数理解析研究所講究録
697 巻 1989 年 90-99

Strongly compatible total orders on free monoids

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In this paper, we deal with a combinatorial problem which the first author raised in relation to term-rewriting systems. To 'complete' a grammar (as a term-rewriting system), one uses a compatible total order on the set X^* of all words (an order $\leq$ on X^* is said compatible if $x < y$ implies $uxv < uyv$ for all u,v) which does not have an infinite descending sequence. An order $\leq$ on X^* is said length-sensitive if $|x| < |y|$ implies $x < y$, where $|x|$ for a word x denotes the length of x. The length-sensitivity assures the non-existence of an infinite descending sequence. Usually, a total order that is used for the above purpose is the length-sensitive lexicographic order (which is clearly compatible). The problem raised by the first author was to describe the all length-sensitive compatible total orders on X^*. He soon noticed that if we replace the compatibility by a somewhat stronger property which we shall call the strong compatibility, then the description of the orders becomes strikingly simple especially when Card(X) = 2 (Card(X) denotes the cardinality of the set X). In this paper, we take the strong compatibility as a basic property on orders, and present some results on such orders.

Let $X = \{a_1,a_2, \cdots ,a_n\}$ be a set. We call X the alphabet and its elements letters. The free monoid X^* on X is the set of all words over X (that is, finite sequences of letters including the empty sequence, denoted 1, with concatenation as the operation). We treat X itself as a subset of X^* by identifying each letter a with the word consisting of 'a' alone. For any word x in X^*, the length of a word x is the number of letters occurring in x, and denoted by $|x|$. For each natural number n, X^n denotes the set $\{w \in X^* : |x| = n\}$, while $X^{[n]}$ denotes the set $\{w \in X^* : |x| \leq n\}$. We basically follow the notations in [B-P].

We consider an order relation $\leq$ on X^* with the following conditions:
(P1) for letters, $a_1 < a_2 < \cdots < a_n$ ($x < y$ means that $x \leq y$ and $x \neq y$.), and for any words $x,y \in X^*$,
(P2) (The length sensitivity) if $|x| < |y|$, then $x < y$.
(P3) (The compatibility) if $x < y$, then $uxv < uyv$ for all $u,v \in X^*$.
(P4) (The strong compatibility) if $x = x_1x_2$, $y = y_1y_2$ and $|x_1| = |y_1|$, $|x_2| = |y_2|$, then $x < y$ implies $x_1ux_2 < y_1uy_2$ for all $u \in X^*$, where $x_1,x_2,y_1,y_2 \in X^*$.
(P5) $\leq$ is a total order on X^*.

By (P1), we mean that we may (and do) fix 'the' ordering on the letters of X without loss of generality. The length-sensitivity (P2) is our another presumption on orders on X^* and will be implicit on any orders which will appear in the paper. It is easy to check that (P4) implies (P3).

By <u>the lexicographic order</u>, we mean the order $\leq$ on X^* determined by the rule: if $x = x_1x_2\cdots x_m$, $y = y_1y_2\cdots y_m$ and $x_1\cdots x_{i-1} = y_1\cdots y_{i-1}$ and $x_i < y_i$, then $x < y$, where $x_i, y_i \in X$ $(i = 1,\cdots,m)$. We denote this order by $\leq_{lex}$.

If $x = x_1x_2\cdots x_m \in X^*$, where $x_i \in X$ $(i = 1,\cdots,m)$, then we denote $x_m\cdots x_2x_1$ by $\tilde{x}$.

By <u>the anti-lexicographic order</u>, we mean the order $\leq$ on X^* such that for any $x,y \in X^*$, $x <_{lex} y$ implies $\tilde{x} < \tilde{y}$. We denote this order by $\leq_{al}$.

It is easy to see that both $<_{lex}$ and $\leq_{al}$ are strongly compatible.

