SOME FORCED NONLINEAR EQUATIONS
AND THE TIME EVOLUTION OF SPECTRAL DATA

ROBERT CARROLL*

1. Introduction. We consider here mainly the forced NLS (nonlinear Schrödinger) and KdV (Korteweg-de Vries) equations with data $q(x,0) = q_0(x)$ and $q(0,t) = Q(t)(q(x,t) \rightarrow 0$ suitably as $x \rightarrow \infty)$. Thus one has (∗) $iq_t = q_{xx} \pm 2|q|^2 q$ (NLS) or $q_t + q_{xxx} - 6qq_x = 0$ (KdV).

For the full line problem $-\infty < x < \infty$ there are many results. Firstly one has general abstract existence uniqueness type theorem as in [21, 41, 24, 22, 42] for example (for NLS) or [26,25] (for KdV) and solutions by inverse scattering as in [38, 23, 16, 17] for KdV (cf. also [3, 44, 43]). For forced problems as above there are not too many results known. For KdV one has theorems of an abstract type in [4, 39, 40] (cf. also [36, 37]) where in [4] for example one requires $q_0 \in H^4$ and $Q \in H^2_{loc}$ with compatibility at $(0,0)$ (plus more smoothness for a classical solution). There are as yet apparently no global abstract theorems for forced NLS (see however [13] for some preliminary results). For forced NLS by inverse scattering one has results in [6, 7, 8, 9] and for forced KdV by inverse scattering one now has [14] (recently M. Ablowitz informed me that he also has a result for forced KdV using half line spectral data). Let us remark that D. Kaup and collaborators initiated much work on forced problems, of the type indicated, in [27, 28, 29, 30] and there is essentially a complete theory via inverse scattering for forced Toda lattice and Sine-Gordon. In the present article we will indicate results on forced NLS and KdV based on [9, 19, 14] in hopes of stimulating interest in such techniques and problems. There are many obvious open questions and opportunities for further investigation. In particular one wants to find the formulation involving spectral data with the "simplest" or most natural time evolution.

2. Forced KdV by inverse scattering. We sketch here a new result based on [14]. Thus consider $Q = D^2 - q(x)(q$ real) on $[0,\infty)$ where $q(x) \rightarrow 0$ as $x \rightarrow \infty$ as rapidly as needed (we will generally not spell out precise hypotheses here – cf. [11, 9, 33, 32, 15, 18, 10] for details). We will work in the context of half line Sturm-Liouville theory and construct generalized eigenfunctions ϕ, θ, and f_+ of Q satisfying $Q\psi = -k^2\psi$ with $\phi(0, k) = \theta'(0, k) = 1, \phi'(0, k) = \theta(0, k) = 0$, and $f_+ \sim \exp(ikx)$ as $x \rightarrow \infty$. Set $f_-(x, k) = f_+(x, -k)$ and then one has $\phi(x, k) = c(k)f_+(x;k) + c(-k)f_-(x, k)$ and $2ik\theta(x, k) = F(-k)f_+(x, k) - F(k)f_-(x, k)$ (we write also $F^- \sim F(-k), c^- \sim c(-k)$, and refer to [11, 9, 33, 32, 15, 18, 10] for properties of ϕ, θ, f_+, c, F, etc). In particular $c^- = \bar{c}$ and $F^- = \bar{F}$ for k real

and in the absence of discrete spectra there are spectral measures $d\omega = dk/2\pi|c|^2$ and $dv = 2k^2 dk/\pi|F|^2$ such that

$$(2.1) \qquad \delta(x-y) = \int_0^\infty \theta(x,k)\theta(y,k)dv(k) =$$

$$\int_0^\infty \phi(x,k)\phi(y,k)d\omega(k)$$

We mention also the properties

$$(2.2) \qquad D_x f_+(0,k) = 2ikc(-k); \quad Fc + \overline{Fc}^- = 1;$$
$$f_+(0,k) = F$$

