数理解析研究所講究録
698巻 1989年 74-93

Ultra-hyperbolic approach to some multi-dimensional inverse problems

By Takashi SUZUKI (Tokyo Metropolitan Univ.)
鈴木 貴

§1. Introduction. Our aim is to describe the basic idea of [16] to show the uniqueness of multi-dimensional inverse problems of some kind.

To fix the idea, let us recall the work [11] by A. Pierce in 1979, where uniqueness of some parabolic inverse problem was established via the theory of Gel'fand-Levitan [6]. Namely, for (p, h, H) and (q, j, J) in $C^1[0,1]\times\mathbb{R}\times\mathbb{R}$, let $u = u(x,t)$ and $v = v(x,t)$ solve

$$(1.1)\quad \begin{cases} u_t - u_{xx} = p(x)u \quad (0<x<1,\ 0<t<T), & u|_{t=0} = 0 \quad (0<x<1) \\ -u_x + hu|_{x=0} = 0 \quad (0<t<T), & u_x + Hu|_{x=1} = f(t) \quad (0<t<T) \end{cases}$$

and

$$(1.2)\quad \begin{cases} v_t - v_{xx} = q(x)v \quad (0<x<1,\ 0<t<T) & v|_{t=0} = 0 \quad (0<x<1) \\ -v_x + jv|_{x=0} = 0 \quad (0<t<T), & v_x + Jv|_{x=1} = f(t) \quad (0<t<T), \end{cases}$$

respectively. Then, the identity

$$(1.3) \qquad v - u\big|_{x=1} = 0 \qquad (0<t<T)$$

implies

$$(1.4) \qquad (q, j, J) = (p, h, H),$$

provided that $f \not\equiv 0$.

The proof is carried out in the following manner. First, in the case of $f \not\equiv 0$ the function $g(t) \equiv u\big|_{x=1}$ $(0<t<T)$ determines the spectral characteristics $\{\lambda_j, c_j\}_{j=0}^{\infty}$ of $A = A_{p,h,H}$, the differential operator $-\frac{d^2}{dx^2} + p(x)$ under the boundary condition $(-\frac{d}{dx} + h)\cdot\big|_{x=0} = (\frac{d}{dx} + H)\cdot\big|_{x=1} = 0$. Then, the conclusion (1.4) follows from the Gel'fand-Levitan theory.

Here, $\{\lambda_j\}_{j=0}^{\infty}$ $(-\infty < \lambda_0 < \lambda_1 < \cdots)$ denotes the set of eigenvalues of $A = A_{p,h,H}$, while $c_j > 0$ is the norming constant ; $c_j = \|\varphi_j\|^2_{L^2(0,1)}$, where $\varphi_j = \varphi_j(x)$ is the eigenfunction of A corresponding to λ_j and normalized as $\varphi_j\big|_{x=0} = 1$. The Gel'fand-Levitan theory implies that the spectral characteristics $\{\lambda_j, c_j\}$ determine the operator $A_{p,h,H}$.

Motivated by this, R. Murayama and the author have studied the equation

$$(1.5)\quad \begin{cases} u_t - u_{xx} = p(x)u \quad (0<x<1,\ 0<t<T), & u|_{t=0} = a(x) \quad (0<x<1) \\ -u_x + hu|_{x=0} = u_x + Hu|_{x=1} = 0 \quad (0<t<T) \end{cases}$$

to determine $(p, h, H, a) \in C^1[0,1] \times \mathbb{R} \times \mathbb{R} \times L^2(0,1)$ through the boundary value $f_j(t) \equiv u|_{x=j}$ $(0<t<T;\ j=0,1)$ of the solution ([17], [9], [13]). For this problem the uniqueness holds in a generic situation.

