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**The initial-boundary value problem for
a nonlinear degenerate parabolic equation**

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1. Introduction and main results.

Let $a < b$ and $\lambda > 0$. We consider nonnegative solutions of the initial-boundary value problem

$$\begin{cases} u_t = uu_{xx} - \lambda |u_x|^2 & (a < x < b, t > 0) \end{cases} \quad (1.1)$$

$$\begin{cases} u(a, t) = u(b, t) = 0 & (t > 0) \end{cases} \quad (1.2)$$

$$\begin{cases} u(x, 0) = u_0(x) & (a < x < b) \end{cases} \quad (1.3)$$

where initial data u_0 satisfy

$$(H.1) \quad u_0 \in W^{1, \infty}(a, b) \quad \text{and} \quad u_0(x) \geq 0 \quad (a \leq x \leq b)$$

In order to construct a solution to the problem (1.1)-(1.3), it might be natural to employ the well-known viscosity method: Let $\varepsilon > 0$ and let $u_\varepsilon(x, t)$ be an unique classical solution of the initial-boundary value problem for the uniformly parabolic equation:

$$\begin{cases} u_{\varepsilon t} = (u_\varepsilon + \varepsilon) u_{\varepsilon xx} - \lambda |u_{\varepsilon x}|^2 & (a < x < b, t > 0) \end{cases} \quad (1.1)_\varepsilon$$

$$\begin{cases} u_\varepsilon(a, t) = u_\varepsilon(b, t) = 0 & (t > 0) \end{cases} \quad (1.2)_\varepsilon$$

$$\begin{cases} u_\varepsilon(x, 0) = u_0(x) & (a < x < b) \end{cases} \quad (1.3)_\varepsilon$$

We call u the viscosity solution of the problem (1.1)-(1.3)

$$\text{if } u(x, t) = \lim_{\varepsilon \rightarrow 0} u_{\varepsilon}(x, t).$$

Let us consider solutions with compact support and define the interface $\xi_{\pm}(t)$ by

$$\xi_{\pm}(t) = \pm \sup \{ \pm x : u(x, t) > 0 \} \quad \text{for } t > 0$$

Differentiating $u(\xi_{\pm}(t), t) = 0$ with respect to t and using eq. (1.1), we easily see that the interface $\xi_{\pm}(t)$ satisfies formally

$$\frac{d\xi_{\pm}(t)}{dt} = \lambda u_x(\xi_{\pm}(t), t), \quad (1.4)$$

provided $u_x(\xi_{\pm}(t), t) \neq 0$. Thus we might expect that the support of solutions shrinks if $u_x(\xi_{\pm}(t), t) \neq 0$. Indeed, for $\lambda > \frac{1}{2}$ we have a special weak solution of the form

$$u(x, t) = (T_0 - t)^{\frac{1}{2\lambda-1}} \left[C_0^2 - \frac{1}{2(2\lambda-1)} x^2 (T_0 - t)^{\frac{2\lambda}{2\lambda-1}} \right]_+ \quad (1.5)$$

where T_0 and C_0 are positive constants such that

$$(-\sqrt{2(2\lambda-1)} C_0 T_0^{\frac{\lambda}{2\lambda-1}}, \sqrt{2(2\lambda-1)} C_0 T_0^{\frac{\lambda}{2\lambda-1}}) \subset [a, b]$$

and $[\cdot]_+ = \max(\cdot, 0)$.

Apparently its support shrinks to one point. But this conjecture is not true for viscosity solutions. In [1], Bertch, Dal Passo and Ughi show that every viscosity solution of the Cauchy problem for (1.1) has a property that

$$\text{supp } u(t) = \text{supp } u_0 \quad \text{for } t > 0. \quad (1.6)$$

It is a striking result. If $\lambda < 0$, equation (1.1) is called the pressure equation, related to the porous medium equation and

the support of solutions spreads out as time goes, as is suggested by the interface equation (1.4).

Another curious property of eq. (1.1) is the nonuniqueness phenomenon which was discovered by Dal Passo and Luckhaus [2] ($\lambda = 0$), Ughi [5] ($\lambda = 0$) and Bertch, Dal Passo and Ughi [1] ($\lambda \geq 0$). The existence of our special weak solution u also suggests the nonuniqueness phenomenon.

