数理解析研究所講究録
698巻 1989年 146-170



# Free Boundary Problems for General Fluids

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## §1. Introduction

In this communication we are concerned with free boundary problems, one-phase and multi-phase, for compressible viscous isotropic Newtonian fluids (say, general fluids). In this one-phase problem, the domain $\Omega(t)\subset\boldsymbol{R}^3$ occupied by the fluid at the moment $t>0$ is to be determined together with the density $\rho=\rho(x,t)$, with the velocity vector field $v=v(x,t)=(v_1,v_2,v_3)$ and with the absolute temparature $\theta=\theta(x,t)$ satisfying the so-called compressible Navier-Stokes equations:

$$(1)\quad \begin{cases} \dfrac{D\rho}{Dt} = -\rho\ \nabla\cdot v, \\ \rho\ \dfrac{Dv}{Dt} = \nabla\cdot\boldsymbol{P} + \rho f, \qquad x\in\Omega(t),\ \ t>0, \\ \rho\,\theta\dfrac{DS}{Dt} = \nabla\cdot(\kappa\ \nabla\theta) + \mu'(\nabla\cdot v)^2 + 2\mu\boldsymbol{D}(v):\boldsymbol{D}(v), \end{cases}$$

and the initial and boundary conditions

$$
(2)\begin{cases}
(\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0)(x), & x \in \Omega(0) \equiv \Omega, \\
(v, \theta) = (0, \theta_a(x, t)), & x \in \Sigma, \\
\boldsymbol{P} n = -p_e n + \sigma H n, \quad \kappa \nabla\theta \cdot n = \kappa_e(\theta_e - \theta), & x \in \Gamma(t),\ t > 0, \\
\dfrac{DF}{Dt} = 0, & x \in \Gamma(t),\ t > 0,
\end{cases}
$$

if $\Gamma(t)$ is given by $F(x, t) = 0$.

Here $f = f(x, t)(x \in \boldsymbol{R}^3, t > 0)$ is a vector field of external forces, $p_e = p_e(x, t)(x \in \boldsymbol{R}^3, t > 0)$ is an outer pressure, $\Sigma$ and $\Gamma(t)$ are two disjoint components of the boundary $\partial\Omega(t)$($\Sigma$ is fixed and $\Gamma(t)$ is free), $n = n(x, t)$ is the unit outward normal vector to $\Gamma(t)$ at the point $x$, $\boldsymbol{P} = (-p + \mu' \nabla\cdot v)\boldsymbol{I} + 2\mu \boldsymbol{D}(v)$ is the stress tensor, $\boldsymbol{D}(v)$ is the velocity deformation tensor with the element

$$
D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \boldsymbol{D}(v):\boldsymbol{D}(v) = D_{jk} D_{jk}
$$

(Here and in what follows we use the summation convention), $p = p(\rho, \theta)$ is a pressure, $S = S(\rho, \theta)$ is an entropy, $\mu, \mu', \kappa, \sigma, \kappa_e$ are, respectively, coefficient of viscosity, second coefficient of viscosity, coefficient of heat conductivity, coefficient of surface tension and coefficient of outer heat conductivity, which are all assumed to be constants satisfying $\mu > 0$, $2\mu + 3\mu' \geq 0$, $\kappa > 0$, $\sigma > 0$, $\kappa_e > 0$, $D/Dt = \partial/\partial t + v \cdot \nabla$, $H/2$ is the mean curvature of $\Gamma(t)$.

The sign of $H$ is chosen in such a way that $Hn = \Delta(t)x$, where $\Delta(t)$ is the Laplace-Beltrami operator on $\Gamma(t)$.

One-phase free boundary problem without surface tension(i.e., $\sigma = 0$) was discussed in Sobolev space by P. Secchi and A. Valli[6] when $\Omega \subset \mathbf{R}^3$ is bounded and $\Sigma = \phi$ and in Hölder space by A. Tani[14] in the case of general domain $\Omega$.

For such problems for the incompressible ones we have better results than our problem (1)-(2). When $\sigma = 0$, the existence of solution, local in time, was proved by V.A. Solonnikov[8] in Hölder space when $\Omega$ is bounded and $\Sigma = \phi$ and by J.T. Beale[2] in Sobolev space when $\Omega$ is an infinite slab. On the other hand when $\sigma > 0$, we have some interesting results on a temporally global solution and its large-time behavior in Sobolev space under some smallness conditions on data: Beale[3], Beale-Nishida[4], Solonnikov[10-12], in each case. Without smallness conditions on data, we have only the local existence results proved by Solonnikov[9] when $\Omega \subset \mathbf{R}^3$ is bounded and $\Sigma = \phi$ and by G. Allain[1] when $\Omega \subset \mathbf{R}^2$ is an infinite slab.

Notation. Throughout this paper we use Sobolev - Slobodetskiĭ spaces defined as follows. For any $r > 0$, $r \notin \mathbf{Z}$ we define

$$W_2^{r,r/2}(Q_T \equiv \Omega \times (0,T)) = \left\{ u, \text{ defined on } Q_T \,|\, \|u\|_{W_2^{r,r/2}(Q_T)} < \infty \right\},$$

where

$$\|u\|_{W_2^{r,r/2}(Q_T)}=\left(\|u\|^2_{W_2^{r,0}(Q_T)}+\|u\|^2_{W_2^{0,r/2}(Q_T)}\right)^{\frac{1}{2}},$$

$$\|u\|^2_{W_2^{r,0}(Q_T)}=\int_0^T\|u\|^2_{W_2^{r}(\Omega)}\,dt,$$

$$\|u\|^2_{W_2^{0,r/2}(Q_T)}=\int_\Omega\|u\|^2_{W_2^{r/2}(0,T)}\,dx,$$

$$\|u\|^2_{W_2^{r}(\Omega)}=\sum_{|s|<r}\|D^s u\|^2_{L_2(\Omega)}+$$

$$+\sum_{|s|=[r]}\int_\Omega\int_\Omega\frac{|D_x^s u(x,t)-D_y^s u(y,t)|^2}{|x-y|^{3+2(r-[r])}}\,dx\,dy,$$

$$\|u\|^2_{W_2^{r/2}(0,T)}=\sum_{j=0}^{[r/2]}\|D_t^j u\|^2_{L_2(0,T)}+$$

$$+\int_0^T\int_0^T\frac{|D_t^{[r/2]}u(x,t)-D_\tau^{[r/2]}u(x,\tau)|^2}{|t-\tau|^{1+2(r/2-[r/2])}}\,dt\,d\tau.$$

We also define the space $W_2^{r,r/2}(\Gamma_T)$ on the manifold $\Gamma_T=\Gamma\times(0,T)$ as $L_2((0,T);\,W_2^r(\Gamma))\cap L_2(\Omega;W_2^{r/2}(0,T))$.