There, we modified the idea of Gel'fand-Levitan to introduce the following deformation formula

$$(1.6)\quad \varphi(x,\lambda;q,j) = \varphi(x,\lambda;p,h) + \int_0^x K(x,y;q,j;p,h)\varphi(y,\lambda;p,h)dy \quad (0 \leqq x \leqq 1)$$

for $\lambda \in \mathbb{R}$. Here, $\varphi = \varphi(x,\lambda;p,h)$ denotes the solution of

$$(1.7)\quad \left(-\frac{d^2}{dx^2} + p(x)\right)\varphi = \lambda\varphi \quad (0 \leqq x \leqq 1) \quad \text{with} \quad \varphi|_{x=0} = 1 \quad \text{and} \quad \frac{d}{dx}\varphi|_{x=0} = h.$$

The kernel $K = K(x,y) = K(x,y;q,j;p,h)$ is independent of λ and is characterized as the solution of the hyperbolic boundary value problem

$$(1.8)\quad \begin{cases} K_{xx} - K_{yy} + p(y)K = q(x)K \quad (\text{in } \Omega), \quad K_y(x,0) = hK(x,0) \quad (0 \leqq x \leqq 1), \\ K(x,x) = (j-h) + \frac{1}{2}\int_0^x (q(s) - p(s))ds \quad (0 \leqq x \leqq 1), \end{cases}$$

where $\Omega = \{(x,y) \mid 0<y<x<1\}$.

This method of integral transformation, sometimes is referred to as the transmutation theory (Carroll [5], e.g.), has been useful in the study of one-space dimensional inverse problems ([14]). However, it seemed to be quite difficult to extend the idea to multi-dimensional cases. In 1986, the author showed a uniqueness result for those cases with analytic coefficients, utilizing Holmgren's theorem ([15]). Now we have established it within the C^∞-category, noting some key identity. This is the object of the present article.

We refer to some work by the Nobosibirsk school for other approaches to multi-dimensional inverse problems, especially for hyperbolic equations (Bukgeim-Jakno [2], Romanov [12], Bukgeim-Klibanov [1]).

Henceforth, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain whose boundary $\partial\Omega$ is smooth. The differential operator $Pu \equiv \nabla \cdot (a\nabla u) + cu$ is symmetric, uniformly elliptic, and second order with smooth real coefficients $a = (a_{ij}(x))$ and $c = c(x)$ on $\overline{\Omega}$, and $d = d(x)$ is a given smooth function on $\partial\Omega$. To fix the idea, consider the parabolic problem

$$(1.9)\quad \begin{cases} \dfrac{\partial u}{\partial t} - Pu = 0 \quad (x \in \Omega,\ 0<t<T), \quad u\big|_{t=0} = 0 \quad (x \in \Omega), \\ \left(\dfrac{\partial}{\partial \nu_P} + d\right) u\Big|_{\partial\Omega} = f \quad (0<t<T) \end{cases}$$

where $\partial/\partial\nu_P = \sum \nu_i a_{ij}\, \partial/\partial x_j$ denotes the differentiation along the co-normal vector, $\nu = (\nu_i)$ being the outer unit normal vector on $\partial\Omega$.

Regarding the function $F = F(\xi, t)$ as an input, we wish to determine the coefficients (a, c, α) through the boundary output $g = g(\xi, t) = u(\xi, t)$ $(\xi \in \Gamma,\ 0 < t < T)$, where $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$. Our conclusion assures a generic uniqueness result provided that the input $F = f(\xi) h(t)$ is given in the some area as the output g, that is, $h \not\equiv 0$ and $\operatorname{supp} f \subset \Gamma$. Thus, we can extend the result by A. Pierce to the multi-space dimensional case.

A similar phenomenon can be seen for the interior input-output problem. Namely, in the parabolic problem

$$
(1.10)\quad \begin{cases} \dfrac{\partial u}{\partial t} - Pu = f(x)\, h(t) \quad (x \in \Omega,\ 0 < t < T), \quad u\big|_{t=0} = 0 \quad (x \in \Omega), \\ \left(\dfrac{\partial}{\partial \nu_P} + \alpha\right) u\big|_{\partial\Omega} = 0 \quad (0 < t < T), \end{cases}
$$

the output $g = u\big|_{\omega}$ $(0 < t < T)$ determines generically (a, c, α), provided that the input $F \equiv f(x) h(t)$ is takes as $h \not\equiv 0$ and $\operatorname{supp} f \subset \omega$, where $\omega \subset \Omega$ is a non-empty open set. However, the cases where inputs and outputs are taken in different areas remain as problems in future.

