(I) $f(t)$ is a $C^\infty$-function on $[0, \lambda_1]$.

(II) $g_0$ and $g_1$ define $\Lambda(s)$.

To show the smoothness of $f(t)$, we define a function $f_n(t)$ for any $n \geq 0$ as follows. Since $f(t)$ is $C^\infty$ on $U = \bigcup_{n \geq 1, \alpha \in \Delta_n} U^n_\alpha$ ( = $[0, \lambda_1] \setminus \Lambda(s)$ ), $f^{(n)}(t)$ exists for all $n \geq 0$ on $U$. We define,

$$
\begin{cases}
  f_n(t) = f^{(n)}(t) & \text{for } t \in U \\
  f_n(t) = 0 & \text{otherwise (i.e. } t \in \Lambda(s)\) }.
\end{cases}
$$

The smoothness is shown by proving that;

**Lemma 6.** For any $n \geq 0$, $f_n$ is differentiable at any $t \in [0, \lambda_1]$ and $f'_n(t) = f_{n+1}(t)$.

For the proof of (II), we need some lemmas. Let $I^n_\alpha = [r^n_\alpha, s^n_\alpha]$.

**Lemma 7.** For all $\alpha, \alpha' \in \Delta_n$,

$$
\int_{I^n_\alpha} f(t) dt = \int_{I^n_\alpha} f(t) dt .
$$
Lemma 8. For all $n \geq 1$,
\[ \int_{0}^{l_n} f(t) dt = \frac{1}{3} m_n l_n + 2 \int_{0}^{l_{n+1}} f(t) dt . \]

Lemma 9. For all $n \geq 1$,
\[ l_{n-1} = g_0(l_n) . \]

We have to prove that,
\[ \Lambda(s) = \bigcap_{n \geq 0} \{ \bigcup_{2, \sigma \in \Sigma_n^2} g_{\sigma(1)}^{-1} g_{\sigma(2)}^{-1} \cdots g_{\sigma(n)}^{-1}(I^0) \} . \]

Recall that $\Sigma_n^2 = \{0,1\}^\{1,\cdots,n\}$ and $I^0 = [0,1]$. This is obtained by showing the following lemma.

Lemma 10. For all $n \geq 0$ and $\alpha \in \Delta_n$,
\[ g_0(I_{0\alpha}^{n+1}) = I_{\alpha}^{n} , \quad g_1(I_{1\alpha}^{n+1}) = I_{\alpha}^{n} . \]

References