Associated with the ball line one has a Marčenko (M) equation (cf [11, 9, 18])

$$(2.3) \qquad 0 = K(x,y) + \Omega(x,y) + \int_x^\infty K(x,\xi)\Omega(\xi,y)d\xi$$

for $y > x$ where $\Omega(x,y) = -\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^\infty \mathcal{S}e^{ik(x+y)}dk$ (we assume there are no bound states), $\mathcal{S} = F^-/F$, and $q(x) = -2D_x K(x,x)$ can be written formally as (cf. [11, 10])

$$(2.4) \qquad q(x) = 2D_x \int_{-\infty}^\infty (1-\mathcal{S})f_+(x,k)e^{ikx}dk$$

Now the KdV equation $q_t + q_{xxx} - 6qq_x = 0$ arises in the context of Lax pairs etc as $L_t = [B,L]$ with $L = D^2 - q$ and $B = -4D^3 + 6qD + 3q_x$. Thus we write $L\psi = -k^2\psi$ and $\psi_t = B\psi$ etc. ($L_t \sim -q_t$). For large $x, \psi_t = B\psi$ becomes $\psi_t = -4\psi_{xxx}$ since $q \to 0$ and one takes then $\psi = f_+(x,k) \cdot \exp(4ik^3 t)$ as a time dependent Jost solution. Using $\psi_{xx} = -k^2\psi + q\psi$ to get $(*)\psi_{xxx} = -k^2\psi_x + q_x\psi + q\psi_x$ we have $\psi_t = (4k^2 + 2q)\psi_x - q_x\psi$. The data for the forced problem are $q(x,0) = q_0(x)$ and $q(0,t) = Q(t)$ but we will need another quantity $q_x(0,t) = P(t)$ which overdetermines the problem (cf [11, 8, 6, 7, 27, 28, 29, 30]). Thus from $(*)$ for $\psi = f_+\exp(4ik^3 t)$ one evaluates ψ_t and lets $x \to 0$ (using (2.2)) to obtain

$$(2.5) \qquad F' + 4ik^3 F = (4k^2 + Q)2ikc^- - PF$$

Multiply (2.5) by $\overline{F} = F^-$ for k real and take real and imaginary parts to get

(2.6)
$$F^- F' - F\overline{F}' + 8ik^3|F|^2 = 8ik^3 + 4ikQ;$$

$$D_t|F|^2 = -2P|F|^2 + (4k^2 + 2Q)(2ik)(F^-c^- - Fc)$$

There are various spectral quantities of importance in various roles in the half line theory (cf [11, 9, 10]; we mention $F, c, |F|^2, |c|^2, \mathcal{S}, \mathcal{S} = c/c^-$, and $\mathcal{R} = F/2c^-$. A little calculation, using $Fc + F\overline{c}^- = 1$ gives $\mathcal{R}e(1/\mathcal{R}) = 1/|F|^2, F^-c^- - Fc = 2F^-c^- - 1$, and $1/\mathcal{R} - \dfrac{1}{|F|^2} = (2F^-c^- - 1)/|F|^2 = i\,Im(1/\mathcal{R})$. We assume that $1/|F|^2$ is such that $Re(1/\mathcal{R}) = 1/|F|^2 = \Lambda = H[Im(1/\mathcal{R})]$ where H is the Hilbert transform (cf. [20]. Then from (2.6) one obtains

(2.7)
$$D_t \log \mathcal{S} = 8ik^3 - 4ik(2k^2 + Q)\Lambda;$$

$$D_t \log \Lambda = 2P - 2k(4k^2 + 2Q)H\Lambda$$

THEOREM 2.1. *Given P and Q the time evolution of $\mathcal{S}$ and Λ are determined by (2.7) (with $\mathcal{S}(k,0)$ and $\Lambda(k,0)$ determined by $q_0(x)$). From this all spectral quantities $F, c, \mathcal{R}$, etc. can be obtained.*

We now go to (2.4) as a recovery formula for $\widehat{Q}$ and $\widehat{P}$, the recovery data for a given spectral quantity $\mathcal{S}$ (along with F, c, etc). When we equate $Q = \widehat{Q}$ and $P = \widehat{P}$ (thus eliminating P) we will obtain an integro-differential equation for spectral data from which overdetermining factor P has been removed. Thus from (2.4) directly