$W_2^r(\Omega)=\{u(x)$, defined on $\Omega \mid \|u\|_{W_2^r(\Omega)}<\infty\}$

Furthermore we introduce Sobolev-Slobodetskiĭ space with weight $e^{-2ht}$ $(h>0)$.

$H_h^{r,r/2}(Q_T)=\{u$, defined on $Q_T \mid \|u\|_{H_h^{r,r/2}(Q_T)}<\infty\}$

$$\|u\|^2_{H_h^{r,r/2}(Q_T)}\equiv\sum_{2j+|k|=0}^{[r]}\int_0^T e^{-2ht}\|D_t^j D_x^k u\|^2_{L_2(\Omega)}\,dt+$$

$$+\sum_{2j+|k|=[r]}\int_0^T e^{-2ht}\langle\!\langle D_\tau^j D_x^k u\rangle\!\rangle_{2,\Omega}^{(r-[r])2}\,dt+$$

$$+\sum_{r-2<2j+|k|<r}\int_0^T e^{-2ht}dt\int_0^\infty\|\Delta^t{}_{t-\tau}D_t^j D_x^k u_0(\cdot,t)\|_{L_2(\Omega)}\times$$

$$\times\ \tau^{-1-r+2j+|k|}\,d\tau,$$

$$\langle\!\langle u\rangle\!\rangle_{2,\Omega}^{(\delta)2}=\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{3+2\delta}}\,dx\,dy,\quad \delta\in(0,1),$$

$$\Delta^t{}_\tau u(x,t)=u(x,t)-u(x,\tau),\qquad u_0=\begin{cases}u & t>0\\ 0 & t<0\end{cases}.$$

The same notation will be used for the spaces of vector fields, the norms of a vector supposed to be equal to the sum of norms of all its components.

## §2. One-phase problem

Our first result is the following.

Theorem 1. Suppose that

(i) $\Omega\subset\boldsymbol{R}^3$ is a bounded domain with a boundary $\partial\Omega=\Gamma\cup\Sigma$, $\Gamma\cap\Sigma=\phi$, $\Gamma,\ \Sigma\in W_2^{l+5/2}$, $l\in(\frac{1}{2},1)$,

(ii) $(\rho_0,v_0,\theta_0)\in W_2^{1+l}(\Omega)\times W_2^{1+l}(\Omega)\times W_2^{1+l}(\Omega)$, $\rho_0,\theta_0>0$,

(iii) $\mu,\ \mu',\ \kappa,\ \sigma,\ \kappa_e$ are constants satisfying the relations

$$2\mu+3\mu'\geq 0,\qquad \mu,\ \kappa,\ \sigma,\kappa_e>0,$$

(iv) $\theta_a\in W_2^{l+3/2,l/2+3/4}(\Sigma_T)$,

(v) $\nabla\nabla(p_e, \theta_e)$, $\nabla(p_{e,t}, \theta_{e,t})$ are defined in $\boldsymbol{R}^3 \times (0,T)$ and Lipschitz continuous in $x$,

(vi) $f$, $\nabla f$ are defined in $\boldsymbol{R}^3 \times (0,T)$, Lipschitz continuous in $x$ and $1/2$ Hölder continuous in $t$,

(vii) $(S, p) = (S, p)(\rho, \theta)$ are defined on $(0,\infty)\times(0,\infty)$, two times partially differentiable, and their second order derivatives are locally Lipschitz continuous there; moreover $S_\theta > 0$.

Then there exists a unique solution $(\rho, v, \theta)$ of (1)-(2) such that

$$D^k v,\ D^k \theta \in L_2(D_{T'}) \text{ for } k=0,1,2,\quad v_t,\ \theta_t \in L_2(D_{T'}),$$

$$D^k \rho \in L_2(D_{T'}) \text{ for } k=0,1,\quad \rho_t \in L_2(D_{T'}),\quad \Gamma(t) \in W_2^{5/2+l}$$

for some $T' \in (0,T]$, where $D_T = \{(x,t) \in \boldsymbol{R}^4 \mid x \in \Omega(t),\ t \in (0,T)\}$.

The sketch of proof.

$1^\circ$. First of all, we tranform the equations (1) and the initial-boundary conditions (2) by the characteristic transformation $\Pi^x_\xi : x \to \xi \equiv X(0; x, t)$, where $X(\tau; x, t)$ is the solution of the equation

$$(3)\quad \frac{d}{d\tau} X(\tau; x, t) = v(X(\tau; x, t), \tau), \quad X(t; x, t) = x.$$

If $v$ be suitably smooth, then the basic theorem of ordinary differential equations yields that (3) has a unique solution curve, which gives us the relation between $x$ and $\xi$:

$$(4)\qquad x=\xi+\int_0^t u(\xi,\tau)\,d\tau=X(t;\xi,0)\equiv X_u(\xi,t),$$

where $u(\xi,t)=v(X_u,t)$. According to a kinematic boundary condition $(2)^4$, $\prod^x{}_\xi$ is one-to-one mapping from $\{(x,t)\in\boldsymbol{R}^4 \mid x\in\Omega(t),\ t\in(0,T)\}$[resp. $\{(x,t)\in\boldsymbol{R}^4 \mid x\in\Gamma(t),\ t\in(0,T)\}$] onto $Q_T$ [resp. $\Gamma_T$]. Then the problem (1)-(2) takes the form

$$(5)\qquad \begin{cases} \dfrac{\partial\rho^*}{\partial t} = -\rho^*\,\nabla_u\cdot u, \\ \rho^*\,\dfrac{\partial u}{\partial t} = \nabla_u\cdot\boldsymbol{P}_u + \rho^* f^*, \qquad x\in\Omega,\quad t>0, \\ \rho^*\theta^* S_{\theta^*}\dfrac{\partial\theta^*}{\partial t} = \nabla_u\cdot(\kappa\,\nabla_u\theta^*) + \mu'\,(\nabla_u\cdot u)^2 + \\ \qquad + 2\mu\,\boldsymbol{D}_u(u):\boldsymbol{D}_u(u) + \rho^*\theta^* S_{\rho^*}\,\nabla_u\cdot u, \end{cases}$$

$$(6)\qquad \begin{cases} (\rho^*, u, \theta^*)|_{t=0} = (\rho_0, v_0, \theta_0)(\xi), \qquad \xi\in\Omega, \\ (u, \theta^*) = (0, \theta_a^*(\xi, t)), \quad \xi\in\Sigma,\quad t>0, \\ \boldsymbol{P}_u n = -p_e^* n + \sigma\,\Delta_u(t) X_u(\xi, t), \\ \qquad\qquad\qquad\qquad\qquad \xi\in\Gamma,\quad t>0. \\ \kappa\,\nabla_u\theta^*\cdot n = \kappa_e(\theta_e^* - \theta^*), \end{cases}$$

Here $(\rho^*,\theta^*,f^*,p_e^*,\theta_a^*,\theta_e^*)=\prod^x{}_\xi(\rho,\theta,f,p_e,\theta_a,\theta_e)$, $\nabla_u = (\nabla_{u,1},\ \nabla_{u,2},\ \nabla_{u,3}) = G\,\nabla$, $G=({}^t(\partial X_u/\partial t))^{-1}$, $\boldsymbol{P}_u=(-p(\rho^*,\theta^*) + \mu'\ \nabla_u\cdot u)\boldsymbol{I}+2\mu\,\boldsymbol{D}_u(u)$, $\boldsymbol{D}_u(u)=(D_{u,ij})=\frac{1}{2}(\nabla_{u,i}\,u_j+\nabla_{u,j}\,u_i)$.