§2. <u>Reduction to the spectral problem</u>. To state the result, let P and Q be second order symmetric uniformly elliptic differential operators and α and β be smooth functions on $\partial\Omega$. We suppose that those are temporally homogeneous and depend only on space variables. Given a family of inputs $F_\ell = f_\ell(\xi)\, h_\ell(t)$ $(\ell \in S)$, we consider the parabolic equations

$$(2.1)\quad \begin{cases} \dfrac{\partial u_\ell}{\partial t} - P u_\ell = 0 \quad (x \in \Omega,\ 0 < t < T), \qquad u_\ell\big|_{t=0} = 0 \quad (x \in \Omega), \\ \left(\dfrac{\partial}{\partial \nu_P} + \alpha\right) u_\ell \Big|_{\partial\Omega} = F_\ell \quad (0 < t < T), \end{cases}$$

and

$$(2.2)\quad \begin{cases} \dfrac{\partial v_\ell}{\partial t} - Q v_\ell = 0 \quad (x \in \Omega,\ 0 < t < T), \qquad v_\ell\big|_{t=0} = 0 \quad (x \in \Omega) \\ \left(\dfrac{\partial}{\partial \nu_Q} + \beta\right) v_\ell \Big|_{\partial\Omega} = F_\ell \quad (0 < t < T). \end{cases}$$

For a given area $\Gamma \subset \partial\Omega$ with $|\Gamma| > 0$, we suppose that

$$(2.3)\quad v_\ell - u_\ell \big|_{\Gamma} = 0 \qquad (0 < t < T,\ \ell \in S).$$

We wish to establish some criterion for (2.3) to imply the uniqueness

$$(2.4)\quad (Q, \beta) = (P, \alpha).$$

For the moment, we drop the suffix $\ell \in S^1$. Henceforth $\{\lambda_j\}$, $\{\varphi_j\}$, $\{\mu_j\}$, $\{\psi_j\}$ denote the eigenvalues and eigenfunctions of $-P_\alpha$, the differential operator $-P$ with $\left(\frac{\partial}{\partial\nu_P}+\alpha\right)\cdot\big|_{\partial\Omega}=0$ and $-Q_\beta$, the differential operator $-Q$ with $\left(\frac{\partial}{\partial\nu_Q}+\beta\right)\cdot\big|_{\partial\Omega}=0$, respectively. Here, $-\infty<\lambda_1<\lambda_2\leqq\cdots\to+\infty$, $-\infty<\mu_1<\mu_2\leqq\cdots\to+\infty$, and $\|\varphi_j\|_{L^2(\Omega)}=\|\psi_j\|_{L^2(\Omega)}=1$. Then,

$$G(x,y;t)=\sum_j e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$$

is nothing but the Green function of $-\frac{\partial}{\partial t}+P_\alpha$, and the solution $u=u(x,t)$ of (2.1) is given as

$$\begin{aligned} u(x,t) &= \int_0^t d\tau\int_{\partial\Omega} d\sigma_\xi\; G(x,\xi;t-\tau)F(\xi,\tau) \\ &= \int_0^t r(x,t-\tau)h(\tau)d\tau, \end{aligned}$$

where

$$r(x,t)=\sum_j e^{-\lambda_j t}\varphi_j(x)\int_{\partial\Omega}\varphi_j(\xi)f(\xi)d\sigma_\xi \tag{2.5}$$

because of $F(\xi,t)=f(\xi)h(t)$. Similarly, we have

$$v(x,t)=\int_0^t s(x,t-\tau)h(\tau)d\tau$$

with

$$(2.6) \qquad s(x,t) = \sum_j e^{-\mu_j t} \varphi_j(x) \int_{\partial\Omega} \varphi_j(\xi) f(\xi) d\sigma_\xi ,$$

so that (2.3) reads as

$$(2.7) \qquad \int_0^t s(x,t-\tau) h(\tau) d\tau = \int_0^t r(x,t-\tau) h(\tau) d\tau \quad (x \in \Gamma,\ 0<t<T).$$

Assuming

$$(A.1) \qquad h \not\equiv 0,$$

we arrive at

$$s(x,t) = r(x,t) \qquad (x \in \Gamma,\ 0<t<T),$$

which extends to the full-range $t \in (0,\infty)$ by analytic continuation:

$$(2.8) \qquad s(x,t) = r(x,t) \qquad (x \in \Gamma,\ 0<t<\infty).$$

Here, we compare the behavior as $t \to +\infty$ of both sides of (2.8).