(2.8)
$$\widehat{Q} = 2\lim_{x\to 0}\int_{-\infty}^{\infty}(1-\mathcal{S})2ikc^-(1+\mathcal{R})e^{ikx}dk$$

since $D_x(f_+e^{ikx}) = f'_+e^{ikx} + ikf_+e^{ikx} \sim (2ikc^- + ikF)e^{ikx} = 2ikc^-(1+\mathcal{R})e^{ikx}$ as $x \to 0$. Similarly $D^2(f_+e^{ikx}) \sim [(Q-2k^2)F - 4k^2c-]\exp(ikx)$ as $x \to 0$ and

(2.9)
$$\widehat{P} = 2\lim_{x\to 0}\int_{-\infty}^{\infty}(1-\mathcal{S})\left[QF - 2k^2F\left(1+\frac{1}{\mathcal{R}}\right)\right]e^{ikx}dk$$

The factor $1 - \mathcal{S} \to 0$ as $|k| \to \infty$ with $c \to 1$ and $F \to 1$ $(\mathcal{R} \to 1/2)$ and one expects no problems with convergence in (2.8)-(2.9), if $1 - \mathcal{S} \to 0$ rapidly enough. One can write now, combining (2.7)-(2.9), and using a suitable determination of $\left(\overline{\mathcal{S}/\Lambda}\right)^{1/2} = F$,

(2.10)
$$D_t \log \mathcal{S} = 8ik^3 - 4ik(2k^2 + Q)\Lambda;$$

$$D_t \log \Lambda + 2k(4k^2 + 2Q)H\Lambda =$$

$$4\lim_{x\to 0}\int_{-\infty}^{\infty}(1-\mathcal{S})\left(\overline{\mathcal{S}/\Lambda}\right)^{1/2}[Q - 2k^2(1+\Lambda - iH\Lambda)]e^{ikx}dk$$

THEOREM 2.2. *The time evolution of spectral data (given $q(x,0) = q_0(x)$ and $q(0,t) = Q(t)$) is determined by solving (2.10). The resulting $q(x,t)$ given eg by (2.4) satisfies $q_t + q_{xxx} - 6qq_x = 0$. Alternatively one can use the spectral data obtained from (2.10) in forming $\Omega(x,y)$ in (2.3) (no bound states); then determine $q(x,t)$ by solving (2.3) and use $q = -2K'(x,x)$ to get $q(x,t)$.*

REMARK 2.3. Theorem 2.2 seems most natural for this problem since it relies on genuinely half line procedures. Alternatively however in [14] another theorem is developed using full line spectral quantities (one thinks of $q(x,t) = 0$ for $x < 0$). We indicate the (heuristic) results here and refer to [14] for details. Then let $f_1 \sim f_+ \sim e^{ikx}$ as $x \to \infty$ and $f_2 \sim e^{-ikx}$ as $x \to -\infty$ with (**) $Tf_2 = f_1^- + Rf_1$ and $Tf_1 = f_2^- + R_L f_2 (f_1^-(x,k) = f_1(x_1 - k)$, etc). One uses a Marčenko kernel (**)$K_+(x,y) = -(1/2\pi) \int_{-\infty}^{\infty} Re^{iky} f_1(x,k)dk$ $(y > x)$ and $q(x) = -2D_x(K_+(x,x)$ (t is suppressed). Set $S = R/T$ and proceed much as in the development above to obtain

THEOREM 2.4. *The time variation of $S = R/T$ and R_L (hence of all spectral data) can be determined from solving*

(2.11)
$$D_t R_L = -(8ik^3 + 4iQ)R_L;$$

$$D_t \log S + 2ikQ + 8ik^3 =$$

$$\frac{2i}{\pi} \lim_{x \to 0} \int_{-\infty}^{\infty} S[QR_L + (Q - 4k^2)e^{2ikx}]dk$$

Then construct $F(z) = \dfrac{1}{2\bar{u}} \displaystyle\int_{-\infty}^{\infty} Re^{ikz}dk$ *and solve the equation* $0 = K_+(x,y) + F(x+y) + \displaystyle\int_{x}^{\infty} K_+(x,\xi)F(\xi+y)d\xi$ $(y > x; t$ *suppressed) from which* $q(x,t) = -2D_x K_+(x,x,t)$ *(with t restored). Alternatively one can compute $q(x,t)$ from (**) diagonalized.*