By $\Delta_u(t)$, we denote Laplace-Beltrami operator on $\Gamma_T$ parametrized by the relation (4). Of course, $n = n(X_u, t)$ is represented by $n = G n_0(\xi)/|G n_0(\xi)|$ where $n_0(\xi)$ is the unit outward normal to $\Gamma$ at the point $\xi$.

It is easily seen that the solution of the Cauchy problem for $\rho^*$ is given by the formula

$$(7) \qquad \rho^*(\xi, t) = \rho_0(\xi) \exp\left[-\int_0^t \nabla_u \cdot u(\xi, \tau)\, d\tau\right]$$

provided that $u \in W_2^{2+l, 1+l/2}(Q_T)$, $\frac{1}{2} < l < 1$ is given. Therefore the main part of our problem is to solve the initial-boundary value problem (5)-(6) for $(u, \theta^*)$ with $\rho^*$ given by (7) in a fixed domain $Q_T$.

2°. We consider an auxiliary linear initial-boundary value problem

$$(8) \quad \begin{cases} \dfrac{\partial u}{\partial t} = a(x)\Delta u + a_1(x)\nabla(\nabla\cdot v) + \phi(x, t) & \text{in } Q_T, \\ u|_{t=0} = u_0(x) & \text{on } \Omega, \\ B_0(x; \nabla)u - \sigma'(x) B_1(x; \nabla)\displaystyle\int_0^t u\, d\tau = b & \text{on } \Gamma_T, \\ u = 0 & \text{on } \Sigma_T, \end{cases}$$

where $B_0 = (B_{0,jk})_{1 \le j,\, k \le 3}$, $B_1 = (B_{1,jk})_{1 \le j,\, k \le 3}$ are as follows:

$$B_{0,jk} = \begin{cases} -a(\delta_{jk}\, n\cdot\nabla + n_k \nabla_j - 2 n_j n_k\, n\cdot\nabla), & j = 1, 2,\ k = 1, 2, 3, \\ (a - a_1)\nabla_k + 2 a\, n_k\, n\cdot\nabla, & j = 3,\ k = 1, 2, 3, \end{cases}$$

$$B_{1,jk}=\begin{cases} 0, & j=1,2,\ k=1,2,3, \\ -n_k\,\Delta(0), & j=3,\ k=1,2,3. \end{cases}$$

In order to solve the problem (8) in the general domain $Q_T$, it is necessary to solve the following problem (9) with constant coefficients in the half space $D_{++}\equiv \boldsymbol{R}^3_+\times(0,\infty)$:

$$(9)\begin{cases} \dfrac{\partial u}{\partial t}=a\,\Delta u+a_1\,\nabla(\nabla\cdot u), \quad \text{in}\quad D_{++}, \\ u\,|_{t=0}=0\ , \\ a\left(\dfrac{\partial u_3}{\partial x_\gamma}+\dfrac{\partial u_\gamma}{\partial x_3}\right)\Big|_{x_3=0}=b_\gamma\,(x',\ t), \quad \gamma=1,2, \\ (a_1-a)\,\nabla\cdot u+2a\dfrac{\partial u_3}{\partial x_3}+\sigma'\displaystyle\int_0^t\nabla'^2\,u_3\,d\tau\,\Big|_{x_3=0}=b_3. \end{cases}$$

Extending $u$ and $b={}^t(b_1, b_2, b_3)$ to the half space $t<0$ by $0$ and making the Fourier transformation with respect to $x'$ and Laplace transformation with respect to $t$:

$$(10)\qquad \hat{u}(\xi',\ s,\ x_3)=\int_0^\infty e^{-st}\,d\,t\int_{\boldsymbol{R}^2}u(x,\ t)\,e^{-ix'\cdot\xi'}\,d\,x',$$

we get the boundary value problem for the system of ordinary differential equations

$$\left\{(a\,r^2+a_1\,\xi_1{}^2-a\frac{d^2}{d\,x_3{}^2})\,\hat{u}_1+a_1\,\xi_1\,\xi_2\,\hat{u}_2-i\,a_1\,\xi_1\frac{d}{d\,x_3}\,\hat{u}_3=0,\right.$$

$$(11)\begin{cases} a_1\xi_1\xi_2\hat{u}_1+(a\,r^2+a_1\xi_2{}^2-a\dfrac{d^2}{d\,x_3{}^2})\ \hat{u}_2-i\,a_1\xi_2\dfrac{d}{d\,x_3}\ \hat{u}_3=0, \\ -i\,a_1(\xi_1\dfrac{d}{d\,x_3}\ \hat{u}_1-\xi_2\dfrac{d}{d\,x_3}\ \hat{u}_2)+(a\,r^2-(a+a_1)\dfrac{d^2}{d\,x_3{}^2})\,\hat{u}_3=0, \\ a(\dfrac{d}{d\,x_3}\ \hat{u}_\gamma+i\,\xi_\gamma\hat{u}_3)\Big|_{x_3=0}=\hat{b}_\gamma\quad(\gamma=1,2), \\ (a_1-a)(i\,\xi_1\hat{u}_1+i\,\xi_2\hat{u}_2)+(a_1+a)\dfrac{d}{d\,x_3}\,\hat{u}_3- \\ \qquad -\dfrac{\sigma'}{s}\,\xi'^2\,\hat{u}_3\Big|_{x_3=0}=\hat{b}_3, \\ \hat{u}\to 0\quad\text{as}\quad x_3\to\infty, \end{cases}$$

where $r^2=s/a+\xi'^2$, $\xi'^2=\xi_1{}^2+\xi_2{}^2$, arg $r\in(-\pi/4,\pi/4)$.