To this end we introduce the sets $J_\lambda = \{j \mid \lambda_j = \lambda\}$ and $L_\lambda = \{j \mid \mu_j = \lambda\}$ for real numbers $\lambda \in \mathbb{R}$ and impose that

(A.2) for each $\lambda \in \mathbb{R}$ with $J_\lambda \neq \phi$ and $L_\lambda \neq \phi$, the matrices $(A_{je})_{j \in L_\lambda, \ell \in S'}$ and $(B_{je})_{j \in L_\lambda, \ell \in S'}$ are of full-rank, respectively, where $A_{je} \equiv \int_{\partial\Omega} \varphi_j(\xi) f_e(\xi)\, d\sigma_\xi$ and $B_{je} \equiv \int_{\partial\Omega} \psi_j(\xi) f_e(\xi)\, d\sigma_\xi$.

Then, in particular for each $\lambda \in \mathbb{R}$ with $J_\lambda \neq \phi$, there exist some $j \in J_\lambda$ and $\ell \in S'$ such that $A_{j\ell} \neq 0$ and so is true for L_λ and $B_{j\ell}$. On the other hand, by virtue of Calderón's uniqueness theorem ([4]), we have the following

<u>Proposition</u>. Each of $\{\varphi_j\}_{j \in J_\lambda}$ and $\{\psi_j\}_{j \in L_\lambda}$ forms a system of linearly independent functions on Γ.

In fact, φ_j's $(j \in J_\lambda)$ satisfy the same equation $(L + \lambda)\varphi = 0$ in Ω. Therefore, supposing $\varphi \equiv \sum_{j \in J_\lambda} a_j \varphi_j = 0$ on Γ for real constants a_j $(j \in J_\lambda)$ we have $\varphi|_\Gamma = \frac{\partial}{\partial \nu_L} \varphi|_\Gamma = 0$ and hence $\varphi = 0$ near Γ by Calderón's theorem. Now the unique continuation property assures us of $\varphi \equiv 0$ in Ω and hence $a_j = 0$ $(j \in J_\lambda)$.

Therefore, for each λ with $J_\lambda \neq \phi$ there exists some ℓ such that

$\sum_{j\in J_x} \varphi_j(x) A_{j\ell} \not\equiv 0$ on Γ and so is true for L_x and $\sum_{j\in L_x} \psi_j(x) B_{j\ell}$.
In particular, (2.8) implies that

$$(2.9) \qquad \{\lambda_j \mid j \in J_x\} = \{\mu_j \mid j \in L_x\}$$

and also

$$(2.10) \qquad \sum_{j\in J_x} \varphi_j(x) A_{j\ell} = \sum_{j\in L_x} \psi_j(x) B_{j\ell} \qquad (x \in \Gamma,\ \ell \in S').$$

Again, the assumption (A2) improve (2.9) as

$$(2.11) \qquad J_x = L_x \qquad (x \in \mathbb{R})$$

via Proposition and (2.10). Furthermore, (2.10) reduces to the relation

$$(2.12) \qquad \varphi_j(x) = \sum_{k\in L_x} \gamma_{jk} \psi_k(x) \qquad (x \in \Gamma,\ j \in J_x)$$

for some real numbers $\{\gamma_{jk}\}$.