3. Forced NLS by inverse scattering. We will sketch the author's development in [11, 8, 6, 7] and indicate the result of Fokas [19] for comparison purposes. Thus consider the AKNS system ($q = 0$ for $x < 0$, $q \to 0$ as $x \to \infty$)

(3.1)
$$\psi_{1x} + i\zeta\psi_1 = q\psi_2;$$

$$\psi_{2x} - i\zeta\psi_2 = r\psi_1 \qquad (r = \pm\bar{q})$$

We take $r = \bar{q}$ for convenience (so $iq_t = q_{xx} - 2|q|^2 q$ which "classically" corresponds to no solitons but $r = -\bar{q}$ can easily be accommodated by suitable addition of discrete spectral terms (cf [8, 7, 12]). The time evolution (for $x \geq 0$) is given via $\psi_{1t} = A\psi_1 + B\psi_2$

and $\psi_{2t} = Cv_1 - Av_2$ where $A = irq + 2i\zeta^2, B = -iq_x - 2\zeta q$, and $C = ir_x - 2\zeta r$. The compatibility condition $\psi_{xt} = \psi_{tx}$ leads to the NLS equation and one introduces generalized eigenfunctions $\phi \sim \binom{1}{0}e^{i\zeta x}, \hat\phi \sim \binom{0}{-1}e^{-i\zeta x}$ $(x \to -\infty)$ and $\psi \sim \binom{0}{1}e^{i\zeta x}, \hat\psi \sim \binom{1}{0}e^{-i\zeta x}(x \to \infty)$ Then $\psi = \hat b\phi - a\hat\phi$ and $\psi = -\hat\phi/\hat a + \hat b/\hat a\hat\psi$ for example and since $\phi(0,\zeta) = \binom{1}{0}, \hat\phi(0,\zeta) = \binom{0}{-1}$ one obtains $\psi(0,\zeta) = \binom{\hat b}{a}$ (t is suppressed again when not needed). Writing out the time variation of ψ at $x = 0$ now leads to

$$(3.2) \qquad \hat b_t = (i|Q|^2 + 4i\zeta^2)\hat b - (iP + 2\zeta Q)a;$$
$$a_t = (i\overline P - 2\zeta\overline Q)\hat b - i|Q|^2 a$$

where again $P = q_x(0,t)$ overdetermines the system. The idea here is based on Kaup [27, 28, 29, 30], and is equivalent, but our formulation is slightly different. Again we look for recovery formula via Marčenko kernels $K, \widehat K$ where $\psi = \binom{0}{1}e^{i\zeta x} + \int_x^\infty K(x,s)e^{i\zeta s}ds$ and

$\hat\psi = \binom{1}{0}e^{-i\zeta x} + \int_x^\infty \widehat K(x,s)e^{-i\zeta s}ds$ (cf. 1, 35, 11]). In the case $r = \overline q$ one has formally

$$q(x) = -2K_1(x,x) = \lim_{y\to x}\left(-\frac{1}{\pi}\right)\int_{-\infty}^\infty \psi_1(\zeta,x)e^{i\zeta y}d\zeta = -\frac{1}{\pi}\lim_{y\to x}\int_{-\infty}^\infty \left(\frac{\hat b}{\hat a}\right)\hat\psi_1(\zeta,x)e^{-i\zeta y}d\zeta$$

from which recovery formulas for $\widehat Q = q(0,t)$ and $\widehat P = q_x(0,t)$ can be determined. Putting this together as in §2 one obtains

THEOREM 3.1. *The time evolution of spectral data $(a,\hat b)$ for the forced NLS $iq_t = q_{xx} - 2|q|^2q$ with $q(x,0) = q_0(x)$ (determining $a,\hat b(\zeta,0)$) and $q(0,t) = Q(t)$ is governed by*

$$(3.3) \qquad \hat b_t = (i|Q|^2 + 4i\zeta^2)\hat b - 2\zeta Qa - iaJ(a,\hat b);$$
$$a_t = -i|Q|^2 a - 2\zeta\overline Q\hat b + i\hat bJ(a,\hat b);$$
$$J(a,\hat b) = \frac{1}{\pi}\lim_{x\to 0}\int_{-\infty}^\infty [2i\zeta\hat b - Qa]e^{-2i\zeta x}d\zeta$$

REMARK 3.2. More details can be found in [8, 12] along with some examples. Some smoothness of Q is to be anticipated as necessary in order to solve (3.3) (cf [8]).