It is not so difficult to solve the problem (11); Indeed

$$(12)\ \hat{u}=-\frac{\exp[-r\,x_3]}{a\,r}\,\hat{b}'+\frac{\exp[-r\,x_3]}{\Delta\,r(r+r_1)}U\,\hat{b}+\frac{a_1\,e_1(x_3)}{\Delta\,r(r+r_1)(a+a_1)}V\,\hat{b}$$

is a solution of (11) where $\hat{b}'={}^t(\hat{b}_1,\ \hat{b}_2,\ 0)$,

$$\Delta=-s\left[\ s+\frac{4\,a\,a_1}{a+a_1}\,\xi'^2(1-\frac{r_1}{r+r_1})+\frac{\sigma\,\xi'^2}{s}\,r_1\right]$$

$$e_1(x_3)=\frac{e^{-r_1x_3}-e^{-r\,x_3}}{r_1-r},\qquad r_1=\frac{s}{a+a_1}+\xi'^2,$$

$$U=(U_{jk})_{1\leq j,\,k\leq 3},\qquad V=(V_{jk})_{1\leq j,\,k\leq 3},$$

$$\begin{cases} -\xi_j\xi_k\left[s(\dfrac{3\,a_1-a}{a+a_1}\,r-r_1)+\dfrac{a_1\sigma'\,\xi'^2}{a(a+a_1)}\right], & j,\,k=1,2, \\ i\,\xi_j\,s\left[r_1(r+r_1)-\dfrac{a_1}{a+a_1}(r^2+\xi'^2)\right], & j=1,2,\quad k=3, \end{cases}$$

$$U_{jk}=\begin{cases} \\ -i\xi_k r s\left(\dfrac{a-a_1}{a+a_1}r+r_1\right), & j=3,\quad k=1,2, \\ r r_1(r+r_1)s, & j=k=3, \end{cases}$$

$$V_{jk}=\begin{cases} \xi_j\xi_k\left(2rs+\dfrac{\sigma'\xi'^2}{a}\right), & i,\ j=1,2, \\ i\xi_j s(r^2+\xi'^2), & j=1,2,\quad k=3, \\ i\xi_k r_1\left(2rs+\dfrac{\sigma'\xi'^2}{a}\right), & j=3,\quad k=1,2, \\ -r_1(r^2+\xi'^2)s, & j=k=3. \end{cases}$$

After some calculations, we can prove the follwing

Lemma. If $\mathrm{Re}\, s=h>0$, $\xi'\in\mathbf{R}^2$, then the estimates

$$|\Delta|\geq|s|\left[h+\frac{a a_1}{2(a+a_1)}\xi'^2+\frac{\sigma'\xi'^2|r_1|h}{2|s|^2}\right],$$

$$|s|^2,\ \sigma'\xi'^2|r_1|\leq 4\left[4+\frac{(a+a_1)^2}{4aa_1}+\frac{(a+a_1)^{1/2}}{aa_1h^{1/2}}\sigma'\right]$$

are valid.

This is essential for our investigation.

Since $\|u\|^2_{H_h^{l,l/2}(D_+)}$ $(D_+=\mathbf{R}^2\times(0,\infty))$ and $\|u\|^2_{H_h^{l,l/2}(D_{++})}$

are equivalent to

$$\|u\|^2_{l,h,D_+}=\int_{\mathbf{R}^2}d\xi'\int_{-\infty}^{\infty}|\hat{u}(\xi',h+i\xi_0)|^2|r_0|^{2l}d\xi_0$$

and

$$\| u \|^2_{l,\,h,\,D_{++}} =$$

$$\sum_{j<l} \int_{\mathbf{R}^2} d\xi' \int_{-\infty}^{\infty} \| (\frac{\partial}{\partial x_3})^j \hat{u}(\xi', h+i\xi_0, x_3) \|^2_{L_2(\mathbf{R}_+)} | r_0 |^{2l-2j} d\xi_0 +$$

$$+ \int_{\mathbf{R}^2} d\xi' \int_{-\infty}^{\infty} \langle\langle (\frac{\partial}{\partial x_3})^{[l]} \hat{u}(\xi', h+i\xi_0, x_3) \rangle\rangle^{(l-[l])2}_{2,\mathbf{R}_+} d\xi_0$$

($r_0 = s + \xi'^2$), respectively, by Parseval relation, we get the following result.

Proposition 1. Let $l \in (1/2, 1)$, $h > 0$.

If $b_1, b_2 \in H_h^{l+1/2,\,l/2+1/4}(D_+)$, $b_3 = b_3' + \int_0^t B\, d\tau$,

$b_3' \in H_h^{l+1/2,\,l/2+1/4}(D_+)$, $B \in H_h^{l-1/2,\,l/2-1/4}(D_+)$, and $b|_{t=0} = 0$,

then the solution $u$ of the problem (9) is estimated as follows:

$$\| u \|^2_{l+2,\,h,\,D_{++}} \leq c(h) (\| \tilde{b} \|^2_{l+1/2,\,h,\,D_+} + \| B \|^2_{l+1/2,\,h,\,D_+})$$

$$(\tilde{b} \equiv {}^t(b_1, b_2, b_3')).$$

From this proposition and the same method as that in [14] it follows that

Proposition 2. Suppose that

(i) $\Gamma, \Sigma \in W_2^{l+3/2}$, $l \in (1/2, 1)$, $\Gamma \cap \Sigma = \phi$,

(ii) $a, a_1 \in W_2^{1+l}(\Omega)$, $a > 0$, $a_1 > 0$,

(iii) $\phi \in H_h^{l, l/2}(Q_T)$ $(h > 0)$,

(iv) $b_3 = b_3' + \int_0^t B \, d\tau$, $\tilde{b} = {}^t(b_1, b_2, b_3') \in H_h^{l+1/2, l/2+1/4}(\Gamma_T)$;

$B \in H_h^{l-1/2, l/2-1/4}(\Gamma_T)$, $\tilde{b}|_{t=0} = u_0|_\Gamma$ (compatibility condition),

(v) $u_0 \in W_2^{1+l}(\Omega)$,

(vi) $\sigma' \in W_2^{l+1/2}(\Gamma)$, $\sigma' > 0$.