Proposition 1. Suppose that $\leq$ is a strongly compatible total order on X^*. If $\leq$ coincides with $\leq_{lex}$ on $X^{[3]}$, then $\leq$ in fact equals $\leq_{lex}$ on the entire X^*. In particular, if Card(X)≥ 3, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

Proof. We show the first assertion by induction on the length of words. Suppose that on $X^{[m-1]}$, $\leq$ coincides with $\leq_{lex}$ $(m \geq 4)$. Let $x,y \in X^m$ be $x = x_1x_2\cdots x_m$ and $y = y_1y_2\cdots y_m$ and let i be the least such that $x_i \neq y_i$. WLOG, we may assume that $x_i < y_i$. Then, since $x <_{lex} y$, we have to show that $x < y$.
Case 1 : $i \geq 2$. Then, $x_i\cdots x_m < y_i\cdots y_m$ by the induction hypothesis.

So, using the strong compatibility, we have that $x = x_1 \cdots x_{i-1} x_i \cdots x_m < y_1 \cdots y_{i-1} y_i \cdots y_m = y$.

Case 2 : $i = 1$ and $x_j \leq y_j$ for some $j \neq i$. Then, $x_1 \cdots x_{j-1} x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_{j+1} \cdots y_m$ by the induction hypothesis. So, using the strong compatibility, we have that $x = x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_j y_{j+1} \cdots y_m = y$.

Case 3 : $i = 1$ and $x_j > y_j$ for all $j \neq i$. Since by the case 1, we have that $x \leq x_1 a_n{}^{m-1}$ and $y_1 a_1{}^{m-1} \leq y$, it suffices to show that $x_1 a_n{}^{m-1} < y_1 a_1{}^{m-1}$. Suppose on the contrary, that $y_1 a_1{}^{m-1} < x_1 a_n{}^{m-1}$. Then, by the strong compatibility, we have that $y_1 x_1 a_1{}^{m-1} < x_1{}^2 a_n{}^{m-1}$. But, on the other hand, we have:

$$\begin{aligned} x_1{}^2 a_n{}^{m-1} &= x_1(x_1 a_n) a_n{}^{m-2} \\ &< x_1(a_n x_1) a_n{}^{m-2} = x_1 a_n (x_1 a_n{}^{m-2}) \\ &< x_1 a_n (a_n{}^{m-3} a_1{}^2) = (x_1 a_n{}^{m-2}) a_1{}^2 \\ &< (y_1 x_1 a_1{}^{m-3}) a_1{}^2 = y_1 x_1 a_1{}^{m-1}, \quad \text{a contradiction.} \end{aligned}$$

We have proved the first assertion.

To show the second assertion, suppose that $\leq$ coincides with $\leq_{lex}$ on $X^{[2]}$. It suffices to show that $\leq$ coincides with $\leq_{lex}$ on $X^{[2]}$. Since in the above argument, m could be 3 until we showed the claim that $x_1 a_n{}^{m-1} < y_1 a_1{}^{m-1}$, we need only show the claim for $m = 3$, i.e., that $x_1 a_n{}^2 < y_1 a_1{}^2$. Suppose that $y_1 a_1{}^2 < x_1 a_n{}^2$. Then, we have that $y_1 a_n a_1{}^2 < x_1 a_n{}^3$. But, we have, if $x_1 > a_1$,

$$\begin{aligned} x_1 a_n{}^3 &= (x_1 a_n) a_n{}^2 \\ &< (y_1 a_1) a_n{}^2 = y_1 (a_1 a_n) a_n \\ &< y_1 (x_1 a_1) a_n = y_1 x_1 (a_1 a_n) \\ &< y_1 x_1 (a_n a_1) = y_1 (x_1 a_n) a_1 \\ &< y_1 (a_n x_1) a_1 = y_1 a_n a_1{}^2, \quad \text{a contradiction,} \end{aligned}$$

and if $x_1 = a_1$,

$$\begin{aligned} x_1 a_n{}^3 &= (x_1 a_n) a_n{}^2 \\ &< (a_2 a_1) a_n{}^2 = a_2 (a_1 a_n) a_n \\ &< a_2 (a_2 a_1) a_n = a_2{}^2 (a_1 a_n) \\ &< a_2{}^2 (a_n a_1) = a_2 (a_2 a_n) a_1 \\ &< a_2 (a_n a_1) a_1 \leq y_1 a_n a_1{}^2, \quad \text{again a contradiction.} \end{aligned}$$

We have proved the proposition.