Let us now sketch briefly the procedure of [19], which leads to the following singular nonlinear integro-differential equation for $\hat\beta$.

$$(3.4) \qquad \frac{\widehat\beta_t - 4ik^2\widehat\beta}{4k} = -Q(t)+$$
$$+\frac{1}{\pi}\int_{-\infty}^\infty \frac{(\widehat\beta(k')/8k') - (\widehat\beta(k)/8k)}{k' - k}D_tH\log(1 - |\widehat\beta|^2)dk'$$

where again H is the Hilbert transform. We emphasize here that $\alpha, \widehat{\beta}$ etc. are based on different generalized eigenfunctions $\phi, \widehat{\phi}, \psi, \widehat{\phi}$ and are not the same as $a, \widehat{b}$, above. The "neat" form of (3.4) suggest that $\alpha, \widehat{\beta}$, etc. will be a "best" version of spectral data. The procedure involves extending $q(x)$ to $(-\infty, 0)$ as an odd function (instead of $q(x) = 0$ for $x < 0$). This produces certain analogies to a Sine transform theory and has a half line flavor. Then write the x problem as ($-\infty < x < \infty$ and Y is the Heavyside function)

$$(3.5) \qquad \phi_x = ikJ\phi + \widetilde{Q}\phi; \widetilde{Q} = Q(x)Y(x) -$$

$$Q(-x)Y(-x); J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \widetilde{Q} = \begin{pmatrix} 0 & \tilde{q} \\ \tilde{r} & 0 \end{pmatrix}$$

where $Q(x) = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ is defined only for $x \geq 0$ and $\tilde{q}, \tilde{r} \sim$ odd extensions of q, r. For $\phi = \Phi \exp(ikxJ)$ (3.5) becomes $\Phi_x = ik[J, \Phi] + \widetilde{Q}\Phi$ and one constructs solutions $\Psi = I - \int\limits_x^\infty d\xi e^{ik(x-\xi)\widehat{J}}\widetilde{Q}\Phi$ and $\Phi = I + \int\limits_{-\infty}^x d\xi e^{ik(x-\xi)\widehat{J}}\widetilde{Q}\Phi$ where $e^{\widehat{y}}F = e^y F e^{-y}, \Psi = (\Psi^- \Psi^+)$, and $\Phi = (\Phi^+ \Phi^-)$. The scattering matrix $S = \begin{pmatrix} \alpha & \widehat{\beta} \\ \beta & \widehat{\alpha} \end{pmatrix}$ arises from $\Psi = \Phi\, e^{ixk\widehat{J}}S(k)$ and here $\widehat{\alpha}, \Psi^+, \Phi^+$ (resp. α, Ψ^-, Φ^-) are analytic in the upper (resp. lower k half plane (we assume no zeros of $\alpha, \widehat{\alpha}$). For real $k \to \infty$ one has $\alpha, \widehat{\alpha} \to 1, \widehat{\beta} \to q(0)/ik$, and $\beta \to -r(0)/ik$ while eg. $\Psi \sim \begin{pmatrix} 1 & q(x)/2ik \\ -r(x)/2ik & 1 \end{pmatrix}$. One can write $S(k) = I - \int\limits_{-\infty}^\infty d\xi e^{ik\xi \widehat{J}}\widetilde{Q}\Psi$ and the scattering equation implies $(*)\dfrac{\Psi^-}{\alpha} = \Phi^+ + \dfrac{\beta}{\alpha}e^{2ikx}\Phi^-$ and $\Psi^+/\widehat{\alpha} = \Phi^- + \widehat{\beta}/\widehat{\alpha}e^{-2ikx}\Phi^+$ One can use $(*)$ in the context of Riemann-Hilbert problems to prove eg that $q(x) = -(1/\pi)\int\limits_{-\infty}^\infty (\widehat{\beta}/\alpha)e^{-2ikx}\psi_1^-(k, x)dk$. We sketch this since (when $\widetilde{Q} = 0$ for $x < 0$) it provides another proof of the formula we used for recovery of Q in Theorem 3.1. Thus from $(*)$ one has