Henceforth, we suppose the following important assumption:

$$\text{(A.3)} \qquad \operatorname{supp.} f_\ell \subset P \qquad (\ell \in S').$$

Then, we have

$$A_{j\ell} = \int_P \varphi_j(\gamma)\, f_\ell(\gamma)\, d\sigma_\gamma$$

and hence can substitute (2.12) into (2.10). Noting Proposition, we get

$$\sum_{j,m \in J_x} \gamma_{jk}\, \gamma_{jm}\, B_{m\ell} = B_{k\ell} \qquad (k \in L_x,\ \ell \in S')$$

or

$${}^{T}(\gamma_{jk})\,(\gamma_{jk}) = (\delta_{jk})$$

by (A.2). Hence $\{\tilde{\psi}_j\}_{j \in L_x}$ forms an ortho-normal system in $L^2(\Omega)$ where $\tilde{\psi}_j = \sum_{k \in L_x} \gamma_{jk} \psi_k$. Without loss of generality we may suppose that

$$\text{(2.13)} \qquad \mu_j = \lambda_j \quad \text{and} \quad \psi_j(x) = \varphi_j(x) \quad (x \in P) \quad \text{for all } j.$$

§3. Iso-spectral deformation.. Now, we shall show that the condition (2.13) implies the uniqueness

$$(3.1) \qquad (Q, \beta) = (P, \alpha),$$

which corresponds exactly to Gel'fand-Levitan's uniqueness result in one-space dimension. The author has shown the fact in a rather special case in [15] of analytic coefficients operators. An extension to C^∞ coefficients operators has been performed by A. Nachman, J. Sylvester and G. Uhlmann in [10] motivated by Borg-Levinson's work ([3], [8]), for $-P = -\Delta + p(x)$, $-Q = -\Delta + q(x)$, $\beta = \alpha = 0$ and $\Gamma = \partial\Omega$ through a function theoretic method. Here, we take a different approach of deformation formula described in §1 for one-space dimensional case.

The formula (1.6) reads as

$$(3.2) \qquad \psi(x,\lambda) = \varphi(x,\lambda) + \int_0^1 H(x-y)\, K(x,y)\, \varphi(y,\lambda)\, dy,$$

where $H = H(x)$ is the Heaviside function. Then, the commutator $[\frac{d^2}{dx^2}, H]$ will produce the δ-function and the deformation (1.6) is achieved. In spite that there is no reasonable extension of the Heaviside function in multi-dimensional spaces, the above relation means

$$(3.3) \qquad \psi_j = (1 + \mathcal{K})\,\varphi_j$$

for the iso-spectral case $\mu_j = \lambda_j$ $(j = 1, 2, \cdots)$. The operator $\mathcal{K}$ is formally an integral operator with the kernel

$$(3.4) \qquad K(x, y) = \text{"}\sum_j \{\psi_j(x) - \varphi_j(x)\}\,\varphi_j(y)\text{"}.$$

In view of (3.2), $K = K(x, y)$ has the Heaviside-function like discontinuity on the diagonal $D \equiv \{(x, x) \mid x \in \Omega\}$ when the space-dimension is one.

For the moment, we shall develop a formal theory which will be justified later. Hence we note the following key "identity"

$$(3.5') \qquad K(x, y) \equiv \sum_j \{\psi_j(x) - \varphi_j(x)\}\,\varphi_j(y) = \sum_j \psi_j(x)\{\varphi_j(y) - \psi_j(y)\}.$$

In fact we have

$$\sum_j \varphi_j(x)\varphi_j(y) = \sum_j \psi_j(x)\,\psi_j(y) = \delta(x - y).$$

The "function" $K^* = K^*(x, y) = \sum_j \psi_j(x)\,\varphi_j(y)$ satisfies the ultra-hyperbolic equation

$$\square K^* = 0,$$

where

(3.6) $$\square = -Q_x + P_y .$$

In fact, we have $\lambda_j = \mu_j$ $(j=1,2,\cdots)$. Hence

(3.7') $$\square K = 0 \qquad (x \neq y)$$

by $K(x,y) = K^*(x,y) - \delta(x-y)$. On the other hand we have

$$\text{"} K|_{P\times\Omega} = K|_{\Omega\times P} = 0 \text{"}$$

by $\varphi_j(x) = \psi_j(x)$ $(x \in P,\ j=1,2,\cdots)$.