An entirely similar proof shows the following:

Proposition 2. Suppose that $\leq$ is a strongly compatible total order

on X^*. If $\leq$ coincides with $\leq_{al}$ on $X^{[3]}$, then $\leq$ in fact equals $\leq_{al}$ on the entire X^*.
In particular, if Card(X)$\geq$ 3, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

From now on until the end of the proof of Theorem 7, we restrict our attention to orders on X^* with Card(X)= 2. We define four orders other than $<_{lex}$ or $<_{al}$ and will show that these four orders together with $<_{lex}$ and $<_{al}$ exhaust all the strongly compatible total orders on X^* when Card(X)= 2.

Let $c \in X$. For $x \in X^*$, the number of c occurring in x is denoted by $|x|_c$. We consider the following conditions on an order $\leq$ on $X^* = \{a,b\}^*$ $(a < b)$: for $x,y \in X^*$ with $|x| = |y|$,

(a) if $|x|_b < |y|_b$, then $x < y$,
(b) if $|x|_b = |y|_b$ and $x <_{lex} y$, then $x < y$,
(c) if $|x|_b = |y|_b$ and $y <_{al} x$, then $x < y$,
(b') if $|x|_b = |y|_b$ and $x <_{al} y$, then $x < y$,
(c') if $|x|_b = |y|_b$ and $y <_{lex} x$, then $x < y$.

It is clear that an order on X^* satisfying (a) and one of (b),(c),(b'), (c') for all $x,y \in X^*$ is a strongly compatible total order on X^*. The next four lemmas show that similar assertions to Proposition 1,2 are true for these four orders.

Lemma 3. Let X = {a,b} with a < b. Suppose that $\leq$ is a strongly compatible total order on X^*. Then, if $\leq$ coincides with the ordering defined by the conditions (a) and (b) on $X^{[4]}$, then they coincides on the entire X^*.

Lemma 4 Let X , $\leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (c) on $X^{[4]}$, then they coincides on the entire X^*.

Lemma 5. Let X , $\leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (b') on $X^{[4]}$, then they coincides on the entire X^*.

Lemma 6. Let X , $\leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (c') on $X^{[4]}$, then they coincides on the entire X^*.

Because the proofs of these lemmas are similar, we exhibit only the proof of Lemma 3.

Proof of Lemma 3.

We show by induction on the length of a word. Suppose that on $X^{[m-1]}$, $\leq$ coincides with the order defined by the conditions (a) and (b) ($m \geq 5$). Let $x,y \in X^m$ be $x = x_1x_2\cdots x_m$ and $y = y_1y_2\cdots y_m$. We have to show that if either $|x|_b = |y|_b$ and $x <_{lex} y$, or $|x|_b < |y|_b$, then $x < y$.

Case 1 : $|x|_b = |y|_b = k$ and $x <_{lex} y$. If $x_1\cdots x_{i-1} = y_1\cdots y_{i-1}$ and $x_i < y_i$ ($2 \leq i \leq m$), then $|x_i\cdots x_m|_b = |y_i\cdots y_m|_b$ and $x_i\cdots x_m <_{lex} y_i\cdots y_m$, so that by the induction hypothesis, $x_i\cdots x_m < y_i\cdots y_m$. By the compatibility of $\leq$, $x = x_1\cdots x_{i-1}x_i\cdots x_m < y_1\cdots y_{i-1}y_i\cdots y_m = y$. From this fact, we can assume that x,y do not have a common initial segment, and obtain that $x < ab^ka^{m-k-1}$ and $ba^{m-k}b^{k-1} < y$. Therefore, it suffices to show that $ab^ka^{m-k-1} < ba^{m-k}b^{k-1}$. If $2k > m$, then we have :

