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<th>Title</th>
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</thead>
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Maximal avoidable sets of words

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We use the following notations.

Σ : an alphabet (a finite set of letters),
Σ* : the set of words over Σ,
Σω : the set of infinite words (sequences),
Σ# := Σ* U Σω,
Σ+ := Σ* - {11}, where 1 is the empty word.

For x = a1a2... , y = b1b2... ∈ Σ# define the distance of x and y by

\[ d(x, y) = \frac{1}{\min(n : a_n \neq b_n)} \]

As is well known ([41]), (Σ#, d) is a compact totally disconnected metric space.

Let x, y ∈ Σ* and X ⊂ Σ#. We say y avoids x, if y does not contain x as subword, and y avoids X, if y avoids every x in X. X is called avoidable, if there is an infinite word y avoiding X, otherwise X is called unavoidable. Avoidability of sets of words called patterns were studied in [11].

Example 1. Let X = \{ v^2 | v ∈ Σ^+ \}. Then y avoids X if and only if y is square-free. It is a famous fact that X is avoidable if |Σ| ≥ 3

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An avoidable set $X$ is **maximal**, if any set properly containing $X$ is unavoidable.

**Theorem 1.** For any avoidable set $X$, there is a maximal avoidable set containing $X$.

For a given $X \subseteq \Sigma^*$, the set

$$
\text{Min}(X) = \{ x \in X \mid \text{any } x' \in X \text{ is not a proper subword of } x \}
$$

is called the **base** of $X$. Easily we see $y$ avoids $X$ if and only if $y$ avoids $\text{Min}(X)$. $X$ is **finitely based** if $\text{Min}(X)$ is a finite set. The base of a maximal avoidable set is called a **critical set** of words.

**Corollary.** For any avoidable set $X$ which is factor-free, that is, any word in $X$ is not a subword of another word in $X$, there is a critical set containing $X$.

**Example 2.** Let $\Sigma = \{a, b\}$. Then, $(a^2, ab, ba)$ and $(a^2, b^2)$ are critical sets.

An infinite word $x$ is recurrent, if for any subword $v$ of $x$, there is an integer $k(v) > 0$ such that any subword of length $k$ of $x$ contains $v$ as subword. In this situation $v$ is said to be recurrent in $x$. If $x = v^\omega$ for some $v \in \Sigma^*$, $x$ is called periodic; the shortest such $v$ is the **period** of $x$. A periodic infinite word is recurrent, but the converse is not true.

The **shift transformation** $\tau$ is a mapping from $\Sigma^\omega$ to itself defined by

$$
\tau(x) = a_2a_3 \cdots \text{ for } x = a_1a_2 \cdots .
$$
Obviously, \( \tau \) is a surjective continuous mapping.

A subshift \( S \) is a non-empty closed subset of \( \Sigma^\omega \) invariant under \( \tau \). \( S \) is minimal, if it does not contain a subshift properly. For a given set \( X \subseteq \Sigma^* \) of words, \( S(X) \) is the set of infinite words avoiding \( X \). For a given subshift \( S \subseteq \Sigma^\omega \), \( X(S) \) is the set of words which do not appear as subwords of elements of \( S \).

Theorem 2. For an avoidable set \( X \), \( S(X) \) is a subshift. If \( X \) is maximal, then \( S(X) \) is minimal. Conversely, if \( S \) is a subshift, then \( X(S) \) is an avoidable set. If \( S \) is minimal, then \( X(S) \) is maximal. This gives a 1-1 correspondence between maximal avoidable sets and minimal subshifts.

Lemma 1. An avoidable set \( X \) is maximal if and only if any word out of \( X \) is recurrent in any infinite word avoiding \( X \).

Theorem 3 (Morse-Hedlund [4]). \( S \subseteq \Sigma^\omega \) is a minimal subshift if and only if

\[
S = \{ \tau^n(x) \mid n = 0, 1, 2, \ldots \}
\]

for some recurrent infinite word \( x \). Moreover,

1. \( S \) is perfect, if \( x \) is non-periodic. In this case every element in \( S \) is non-periodic.
2. \( S \) is finite, if \( x \) is periodic. In this case every element in \( S \) is periodic.

Corollary (c.f. [3, Theorem 4.2]). Let \( X \) be an avoidable set such that for any \( v \in \Sigma^* \), \( v^n \) does not avoid \( X \) for \( x \gg 0 \), then \( S(X) \) contains a perfect subset.
Theorem 4. Let $X$ be a maximal avoidable set. Then, $S(X)$ is finite if and only if $X$ is finitely based.

For an avoidable set $X$, the radical $\text{rad}(X)$ of $X$ is the intersection of all the maximal avoidable sets containing $X$. $X$ is called reduced, if $X = \text{rad}(X)$.

Lemma 2. A word $v$ is in $\text{rad}(X)$, if and only if any recurrent infinite word avoiding $X$ avoids $v$.

Corollary. Any word out of $\text{rad}(X)$ is extensible to a recurrent infinite word avoiding $X$.

Theorem 5. If $X$ is a reduced avoidable set, then every isolated point of $S(X)$ is periodic.

Corollary. If $X$ is a reduced avoidable set such that for any $v \in \Sigma^+$, $v^n$ does not avoid $X$ for $n \gg 0$, then $S(X)$ is perfect.

A set $X$ of words is quasi-maximal, if $\text{rad}(X)$ is maximal.

Theorem 6. Let $X$ be an avoidable set. Then following statements are equivalent.

1. $X$ is quasi-maximal.
2. $S(X)$ contains a unique minimal subshift.
3. For any $n > 0$, there is a word $v$ of length $n$ such that $X \cup \{v\}$ is unavoidable.
4. For any word $w$ such that $X \cup \{w\}$ is unavoidable and for any $n > 0$, there is a word $v$ of length $n$ such that $X \cup \{vw\}$ is unavoidable.

Example 3. Let $X = \{(a^2, bab) \in \{a, b\}^\infty \}$. Then, $b^\omega$ and $ab^\omega$ are only infinite words avoiding $X$, and $X \cup \{b^n\}$ is unavoidable for any $n > 0$. 

4
Thus X is quasi-maximal.

An unavoidable set X is said to be minimal, if X - \{v\} is avoidable for any v \in X. As is easily seen ([2]), a minimal unavoidable set is finite.

Conjecture I (Ehrenfeucht, see [2]). For any unavoidable set X, there is a word x \in X and a letter a \in \Sigma such that (X - \{x\}) \cup \{xa\} is unavoidable.

Conjecture II. For any minimal unavoidable set X, there is a word x in X such that X - \{x\} is a quasi-maximal avoidable set.

Theorem 7. Conjecture I and Conjecture II are equivalent.

References