However, noting (3.7) we have

(3.8') $$Q_x^m K|_{P\times\Omega} = P_y^m K|_{P\times\Omega} = 0,$$

where $m=0,1,2,\cdots$. In other words,

$$\text{"} \sum_j \lambda_j^m \psi_j(x) \{\varphi_j(y) - \psi_j(y)\} = 0 \quad (m=0,1,2,\cdots;\ x \in P,\ y \in \Omega) \text{"}.$$

From the "Weierstrass approximation theorem" we get

$$\sum_{j \in L_x} \chi_j(x) \{ \varphi_j(y) - \psi_j(y) \} = 0 \qquad (x \in \Gamma,\ y \in \Omega)$$

for every $\lambda \in \mathbb{R}$. Again by Proposition in the previous section, we have $\varphi_j \equiv \psi_j$ $(j = 1, 2, \cdots)$. Now, both of eigenvalues and eigenfunctions coincide, we arrive at $P_\alpha = Q_\beta$, which means that $Q \equiv P$ and $\beta \equiv \alpha$.

The above formal argument can be justified in the following way. First, for any given integer $k \geq 0$, we chose large numbers λ and s so that

$$L_s(x, y; \lambda) \equiv \sum_j \{ \psi_j(x) - \varphi_j(x) \} \varphi_j(y) (\lambda_j + \lambda)^{-s} \in C^k(\overline{\Omega} \times \overline{\Omega}).$$

Then,

$$L(x, y) \equiv (-P_y + \lambda)^s L_s(x, y; \lambda) \in C^k(\overline{\Omega}_x \to \mathcal{D}'_y(\Omega))$$

is independent of λ and s, and $L = L(x, y) \in C^\infty(\overline{\Omega}_x \to \mathcal{D}'_y(\Omega)) \subset \mathcal{D}'(\Omega \times \Omega)$ can be defined. Similarly, from the function

$$M_s(x,y;\lambda) \equiv \sum_j \chi_j(x)\{\varphi_j(y) - \chi_j(y)\}(\mu_j+\lambda)^{-s} \in C^k(\bar{\Omega}\times\bar{\Omega})$$

we can define $\quad M = M(x,y) \in C^\infty(\bar{\Omega}_y \to \mathcal{D}'_x(\Omega)) \subset \mathcal{D}'(\Omega\times\Omega)$ through

$$M(x,y) \equiv (-Q_x+\lambda)^s M_s(x,y;\lambda).$$

Now the key identity (3.5) is justified as

$$(3.5) \qquad L(x,y) = M(x,y) \quad (:= K(x,y)) \quad \text{as } \mathcal{D}'(\Omega\times\Omega).$$

Namely, the distribution $K = K(x,y) \in \mathcal{D}'(\Omega\times\Omega)$ has an "uncertain" character $\quad K(x,y) = L(x,y) \in C^\infty(\bar{\Omega}_x \to \mathcal{D}'_y(\Omega))$ and $K(x,y) = M(x,y) \in C^\infty(\bar{\Omega}_y \to \mathcal{D}'_x(\Omega))$. By the hypothesis of iso-spectral: $\lambda_j = \mu_j \ (j=1,2,\cdots)$, ultra-hyperbolic relation

$$(3.7) \qquad \Box K = 0$$

holds in $\bar{\Omega}\times\Omega\setminus D$ and $\Omega\times\bar{\Omega}\setminus D$ as $K = L$ and $K = M$, respectively, where $D = \{(x,x) \mid x \in \Omega\}$. Hence from $L_s|_{\Gamma\times\Omega} = 0$ we obtain

$$Q_x^m K\big|_{P\times\Omega} = 0 \qquad (m=0,1,2,\cdots) \tag{3.8}$$

as elements in $C^\infty(\overline{\Omega}_x \to \mathcal{D}'_y(\Omega))$.