$$|ab^{m-k}a^{m-k-1}| = |ba^{m-k}b^{m-k-1}| = 2m - 2k < m.$$

Since $|ab^{m-k}a^{m-k-1}|_b = |ba^{m-k}b^{m-k-1}|_b = m-k$ and $ab^{m-k}a^{m-k-1} <_{lex} ba^{m-k}b^{m-k-1}$, we have that $ab^{m-k}a^{m-k-1} < ba^{m-k}b^{m-k-1}$. Then, from the strong compatibility of $\leq$, we have :

$$ab^ka^{m-k-1} = (ab^{m-k})b^{2k-m}a^{m-k-1} < (ba^{m-k})b^{2k-m}b^{m-k-1} = ba^{m-k}b^{k-1}.$$

If $2k < m$, then we similarly obtain that $ab^ka^{k-1} < ba^kb^{k-1}$, so that

$$ab^ka^{m-k-1} = (ab^k)a^{m-2k}a^{k-1} < (ba^k)a^{m-2k}b^{k-1} = ba^{m-k}b^{k-1}.$$

In the case of $2k = m$, suppose that $ba^{m-k}b^{k-1} < ab^ka^{m-k-1}$. Then, $ba^kb^{k-1} < ab^ka^{k-1}$, so that $ba^{k+1}b^{k-1} < a^2b^ka^{k-1}$. On the other hand, we have :

$$\begin{aligned} a^2b^ka^{k-1} &= a(ab^{k-2}b^2a)a^{k-2} \\ &< a(bb^{k-2}a^2b)a^{k-2} && \text{(since } abba < baab) \\ &= (ab^{k-1})(a^2b)a^{k-2} \\ &< (ba^{k-1})(a^2b)b^{k-2} && \text{(since } ab^{k-1}a^{k-2} < ba^{k-1}b^{k-2}) \\ &= ba^{k+1}b^{k-1}, && \text{a contradiction.} \end{aligned}$$

Case 2 : $|x|_b < |y|_b$. Put $|x|_b = s$ and $|y|_b = t$. with $s < t$. If $s+t < m$, then $x_i = y_i = a$ for some i ($1 \leq i \leq m$), and if $s + t > m$, then $x_i = y_i = b$ for some i ($1 \leq i \leq m$). In the both cases, we have :

$$|x_1\cdots x_{i-1}x_{i+1}\cdots x_m|_b < |y_1\cdots y_{i-1}y_{i+1}\cdots y_m|_b, \quad \text{so that}$$
$$x_1\cdots x_{i-1}x_{i+1}\cdots x_m < y_1\cdots y_{i-1}y_{i+1}\cdots y_m.$$

Thus, $x = x_1\cdots x_{i-1}x_i x_{i+1}\cdots x_m < y_1\cdots y_{i-1}y_i y_{i+1}\cdots y_m = y$.

So, suppose that $s+t = m$. Then, from the case 1, we obtain that $x \leq b^sa^t$ and $a^sb^t \leq y$. Thus, it suffices to show that $b^sa^t <$

a^sb^t. If $t-s \geq 2$, then since $b^sa^{t-1} < a^sb^{t-1}$ by induction, we have $b^sa^t < a^sb^t$. Hence, let $t = s+1$(so $m = 2s+1$), and assume on the contrary to our purpose that $a^sb^{s+1} < b^sa^{s+1}$. Then, $a^sa^sbb^{s+1} < a^sb^sba^{s+1}$. But, on the other hand, we have that

$$\begin{aligned} a^sb^sba^{s+1} &= (a^sb^{s+1})a^{s+1} \\ &< (b^sa^{s+1})a^{s+1} \\ &= (b^{s-1})ba^{s-1}(a^{s+1})a^2 \\ &< (a^{s-1})ba^{s-1}(ab^s)a^2 \quad \text{(since } b^{s-1}a^{s+1} < a^{s-1}ab^s\text{)} \\ &< (a^{s-1})aa^{s-1}(ab^s)b^2 \quad \text{(since } ba^{s-1}a^2 < aa^{s-1}b^2\text{) ---(*)} \\ &= a^sa^sbb^{s+1}, \quad \text{a contradiction.} \end{aligned}$$

At (*), we can use the induction hypothesis since $|ba^{s-1}a^2| = s+2 < 2s+1$ (= m) by our hypothesis that $m \geq 5$. We have proved the lemma.