Then there exists a unique solusion $u$ to (8) such that

$$\|u\|_{H_h^{l+2, l/2+1}(Q_T)} \leq C(T) \left(\|\phi\|_{H_h^{l, l/2}(Q_T)} + \|u_0\|_{W_2^{1+l}(\Omega)} + \right.$$

$$+ \|\tilde{b}\|_{H_h^{l+1/2, l/2+1/4}(\Gamma_T)} + \|B\|_{H_h^{l-1/2, l/2-1/4}(\Gamma_T)} +$$

$$\left. + \|\sigma'\|_{W_2^{l+1/2}(\Gamma)} \right).$$

3°. Of course it is easier to solve the linear initial-boundary value problem corresponding to the linearized problem for $\theta^*$.

$$(13) \begin{cases} \dfrac{\partial u_4}{\partial t} = a_2(\xi) \Delta u_4 + \phi_4 & \text{in } Q_T, \\ u_4|_{t=0} = u_{4,0} & \text{on } \Omega, \\ u_4 = u_{4,a} & \text{on } \Sigma_T, \\ a_2 \nabla u_4 \cdot n = b_4 & \text{on } \Gamma_T. \end{cases}$$

Proposition 3. Suppose that

(i) $\Gamma,\ \Sigma \in W_2^{l+3/2},\ l\in(1/2,\ 1),\ \Gamma\cap\Sigma=\phi,$

(ii) $a_2\in W_2^{1+l}(\Omega),\ a_2>0,$ (iii) $\phi_4 \in H_h^{l,l/2}(Q_T),$

(iv) $u_{4,0}\in W_2^{1+l}(\Omega),$ (v) $u_{4,a}\in H_h^{l+3/2,l/2+3/4}(\Sigma_T),\ u_{4,0}\Big|_{\Sigma}=u_{4,a}|_{t=0},$

(vi) $b_4\in H_h^{l+1/2,l/2+1/4}(\Gamma_T),\ b_4|_{t=0}=a_2\ \nabla u_{4,0}\cdot n\Big|_{\Gamma}.$

Then there exists a unique solution $u_4$ of (13) satisfying the estimate

$$\|u_4\|_{H_h^{l+2,l/2+1}(Q_T)} \leq C(T)(\|\phi_4\|_{H_h^{l,l/2}(Q_T)}+\|u_{4,0}\|_{W_2^{1+l}(\Omega)}+$$

$$+\|u_{4,a}\|_{H_h^{l+3/2,l/2+3/4}(\Sigma_T)}+\|b_4\|_{H_h^{l+1/2,l/2+1/4}(\Gamma_T)}).$$

4°. Next we construct the sequence $\{(\rho^*_m,\ u_m,\ \theta^*_m)(\xi,t)\}$ of successive approximate solutions as follows:

$(\rho^*_0,\ u_0,\ \theta^*_0)(\xi,t)=(\rho_0,\ v_0,\ \theta_0)(\xi),$

$\rho^*_m$ is defined by (7) with $u=u_{m-1}\in W_2^{2+l,1+l/2}(Q_T);$

$u_m$ is defined as a solution of (8) with

$$a(\xi)=\mu/\rho_0(\xi),\ a_1=(\mu+\mu')/\rho_0(\xi),\ \sigma'=\sigma/\rho_0(\xi),$$

$$\phi=f^*+\frac{1}{\rho^*_{m-1}}\ \nabla_{u_{m-1}}\cdot P_{u_{m-1}}-(\frac{\mu}{\rho^*_{m-1}}-a)\,\Delta u_{m-1}-$$

$$-(\frac{\mu+\mu'}{\rho^*_{m-1}}-a_1)\ \nabla(\nabla\cdot u_{m-1}),$$

$$u_0=v_0,\quad B_0=B_0(\xi;\nabla),\quad B_1=B_1(\xi;\nabla),$$

$$b=\frac{1}{\rho_0}\{-P_{u_{m-1}}\,n(X_{u_{m-1}},t)-p_e{}^*\,n(X_{u_{m-1}},t)+\sigma\,\Delta(t)X_{u_{m-1}}\}+$$

$$+B_0(\xi;\nabla)u_{m-1}-\sigma'\,B_1(\xi;\nabla)\int_0^t u_{m-1}\,d\tau;$$

$\theta^*_m$ is defined as a solution of (13) with

$$a_2=\kappa/(\rho_0\,\theta_0\,S_\theta{}^*(\rho_0,\theta_0))$$

$$\phi_4=\frac{1}{\rho^*_{m-1}\,\theta^*_{m-1}\,S_\theta{}^*(\rho^*_{m-1},\theta^*_{m-1})}\{\kappa\,\nabla^2_{u_{m-1}}\,\theta^*_{m-1}+$$

$$+\mu'(\nabla_{u_{m-1}}\cdot u_{m-1})^2+2\mu\,D_{u_{m-1}}(u_{m-1}):D_{u_{m-1}}(u_{m-1})+$$

$$+\rho^*_{m-1}\,\theta^*_{m-1}\,S_\rho{}^*(\rho^*_{m-1},\theta^*_{m-1})\,\nabla_{u_{m-1}}\cdot u_{m-1}\}-a_2\,\Delta\,\theta^*_{m-1},$$

$$u_{4,0}=\theta_0,\quad u_{4,a}=\theta_a{}^*,\quad b_4=\frac{1}{\rho_0\,\theta_0\,S_\theta{}^*(\rho_0,\theta_0)}\{\kappa_e(\theta_e{}^*-\theta^*_{m-1})+$$

$$+\kappa\,\nabla_{u_{m-1}}\theta^*_{m-1}\cdot n(X_{u_{m-1}},t)\}-a_2\,\nabla\theta^*_{m-1}\cdot n(\xi,t).$$

Propositions 2 and 3 and the interpolation inequality imply that

$$\|(u_m,\theta^*_m)\|_{H_h^{l+2,l/2+1}(Q_T)}\le C_1(T)+$$

$$+C_2(T,\|(u_{m-1},\theta^*_{m-1})\|_{H_h^{l+2,l/2+1}(Q_T)}),$$

where both $C_1(T)$ and $C_2(T,\cdot)$, increase monotonically in each argument and $C_2(T,.)\longrightarrow 0$ as $T\longrightarrow 0$. Hence we choose a constant $M$ greater than $C_1(T)$, then $T'\in(0,T]$ such that $C_2(T',M) < M-C_1(T)$. Concequently, $u_m$, $\theta^*_m(m=0,1,2,\cdots)$ are well-defined and satysfy the estimates

$$\|(u_m,\ \theta^*_m)\|_{H_h^{2+l,\,1+l/2}(Q_{T'})} < M \qquad \text{for } m = 0,1,2,\cdots .$$

Again applying Propositions 2 and 3 to the equations concerning $u_m-u_{m-1}$ and $\theta^*_m-\theta^*_{m-1}$, we can prove that the sequence $\{(u_m,\ \theta^*_m)\}$ converges to $(u,\ \theta^*)$ as $m\to\infty$ uniformly in $H_h^{2+l,\,1+l/2}(Q_{T''})$ for some $T''\in(0,\ T']$. Formula (7) gives that $\rho^*_m$ converges to

$$\rho^*(\xi,\ t)=\rho^0(\xi)\exp[-\int_0^t \nabla_u\cdot u\, d\tau]$$

as $m\to\infty$ uniformly in $W_2^{1+l,\,1/2+l/2}(Q_{T''})$. Moreover, $\frac{\partial}{\partial t}\rho^*_m \longrightarrow \frac{\partial}{\partial t}\rho$ as $m\to\infty$ uniformly in $W_2^{l,\,l/2}(Q_{T''})$. The uniqueness of $(\rho^*, u, \theta^*)$ also follows from Propositions 2 and 3 and (7). Therefore we get

Theorem 2. Under the same assumptions of Theorem 1, there exists a unique solution $(\rho^*,u,\ \theta^*)$ of (5)-(6) such that

$u,\theta^* \in W_2^{2+l,\,1+l/2}(Q_{T'})$, $\qquad \rho^*\in W_2^{1+l,\,1/2+l/2}(Q_{T'})$,

$\rho^*_t\in W_2^{l,\,l/2}(Q_{T'})$ for some $T'\in(0,T]$.