$$(3.6) \qquad \psi_1^+ = \frac{1}{2\pi i}\int\limits_{-\infty}^\infty \frac{(\widehat{\beta}/\alpha)e^{-2ik'x}\psi_1^-}{k' - (k + i0)}dk' =$$

$$\frac{1}{2\pi i}\int\limits_{-\infty}^\infty \frac{(\widehat{\beta}/\alpha)e^{-2ik'x}\psi_1^-}{k' - k}dk' + \frac{1}{2}(\widehat{\beta}/\alpha)e^{-2ikx}\psi_1^-$$

Now one looks at the coefficient of $1/k$ as $k \to \infty$, noting that $\int\limits_{-\infty}^\infty \dfrac{e^{-2ik'x}}{k'(k' - k)}dk' = \dfrac{i\pi}{k}(1 - e^{-2ikx})$ and $\int\limits_{-\infty}^\infty e^{-2ik'x}\dfrac{dk'}{k'} = -i\pi$ ($x > 0$). We obtain then (note $\dfrac{1}{k' - k} =$

$$-\frac{1}{k}(1 - k'/k)^{-1})$$

$$(3.7) \qquad \frac{q(x)}{2ik} = -\frac{1}{2\pi ki} \int\limits_{-\infty}^{\infty} \left(\frac{\widehat{\beta}}{\alpha}\psi_1^- - \frac{q(0)}{ik'} \right) e^{-2ik'x}dk'$$

$$+\frac{q(0)}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{e^{-2ik'x}dk'}{ik'(k'-k)} + \frac{q(0)}{2ik}e^{-2ikx}$$

which leads to the desired result.

Next, to remove discontinuities in Ψ_t at $x = 0$, in [19] one adjusts matters for $x < 0$ via $(\psi = \Psi e^{ikxJ})$

$$(3.8) \qquad \psi_t = \widetilde{U}\psi + 2ik^2\psi J -$$
$$-4kY(-x)\psi\psi^{-1}(0,t,k)Q(0,t)\psi(0,t,k);$$
$$\widetilde{U} = -2ik^2J - i\widetilde{q}\widetilde{r}J - 2k\widetilde{Q} - i\widetilde{Q}_xJ$$

but of course $\psi(0,t,k)$ is not known. This leads to $S_t = -2ik^2[J,S] - 4kS \; M(t,k)$ where $M = \Psi^{-1}(0,t,k)Q(0,t)\Phi(0,t,k)$ is unknown (and this again is a form of overdetermination). It turns out however that, using Riemann-Hilbert ideas, one can express the various factors of concern in terms of $a, \widehat{b}$, etc., and this leads to (3.4). For a philosophy of (3.4) etc we refer to [19] Using different extension of q, r one could in principle determine the time evolution of the corresponding $(a, \widehat{b}), (a, \widehat{\beta})$, etc. via equations such as (3.3), (3.4), etc. One hopes to find a natural choice of such data ~~involving Hamiltonian structure etc.~~ and this seems open.

REFERENCES

[1] M. ABLOWITZ AND H. SEGUR, Solitons and the inverse scattering transform, SIAM, Philadelphia, 1981.

[2] M. ABLOWITZ AND H. SEGUR, Jour. Math Physics, 16 (1975), 1054–1056.

[3] R. BEALS AND R. COIFMANN, Proc. Symp. Pure Math. 43 (1985), 45–70.

[4] J. BONA AND R. WINTHER, SIAM Jour. Math. Anal., 6 (1983), 1056–1106.

[5] F. COLOGERO AND A. DEGASPERIS, Spectral transform and solitons, I, North-Holland, Amsterdam, 1982.