Now, our main theorem:

Theorem 7. Let $X = \{a,b\}$ with $a < b$. Suppose that $\leq$ is a strongly compatible total order on X^*. Then, $\leq$ must be one of the following six orders on X^*:

(1) the lexicographic order,

(2) the anti-lexicographic order,

(3)~(6) one which satisfies the condition (a) and one of the conditions (b),(c),(b'),(c').

Proof. For the sake of Proposition 1-2 and Lemma 3-6, it suffices to check that if $\leq$ is a strongly compatible total order on $\{a,b\}^*$, then it coincides with $<_{lex}$ or $<_{al}$ on $X^{[3]}$, or with one of the other four orders on $X^{[4]}$. So, let $\leq$ be a strongly compatible total order on $\{a,b\}^*$. Then, since $a < b$, by the compatibility of $\leq$, there are only two possibilities on X^2:

I. aa < ab < ba < bb, and

II. aa < ba < ab < bb.

Suppose that the possibility I is the case. Then, the strong compatibility of $\leq$ imposes on the ordering on X^3 that aaa < aab < aba < (abb or baa) < bab < bba < bbb. The strong compatibility alone can not determine if abb < baa or baa < abb. Therefore, we get two possibilities:

I-(i). aaa < aab < aba < abb < baa < bab < bba< bbb, or

I-(ii). aaa < aab < aba < baa < abb < bab < bba < bbb.

If I-(i) is the case, then $\leq$ coincides with $<_{lex}$ on $X^{[3]}$, and we are done. So, suppose that I-(ii) is the case. Then, the strong compatibility of $\leq$ imposes on the ordering on X^4 that

aaaa < aaab < aaba < abaa < baaa < aabb < abab <

$< (abba \text{ or } baab) <$

$< baba < bbaa < abbb < babb < bbab < bbba < bbbb.$

Here, again the strong compatibility fails to determine if $abba < baab$ or $baab < abba$. If $abba < baab$, then $\leq$ coincides on $X^{[4]}$ with the order defined by the conditions (a) and (b), and if $baab < abba$, then $\leq$ coincides on $X^{[4]}$ with the order defined by the conditions (a) and (c). Hence, we have shown the case I above. The case II can be treated similarly.

When Card(X) ≥ 3, a description of the strongly compatible total orders becomes much more complicated even when Card(X) $= 3$. We are going to show that there are infinitely many strongly compatible total orders on X^* when Card(X) ≥ 3.

Let $X = \{a_1, a_2, \cdots, a_n\}$ $(n \geq 3)$. Let $f : X^* \rightarrow \mathbf{R}$ be a mapping from X^* into the set of real numbers satisfying :

(1) for $a_i \in X$, $f(a_1) \leq f(a_2) \leq \cdots \leq f(a_n)$,

(2) for $x = x_1 x_2 \cdots x_m \in X^*$, where $x_i \in X$ $(i = 1, \cdots, m)$, $f(x) = \sum_{i=1}^{m} f(x_i)$.

Then, let $\leq_f$ be the order on X^* defined by:

(i) if $|x| < |y|$, then $x \leq_f y$,

(ii) if $|x| = |y|$ and $f(x) < f(y)$, then $x \leq_f y$,

(iii) if $|x| = |y|$, $f(x) = f(y)$ and $x <_{lex} y$ then $x \leq_f y$.

Then, it is clear that $\leq_f$ is a strongly compatible total order on X^* for every f satisfying (1) and (2) above.

Proposition 8. Let $X = \{a_1, a_2, \cdots, a_n\}$ $(n \geq 3)$. Then, the cardinality of the set of all strongly compatible total orders on X^* is that of the set of real numbers.