Theorem 1 is easily deduced from Theorem 2. Indeed the function $(\rho, v, \theta)(x, t)$ defined by $\prod^{\xi}{}_{x}(\rho^*, u, \theta^*)(\xi, t)$ is the desired solution of (1)-(2) mentioned in Theorem 1. Here $\prod^{\xi}{}_{x}$ is the inverse mapping of $\prod^{x}{}_{\xi}$, which exists for $T_0 \in (0, T]$ satisfying $0 < MT_0 < 1$.

## §3. Multi-phase problem

In this section we consider the multi-phase free boundary problem for general fluids. This problem was discussed by the present author in [15-17] when $\sigma = 0$ and shall be done in detail in [18] when $\sigma > 0$.

For simplicity, we shall investigate only two-phase problem which is formulated as follows. Let $\Omega_1$ and $\Omega_2$ be two bounded domains in $R^3$; $\partial\Omega_1 = \Sigma_1 \cup \Gamma$, $\partial\Omega_2 = \Sigma_2 \cup \Gamma$, $\Sigma_1 \cap \Gamma = \phi$, $\Sigma_2 \cap \Gamma = \phi$, $\Sigma_1 \cap \Sigma_2 = \phi$. And let $\Omega_1(t)$ [resp. $\Omega_2(t)$] be the domain of the general fluid at the moment $t$ which initially occupies $\Omega_1$ [resp. $\Omega_2$].

Then our two-phase free boundary problem consists of finding the domains $\Omega_1(t)$, $\Omega_2(t)$ and the functions $(\rho^{(1)}, v^{(1)}, \theta^{(1)})$ defined on $\Omega_1(t)$ and $(\rho^{(2)}, v^{(2)}, \theta^{(2)})$ defined on $\Omega_2(t)$ satisfying the system of equations

$$(14)\begin{cases} [\dfrac{D}{Dt}]^{(1)} \rho^{(1)} = -\rho^{(1)} \nabla\cdot v^{(1)}, \\ \rho^{(1)} [\dfrac{D}{Dt}]^{(1)} v^{(1)} = \nabla\cdot P^{(1)} + \rho^{(1)} f^{(1)}, \quad x \in \Omega_1(t),\ t > 0, \end{cases}$$

$$\left\{\begin{aligned} &\rho^{(1)}\theta^{(1)}\left[\frac{D}{Dt}\right]^{(1)}S^{(1)} = \nabla\cdot(\kappa^{(1)}\nabla\theta^{(1)}) + \mu^{(1)\prime}(\nabla\cdot v^{(1)})^2 + \\ &\qquad +2\mu^{(1)}\boldsymbol{D}^{(1)}(v^{(1)}):\boldsymbol{D}^{(1)}(v^{(1)}), \end{aligned}\right.$$

$$(15)\left\{\begin{aligned} &\left[\frac{D}{Dt}\right]^{(2)}\rho^{(2)} = -\rho^{(2)}\nabla\cdot v^{(2)}, \\ &\rho^{(2)}\left[\frac{D}{Dt}\right]^{(2)}v^{(2)} = \nabla\cdot\boldsymbol{P}^{(2)}+\rho^{(2)}f^{(2)}, \quad x\in\Omega_2(t),\quad t>0 \\ &\rho^{(2)}\theta^{(2)}\left[\frac{D}{Dt}\right]^{(2)}S^{(2)} = \nabla\cdot(\kappa^{(2)}\nabla\theta^{(2)})+\mu^{(2)\prime}(\nabla\cdot v^{(2)})^2 + \\ &\qquad +2\mu\boldsymbol{D}^{(2)}(v^{(2)}):\boldsymbol{D}^{(2)}(v^{(2)}), \end{aligned}\right.$$

**the initial conditions**

$$(16)\left\{\begin{aligned} &(\rho^{(1)}, v^{(1)}, \theta^{(1)})|_{t=0}=(\rho_0^{(1)}, v_0^{(1)}, \theta_0^{(1)})(x), \quad x\in\Omega_1, \\ &(\rho^{(2)}, v^{(2)}, \theta^{(2)})|_{t=0}=(\rho_0^{(2)}, v_0^{(2)}, \theta_0^{(2)})(x), \quad x\in\Omega_2, \end{aligned}\right.$$

**the boundary conditions**

$$(17)\left\{\begin{aligned} &v^{(1)}=v^{(2)}, \quad \boldsymbol{P}^{(1)}n-\boldsymbol{P}^{(2)}n=-p_e n+\sigma Hn, \\ &\theta^{(1)}=\theta^{(2)}, \quad \kappa^{(1)}\nabla\theta^{(1)}\cdot n= \kappa^{(2)}\nabla\theta^{(2)}\cdot n, \end{aligned}\right. \qquad x\in\Gamma(t),\ t>0,$$

$$(18)\qquad\left\{\begin{aligned} &v^{(1)}=0, \quad \theta^{(1)}=\theta_a^{(1)} \quad \text{on } \Sigma_1, \\ &v^{(2)}=0, \quad \theta^{(2)}=\theta_a^{(2)} \quad \text{on } \Sigma_2, \end{aligned}\right.$$

**and the equation (kinematic boundary condition)**

$$(19)\qquad \frac{D}{Dt}F(x,t)=0 \quad \text{on} \quad \Gamma(t)\ (t>0).$$

Here $\left[\frac{D}{Dt}\right]^{(1)}=\frac{\partial}{\partial t}+v^{(1)}\cdot\nabla$, $\left[\frac{D}{Dt}\right]^{(2)}=\frac{\partial}{\partial t}+v^{(2)}\cdot\nabla$,

$\boldsymbol{P}^{(1)}=(-p^{(1)}(\rho^{(1)}, \theta^{(1)})+\mu^{(1)\prime}\nabla\cdot v^{(1)})\mathrm{I}+2\mu^{(1)}\boldsymbol{D}^{(1)}(v^{(1)})$,

$\boldsymbol{P}^{(2)}=(-p^{(2)}(\rho^{(2)},\theta^{(2)})+\mu^{(2)\prime}\nabla\cdot v^{(2)})\mathrm{I}+2\mu^{(2)}\boldsymbol{D}^{(2)}(v^{(2)})$,

$F(x,t)$ is such as $\Gamma(t)=\{x\in R^3\,|\,F(x,t)=0\}$ and $n=n(x,t)$ is a unit normal vector at $x\in\Gamma(t)$ pointing into the interior of $\Omega_1(t)$.