Here, we introduce the function

$$F_t(x,y) = \sum_j e^{-\lambda_j t}\varphi_j(x)\{\varphi_j(y) - \psi_j(y)\} \in C^\infty(\overline{\Omega}\times\overline{\Omega}) \tag{3.9}$$

for $t>0$. The key identity (3.5) deduces

$$F_t(x,y) = \sum_{m=0}^{\infty} \frac{t^m}{m!} Q_x^m K(x,y) \qquad \text{in } \mathcal{D}'(\Omega\times\Omega). \tag{3.10}$$

In fact, we have in $\mathcal{D}'(\Omega\times\Omega)$ that

$$\begin{aligned} F_t(x,y) &= \sum_j \sum_m \frac{t^m}{m!}(-\lambda_j)^m \varphi_j(x)\{\varphi_j(y) - \psi_j(y)\} \\ &= \sum_m \sum_j \frac{t^m}{m!}(-\lambda_j)^m \varphi_j(x)\{\varphi_j(y) - \psi_j(y)\}, \end{aligned}$$

while

$$Q_x^m K(x,y) = \sum_j (-\lambda_j)^m \varphi_j(x)\{\varphi_j(y) - \psi_j(y)\}$$

in $\mathcal{D}'(\Omega\times\Omega)$ by (3.5). However, the right-hand side of (3.10) converges in $C^0(\overline{\Omega}_x \to \mathcal{D}'_y(\Omega))$ and hence (3.10) holds as a relation there. Hence

(3.11) $\quad F_t(x, y) = 0 \qquad (0 < t < \infty,\ x \in \Gamma,\ y \in \Omega)$

by (3.8).

Comparing the behavior as $t \to \infty$ and utilizing Proposition in the preceding section, we arrive at

(3.12) $\quad \rho_j = \sigma_j \qquad (j = 1, 2, \cdots).$

Then, $(Q, \beta) = (R, \alpha)$ follows.

References

[1] Buhgeim, A.L., Klibanov, M.V., Global uniqueness of a class of multi-dimensional inverse problems, Soviet Math. Dokl., 24 (1981) 244–247.

[2] Buhgeim, A.L., Jakno, V.G., On two inverse problems for differential equations, Soviet Math. Dokl., 17 (1976) 1083–1085.

[3] Borg, G., Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Bestimmung der Differentialgleichung durch die Eigenwerte, Acta Math., 78 (1946) 1–96.

[4] Calderón, A.P., Uniqueness in the Cauchy problem for partial differential operators, Amer. J. Math., 80 (1958) 16–36.

[5] Carroll, R., Transmutation Theory and Applications, North-Holland, Amsterdam / New York / Oxford, 1985.

[6] Gel'fand, I.M., Levitan, B.M., On the determination of a differential equation from its spectral function, AMS Transl., (2) 1 (1955) 253–304.

[7] Isozaki, H., private communication

[8] Levinson, N., The inverse Sturm-Liouville problem, Mat. Tidsskr. B (1949) 25–30.

[9] Murayama, R., The Gel'fand-Levitan theory and certain inverse problems for the parabolic equation, J. Fac. Sci. Univ. Tokyo, Sec. IA 28 (1981) 317–330.

[10] Nachman, A., Sylvester, J., Uhlmann, G., An n-dimensional Borg-Levinson theorem, Comm. Math. Phys., 115 (1988) 595–605.

[11] Pierce, A., Unique identification of eigenvalues and coefficients in a parabolic problem, SIAM J. Control & Optim., 17 (1979) 494–499.

[12] Romanov, V.G., Inverse problems for hyperbolic equations and energy inequalities, Soviet Math. Dokl., 19 (1978) 1154–1158.

[13] Suzuki, T., Uniqueness and nonuniqueness in an inverse problem for the parabolic equation, J. Differential Equations, 47 (1983) 296–316.

[14] ——, Gel'fand-Levitan's theory, deformation formulas and inverse problems, J. Fac. Sci. Univ. Tokyo, Sec. IA 32 (1985) 223–271.

[15] ——, On a multi-dimensional inverse parabolic problem, Proc. Japan Acad., 62 A (1986) 83-86.

[16] ——. Ultra-hyperbolic approach to some multi-dimensional inverse problems, Proc. Japan Acad., 64 A (1988).

[17] Suzuki, T., Murayama, R., A uniqueness theorem in an identification problem for coefficients of parabolic equations, Proc. Japan Acad., 56 A (1980) 259-263.

Note. An important remark has been given by H. Isozaki for multi-dimensional iso-spectral problems ([7]).