Proof. Let a,b,c be three distinct letters in X with $a < b < c$. For each real number $s \in [0,1]$, let f_s be a map : $X^* \rightarrow \mathbf{R}$ satisfying (1),(2) above and such that $f_s(a) = 0$, $f_s(b) = s$, and $f_s(c) = 1$. Denote $\leq_{f_s}$ by $\leq_s$. We claim that if $0 < s < t < 1$, then $\leq_s$ and $\leq_t$ are not the same orders, which will show the theorem. To show the claim, let p,q be natural numbers such that $s < \frac{q}{p} < t$ (hence, $ps < q$

$< pt$). Consider the words $x = b^p$ and $y = c^q a^{p-q}$. Since $f_s(x) = ps < q = f_s(y)$, $x <_s y$. On the other hand, since $f_t(x) = pt > q = f_t(y)$, $y <_t x$. Hence, $\leq_s$ and $\leq_t$ are not the same.

Finally, we answer the following question :

Q. Does there exist a compatible total order which is not strongly compatible?

The answer is 'positive'. To show an example, we modify the above $\leq_f$ so that an evaluation of a word also counts a weight on the 'location' of each letter in the word.

Let the alphabet $X = \{ a_1, a_2, \cdots, a_n \}$. As before, f puts weight (a real number) on each letter:

$$f(a_1) < f(a_2) < \cdots < f(a_n).$$

Let s be a positive real. Then, f_s evaluates a word $x = x_1 x_2 \cdots x_m$ (where $x_i \in X$) by :

$$f_s(x) = \sum_{i=1}^{m} f(x_i)\, s^{i-1}.$$

Then, let $\leq_f^s$ be the order on X^* defined by:

(i) if $|x| < |y|$, then $x <_f^s y$,

(ii) if $|x| = |y|$ and $f_s(x) < f_s(y)$, then $x <_f^s y$.

It is clear that if s is a transcendental real, then $\leq_f^s$ is a total order. To see that $\leq_f^s$ is compatible, it suffices to observe that $f_s(uv) = f_s(u) + f_s(v)s^{|u|}$ for $u,v \in X^*$. But, for some s, $\leq_f^s$ is not strongly compatible. For example, let s be a transcendental number in the interval $(1, \frac{1+\sqrt{5}}{2})$, and let $f(a) = 0$, $f(b) = 1$ for $a,b \in X$. Then, it is easy to see that $s^2 < s+1$ and $aab <_f^s bba$. If $\leq_f^s$ were strongly compatible, then $aaa^{k-2}b <_f^s bba^{k-2}a$ for every k. But, for a sufficiently large k, $f_s(aaa^{k-2}b) = s^k > s+1 = f_s(bba^{k-2}a)$ and $bba^{k-2}a <_f^s aaa^{k-2}b$. Hence, this $\leq_f^s$ is not strongly compatible.

Appendix.

In this appendix, we present some complimentary results on $\leq_f^s$ -orders in the case that the alphabet $X = \{a,b\}$.

First of all, we get rid of an unnecessary complication on f.

Suppose that we are given an order $\leq_f^s$ on $\{a,b\}^*$ for some f and (a transcendental) s. Then, for $x = x_1x_2 \cdots x_m$, (where $x_i \in \{a,b\}$), we have:

$$f_s(x) = \sum_{i=1}^{m} f(x_i)\, s^{i-1} = (f(b)-f(a)) \cdot \left[\sum_{i=1}^{m} \delta_i(x) s^{i-1} \right] + \sum_{i=1}^{m} f(a) s^{i-1},$$

where $\delta_i(x) = 1$ if $x_i = b$, else 0. From this, we conclude that an order $\leq_f^s$ on $\{a,b\}^*$ with $f(a) < f(b)$ is equivalent to the order $\leq_g^s$ with $g(a) = 0$ and $g(b) = 1$. Hence, in the followings, we consider only orders $\leq_f^s$ with $f(a) = 0$ and $f(b) = 1$, and omit the subscript f as $\leq^s$.