The main theorem of two-phase free boundary problem is the following.

Theorem 3 ([18]). Suppose that

(i) $\Omega_1,\Omega_2\subset\boldsymbol{R}^3$ are bounded domains such that $\partial\Omega_1=\Sigma_1\cup\Gamma,\partial\Omega_2=\Sigma_2\cup\Gamma$, $\Gamma,\Sigma_1,\Sigma_2\in W_2^{5/2+l}$, $l\in(1/2,\ 1)$, $\Sigma_1,\Sigma_2,\Gamma$ are mutually disjoint,

(ii) $(\rho_0^{(1)},v_0^{(1)},\theta_0^{(1)})\in W_2^{1+l}(\Omega_1)\times W_2^{1+l}(\Omega_1)\times W_2^{1+l}(\Omega_1)$,

$(\rho_0^{(2)},v_0^{(2)},\theta_0^{(2)})\in W_2^{1+l}(\Omega_2)\times W_2^{1+l}(\Omega_2)\times W_2^{1+l}(\Omega_2)$,

$$\rho_0^{(1)},\theta_0^{(1)},\rho_0^{(2)},\theta_0^{(2)}>0,$$

(iii) $\mu^{(1)},\mu^{(1)\prime},\kappa^{(1)},\mu^{(2)},\mu^{(2)\prime},\kappa^{(2)},\sigma$ are constants satisfying the relations $2\mu^{(1)}+3\mu^{(1)\prime}\geqq0$, $\sqrt{3}\mu^{(1)}-\mu^{(1)\prime}\geqq0$, $2\mu^{(2)}+3\mu^{(2)\prime}\geqq0$, $\sqrt{3}\mu^{(2)}-\mu^{(2)\prime}\geqq0$, $\mu^{(1)},\mu^{(2)},\kappa^{(1)},\kappa^{(2)},\sigma>0$,

(iv) $\theta_a^{(1)}\in W_2^{l+3/2,l/2+3/4}(\Sigma_{1,T})$, $\theta_a^{(2)}\in W_2^{l+3/2,l/2+3/4}(\Sigma_{2,T})$,

(v) Both $\nabla\nabla p_e$ and $\nabla p_{e,t}$ are defined in $R^3\times(0,T)$ and are Lipschitz continuous in $x$,

(vi) $(f^{(1)},f^{(2)})$ and $\nabla(f^{(1)},f^{(2)})$ are defined in $R^3\times(0,T)$ and are Lipschitz continuous in $x$ and $\frac{1}{2}$ Hölder continuous in $t$,

(vii) Both $(S^{(1)}, p^{(1)})=(S^{(1)}, p^{(1)})(\rho^{(1)}, \theta^{(1)})$ and $(S^{(2)}, p^{(2)})=$ $=(S^{(2)}, p^{(2)})(\rho^{(2)}, \theta^{(2)})$ are defined in $(0,\infty)\times(0,\infty)$, and are two times partially differentiable, and their second order derivatives are locally Lipschitz continuous there; moreover $S^{(1)}_{\theta^{(1)}}, S^{(2)}_{\theta^{(2)}}>0$.

Then there exists a unique solution $(\rho^{(1)}, v^{(1)}, \theta^{(1)}, \rho^{(2)}, v^{(2)}, \theta^{(2)})$ of (14)-(19), which has the properties

$$D^k v^{(j)}, D^k \theta^{(j)} \in L_2(D_{j,T'}) \text{ for } k=0,1,2,\ v^{(j)}{}_t, \theta^{(j)}{}_t \in L_2(D_{j,T'}),$$

$$D^k \rho^{(j)} \in L_2(D_{j,T'}) \text{ for } k=0,1, \quad \rho^{(j)}{}_t \in L_2(D_{j,T'}) \ (j=1,2),$$

$$\Gamma(t) \in W_2^{5/2+l} \quad \text{for some} \quad T' \in (0,T] \ (D_{j,T}=\Omega_j \times (0,T)).$$

Similarly to the one-phase problem we also utilize the characteristic transformation $\Pi^x{}_\xi$ in the present problem.

The transfomed problem is as follows:

(5) for $(\rho^{(1)*}, u, \theta^{(1)*})$ in $Q_{1,T}$,

(5) for $(\rho^{(2)*}, w, \theta^{(2)*})$ in $Q_{2,T}$,

$$\begin{cases} (\rho^{(1)*}, u, \theta^{(1)*})|_{t=0}=(\rho_0{}^{(1)}, v_0{}^{(1)}, \theta_0{}^{(1)}) & \text{on } \Omega_1, \\ (\rho^{(2)*}, w, \theta^{(2)*})|_{t=0}=(\rho_0{}^{(2)}, v_0{}^{(2)}, \theta_0{}^{(2)}) & \text{on } \Omega_2, \end{cases}$$

$$\begin{cases} u=w, \quad P_u{}^{(1)} n(X_u, t)-P_w{}^{(2)} n(X_w, t)=-p_e{}^* n(X_u, t) \\ \qquad +\frac{1}{2}\sigma \Delta_u(t) X_u(\xi, t) +\frac{1}{2}\sigma \Delta_w(t) X_w(\xi, t), \quad \text{on } \Gamma_T, \\ \theta^{(1)*}=\theta^{(2)*}, \quad \kappa^{(1)} \nabla_u \theta^{(1)*}\cdot n(X_u, t)=\kappa^{(2)} \nabla_w \theta^{(2)*}\cdot n(X_w, t), \end{cases}$$

$$\begin{cases} u=0, \quad \theta^{(1)*}=\theta^{(1)}{}_a{}^* \quad \text{on } \Sigma_{1,T}, \end{cases}$$

$$\left. w=0, \quad \theta^{(2)*}=\theta^{(2)}{}_{a}{}^{*}, \quad \text{on} \quad \Sigma_{2,T}. \right.$$