[6] R. CARROLL, Proc. Conf. Howard Univ., 1987, to appear.

[7] R. CARROLL, Proc. Conf. Ohio Univ., 1988, to appear.

[8] R. CARROLL, Proc. workshop nonlinear evolution equations, Como, Italy, 1988, to appear.

[9] R. CARROLL, Transmutation theory and applications, North-Holland, Amsterdam, 1985.

[10] R. CARROLL, CR Royal Soc. Canada, 9 (1987), 237–242; Applicable Anal., 26 (1987), 61-85.

[11] R. CARROLL, Mathematical Physics, North-Holland, Amsterdam, 1988.

[12] R. CARROLL, *On the forced nonlinear Schrödinger equation*, to appear.

[13] R. CARROLL AND S. OHARU, *Some results are forced nonlinear problems*, to appear.

[14] R. CARROLL AND S. OHARU, *Some remarks on forced KdV*, to appear.

[15] K. CHADAN AND P. SABATIER, *Inverse problems in quantum scattering theory*, Springer, NY, 1977.

[16] A. COHEN, Comm. PDE, 9 (1984) 751–806.

[17] A. COHEN AND T. KAPPELER, SIAM Jour. Math. Anal., 18 (1987), 991–1025.

[18] L. FADEEV, Uspekhi Mat. Nauk, 14, (1959), 57–119; Sov. Prob. Mat., 31 (1974), 93–(80).

[19] A. FOKAS, *An initial-boundary value problem for the nonlinear Schrödinger equation*, to appear.

[20] F. GAKHOV, *Boundary value problems*, Permagon, 1966.

[21] N. HAYASHI, K. NAKAMITSU, AND M. TSUTSUMI, Math Zeit., 192 (1987), 637–650.

[22] N. HAYASHI, K. NAKAMITSU, AND Y. TSUTSUMI, Jour. Funct. Anal., 71 (1987), 218–245.

[23] T. KAPPELER, Jour. Diff. Eqs., 63 (1986), 306–331.

[24] T. KATO, Ann. Inst. Poincaré, 46 (1987), 113-129.

[25] T. KATO, *Nonlinear PDE*, I, Pitmann, 1979, 239–306.

[26] T. KATO, Advances Math. Supp., 8 (1983), 93–128.

[27] D. KAUP, Lect. Appl. Math, (1986), 195–215; Physica 25 D (1987), 361–368.

[28] D. KAUP, *Wave Phenomena*, Elsevier, (1984), 163–174; *Advances in nonlinear waves*, Pitman, (1984, 197–209.

[29] D. KAUP AND H. NEUBERGER, Jour. Math Physics, 25 (1984), 282–284.

[30] D. KAUP AND D. WYCOFF, *The forced Toda Lattice: An integrable system*, to appear.

[31] D. KAUP AND P. HANSEN, Physica, 8D (1987), 77–84; 25 D (1987), 369–381.

[32] B. LEVITAN, *Inverse Sturm-Lionville problems*, Moscow, 1984.

[33] V. MARČENKO, *Sturm Liouville operators and their applications*, Moscow, (1977).

[34] H. MOSES, Jour. Math Physics, 17 (1976), 73-75.

[35] A. NEWELL, *Solitons*, Springer, 1980, 172–242.

[36] S. OHARU AND T. TAKAHASHI, Lect. Notes Numer. Anal. Appl., 6 (1983), 125–142.

[37] S. OHARU AND T. TAKAHASHI, Proc. Amer. Math Soc., 97 (1986), 139–145.

[38] R. SACHS, Comm. PDE, 10 (1985), 29–98.

[39] R. TEMAM, Jour. Math Pures.Appl., 48 (1969), 159–172.

[40] B.A. TON, Jour. Diff. Eqs, 25 (1977), 288–309.

[41] Y. TSUTSUMI, Funk. Ekvac., 30 (1987), 115–125.

[42] Y. TSUTSUMI, Ann. Inst. of Poincaré, 43 (1985), 321–347.

[43] S. UKAI, *Local solutions of the KP equations*.

[44] M WICKERHAUSER, Comm. Math. Physics, 108 (1987), 67–89.