Proposition A. For $s \geq 2$ (resp. $0 < s \leq \frac{1}{2}$), $\leq^s$ coincides with $\leq_{al}$ (resp. $\leq_{lex}$).

Proof. We only prove the case for $s \geq 2$. It suffices to show that for every $k \geq 1$, $1 + s + s^2 + \cdots + s^{k-1} < s^k$ because L.H.S. $= f_s(b^ka)$ and R.H.S. $= f_s(a^kb)$. But, since $s \geq 2$, $\frac{s^k - 1}{s-1} \leq s^k - 1$, i.e. , $\frac{s^k - 1}{s-1} < s^k$, i.e. , $1 + s + s^2 + \cdots + s^{k-1} < s^k$.

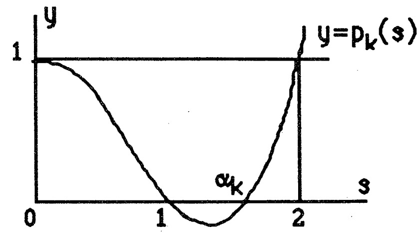
Proposition B. For $s \in (\frac{1}{2}, 1) \cup (1, 2)$, $\leq^s$ is not strongly compatible.

Proof. We prove only the case for $s \in (1, 2)$. We first show that $\leq^s \neq \leq_{al}$, for which it suffices to prove that there is a $k \geq 1$ such that $s^k < 1 + s + s^2 + \cdots + s^{k-1}$ since then $a^kb <^s b^ka$. But, if $s \in (1, 2)$, then for a sufficiently large k, it holds that $1 < (2-s)s^k$, i.e., $1 - s^k < (1-s)s^k$, i.e. , $\frac{s^k - 1}{s-1} > s^k$, i.e. , $1 + s + s^2 + \cdots + s^{k-1} > s^k$. On the other hand, there is an m such that $1 + s + s^2 + \cdots + s^{k-1} < s^{k+m}$. For such an m, we have that $a^ka^mb^s > b^ka^ma$. Hence, $\leq^s$ is not strongly compatible.

Put for each $k \geq 2$, $p_k(s) = s^{k+1} - 2s^k + 1$. Then, as in the above proof, for each $k \geq 2$, and for each $s \in (1, 2)$,

$$p_k(s) > 0 \quad \text{iff} \quad a^kb <^s b^ka$$
$$(<) \qquad\qquad ({}^s>)$$

For each $k \geq 2$, since $p_k(s) = s^k\{(k+1)s - 2k\}$, it is easy to see that there is a unique root α_k of $p_k(s) = 0$ in the interval $(1, 2)$.



Then, $\{\alpha_k\}_{k\geq 2}$ is a strictly increasing sequence (converging to 2). For, if $2 \leq k < m$, then $p_m(\alpha_k) = \alpha_k^{m+1} - 2\alpha_k^m + 1 = \alpha_k^{m-k}(\alpha_k^{k+1} - 2\alpha_k^k + 1) + 1 - \alpha_k^{m-k} = 1 - \alpha_k^{m-k} < 0$, which means that $\alpha_k < \alpha_m$. The convergence is proved by a similar argument to the proof of Proposition B.

Proposition C. There are at least countably many compatible total orders on $\{a,b\}^*$.

Proof. Let $2 \leq k < m$ and $s \in (\alpha_k, \alpha_{k+1})$, $t \in (\alpha_m, \alpha_{m+1})$. Then, it suffices to show that $<^s$ and $<^t$ are distinct orders. Since $s < \alpha_{k+1} \leq \alpha_m$, $p_m(s) < 0$, i.e. , $b^m a <^s a^m b$. But, since $\alpha_m < t$, $p_m(t) > 0$, i.e. , $a^m b <^t b^m a$. Hence, $<^t \neq <^s$.

REFERENCE

1. [B-P] Jean Berstel,Dominique Perrin "THEORY OF CODES", Academic Press, 1985.

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