As we have already pointed out in §2,2°, it is essential to solve the system of ordinary diffenential equations (cf.[11]) reduced by the Fourier-Laplace transformation from the linear initial-boundadry value problem for $u$ and $w$ with constant coefficients in the half spaces $D_{++}$ and $D_{+-}$:

$$\begin{cases} \dfrac{\partial u}{\partial t}=a^{(1)}\Delta u+a^{(1)}{}_{1}\nabla(\nabla\cdot u) \quad \text{in} \quad D_{++}\equiv\mathbf{R}^{3}{}_{+}\times(0,\infty), \\ \dfrac{\partial w}{\partial t}=a^{(2)}\Delta w+a^{(2)}{}_{1}\nabla(\nabla\cdot w) \quad \text{in} \quad D_{+-}\equiv\mathbf{R}^{3}{}_{-}\times(0,\infty), \\ u|_{t=0}=0 \quad \text{on} \quad \mathbf{R}^{3}{}_{+}\equiv\{\xi\in\mathbf{R}^{3}\,|\,\xi_{3}>0\}, \\ w|_{t=0}=0 \quad \text{on} \quad \mathbf{R}^{3}{}_{-}\equiv\{\xi\in\mathbf{R}^{3}\,|\,\xi_{3}<0\}, \\ u-w\Big|_{\xi_{3}=0}=b\equiv{}^{t}(b_{1},b_{2},b_{3}), \\ a^{(1)}\left(\dfrac{\partial u_{3}}{\partial\xi_{\gamma}}+\dfrac{\partial u_{\gamma}}{\partial\xi_{3}}\right)-a^{(1)}\left(\dfrac{\partial w_{3}}{\partial\xi_{\gamma}}+\dfrac{\partial w_{\gamma}}{\partial\xi_{3}}\right)\Big|_{\xi_{3}=0}=b_{3+\gamma} \ (\gamma=1,2), \\ (a^{(1)}{}_{1}-a^{(1)})\nabla\cdot u+2a^{(1)}\dfrac{\partial u_{3}}{\partial\xi_{3}}-(a^{(2)}{}_{1}-a^{(2)})\nabla\cdot u+ \\ \qquad +2a^{(2)}\dfrac{\partial w_{3}}{\partial\xi_{3}}+\sigma\displaystyle\int_{0}^{t}(\nabla'^{2}u_{3}+\nabla'^{2}w_{3})\,d\tau\Big|_{\xi_{3}=0}=b_{6}, \end{cases}$$

especially, to estimate from below the absolute value of the determinant $\Delta$ of the coefficient matrix of the above-mentioned system of ordinary differential equations (cf. Lemma 1).

After lengthy calculations, $\Delta$ is given by the formula

$$\Delta=-s^{2}\times$$

$$\Big\{\rho_{0}^{(1)}\rho_{0}^{(2)}(r^{(1)}r_{1}^{(2)}+r^{(2)}r_{1}^{(1)}-2\xi'^{2})+$$

$$
\times \left\{
\begin{aligned}
&+\rho_0^{(1)2}(r^{(2)}r_1^{(2)}-\xi'^2)+\rho_0^{(2)2}(r^{(1)}r_1^{(1)}-\xi'^2)\ + \\
&+4\xi'^2[\rho_0^{(1)}-(a^{(1)}\rho_0^{(1)}-a^{(2)}\rho_0^{(2)})\frac{r^{(1)}r_1^{(1)}-\xi'^2}{s}]\ \times \\
&\times\ [\rho_0^{(2)}+(a^{(1)}\rho_0^{(1)}-a^{(2)}\rho_0^{(2)})\frac{r^{(2)}r_1^{(2)}-\xi'^2}{s}]\ + \\
&+\frac{\sigma}{s}\xi'^2[\ \rho_0^{(2)}r^{(2)}\frac{r^{(1)}r_1^{(1)}-\xi'^2}{s}+\rho_0^{(1)}r^{(1)}\frac{r^{(2)}r_1^{(2)}-\xi'^2}{s}]
\end{aligned}
\right.
$$

and is estimated from below as follows

$$
(20)\ |\Delta|\geq|s|^2\left\{\frac{1}{2}(\rho_0^{(2)}|r_1^{(1)}|+\rho_0^{(1)}|r_1^{(2)}|)^2+4\frac{a^{(2)}}{a^{(1)}}\rho_0^{(2)2}\xi'^2\ +\right.
$$

$$
\left.+\frac{\sigma}{2\sqrt{2}\ |s|^2}\ h\xi'^2[\frac{\rho_0^{(2)}}{a^{(2)}+a_1^{(2)}}|r_1^{(1)}|^2+\frac{\rho_0^{(1)}}{a^{(1)}+a_1^{(1)}}|r_1^{(2)}|^2]\right\}.
$$

Here
$$
r^{(j)2}=\frac{s}{a^{(j)}}+\xi'^2, \qquad \arg r^{(j)}\in(-\frac{\pi}{4},\ \frac{\pi}{4})
$$

$$
r_1^{(j)2}=\frac{s}{a^{(j)}+a_1^{(j)}}+\xi'^2, \qquad h=\mathrm{Re}\, s>0, \qquad \xi'\in\boldsymbol{R}^2.
$$

Once this is checked, we do as previous section.

Theorem 4. Under the same assumptions of Theorem 3, there exists a unique solution $(\rho^{(1)*}, u, \theta^{(1)*}, \rho^{(2)*}, w, \theta^{(2)*})$ of the transformed equations $\Pi^{*}_{\xi}((14)$-$(19))$, which has the properties

$$
u,\ \theta^{(1)*}\in W_2^{2+l,\,1+l/2}(Q_{1,T'}),\quad w,\ \theta^{(2)*}\in W_2^{2+l,\,1+l/2}(Q_{2,T'}),
$$

$$\rho^{(1)*} \in W_2^{1+l,\,1/2+l/2}(Q_{1,T'}), \quad \rho^{(1)*}{}_t \in W_2^{l,\,l/2}(Q_{1,T'}),$$

$$\rho^{(2)*} \in W_2^{1+l,\,1/2+l/2}(Q_{2,T'}), \quad \rho^{(2)*}{}_t \in W_2^{l,\,l/2}(Q_{2,T'})$$

for some $T' \in (0, T]$ $(Q_{j,T} \equiv \Omega_j \times (0,T),\ j=1,2)$.

Remark. We have not succeeded to get the estimate from below (20) of $|\Delta|$ without the additional conditions

$$\sqrt{3}\mu^{(1)} - \mu^{(1)\prime} \geqq 0, \quad \sqrt{3}\mu^{(2)} - \mu^{(2)\prime} \geqq 0.$$

But in our case the Stokes relations $2\mu^{(1)}+3\mu^{(1)\prime}=0$, $2\mu^{(2)}+3\mu^{(2)\prime}=0$, $\mu^{(1)}>0$, $\mu^{(2)}>0$ are contained.

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