We now define weak solutions of the problem (1.1)-(1.3) as follows:

Definition 1. A nonnegative function $u \in L^\infty([0, \infty); W^{1, \infty}[a, b])$ is called a weak solution of (1.1)-(1.3) if for any $T > 0$

$$u_t \in L^2([a, b] \times [0, T])$$

and for all $t \geq 0$

$$\int_a^b u(x, t) \psi(x, t) dx = \int_a^b u_0(x) \psi(x, 0) dx$$

$$+ \int_0^t \int_a^b \{u(x, s) \psi_t(x, s) - u(x, s) u_x(x, s) \psi_x(x, s) - (\lambda + 1) |u_x(x, s)|^2 \psi(x, s)\} dx ds$$

for any function $\psi \in C^{2,1}([a, b] \times [0, \infty))$ with compact support in (a, b) .

Note that $u \in L^\infty([0, \infty); W^{1, \infty}([a, b]))$ with $u_t \in L^2([a, b] \times [0, T])$ for any $T > 0$ implies that u is continuous in x and t .

In this paper we establish the global existence of (weak) solutions of (1.1)-(1.3) and investigate the uniqueness of solutions. We propose a new uniqueness class of solutions which is different from [1], [2] and [5].

As to the existence theorem, we have

Theorem 1. Let u_0 satisfy (H1). Then the problem (1.1)-(1.3) has at least one weak solution.

Theorem 2. Let $\lambda > \frac{1}{2}$. Assume that u_0 satisfies (H1) and

$$(H2) \quad \lim_{x \downarrow a} \frac{u_0(x)}{(x-a)^2} < \infty \quad \text{and} \quad \lim_{x \uparrow b} \frac{u_0(x)}{(b-x)^2} < \infty.$$

Then u satisfies

$$|u_{xx}(x, t)| \leq \frac{1}{t} \quad (1.8)$$

and, in particular, $u \in L^\infty([\delta, \infty); W^{2, \infty}([a, b]))$ as well as

$u_t \in L^\infty([\delta, \infty); L^\infty([a, b]))$ for any $\delta > 0$. Moreover, if we assume

that u_0 is semiconcave, that is,

$$u_{0xx} \leq C \quad \text{in } D'$$

for some constant C , then u is also semiconcave almost everywhere, that is,

$$u_{xx}(x, t) \leq C \quad \text{for a.e. } (x, t) \in [a, b] \times (0, \infty)$$

where C is also a positive constant.

Remark 1. In theorem 2 the hypotheses (H1) can be weakened as follows:

$$(H1)_w \quad u_0 \in L^\infty([a, b]), \quad u_0(x) \geq 0 \text{ a.e.}$$

Corollary 1. Under the assumption $(H1)_w$ and (H2), the problem (1.1)-(1.3) has at least one weak solution which has properties in Theorem 2.

Concerning the uniqueness and continuous-dependence-on-data of solutions, we have

Theorem 3. Let u and v be two weak solutions corresponding to the initial data u_0 and v_0 , respectively. Assume that u and v are semiconcave almost everywhere. Then the inequality

$$\int_a^b |u(x, t) - v(x, t)| dx \leq e^{ct} \int_a^b |u_0(x) - v_0(x)| dx$$

holds valid for any $t > 0$ and a positive constant c .

Corollary 2. Let u_0 satisfy $(H1)_w$, (H2) and be semiconcave. Then the problem (1.1)-(1.3) has an unique weak solution u which is also semiconcave and depends on initial data continuously in $L^1(a, b)$.

Remark 2. Our special solution (1.5) is not semiconcave.

Uniqueness theorem does not hold valid for the problem (1.1)-(1.3) with initial data

$$u_0(x) = (T_0)^{\frac{1}{2\lambda-1}} \left[C_0^2 - \frac{1}{2(2\lambda-1)} x^2 (T_0)^{\frac{2\lambda}{2\lambda-1}} \right]_+,$$

which does not satisfy (H2).

2. Proof of Theorem 1.

Before proving Theorem 1, we shall obtain a priori estimates of u_ε .

Lemma 1. Let u_0 satisfy (H1). Then

$$\|u_\varepsilon\|_{L^\infty([0, \infty); W^{1, \infty}([a, b]))} \leq C \quad (2.1)$$

and

$$\int_0^\infty \int_a^b (u_\varepsilon(x, t) + \varepsilon) |u_{\varepsilon x}(x, t)|^{p-1} u_{\varepsilon xx}^2(x, t) dx \leq C \quad (2.2)$$

for any $p \geq 1$, where and in the sequel C denotes various positive constants independent of ε .

Proof. The maximum principle gives

$$0 \leq u_\varepsilon(x, t) \leq \max_{a \leq x \leq b} u_0(x). \quad (2.3)$$

Multiplying (1.1)_ε by $\frac{1}{p} (|u_{\varepsilon X}(x, t)|^{p-1} u_{\varepsilon X})_X$ and

integrating by parts on $[a, b]$, we have

$$\begin{aligned} & \frac{1}{p(p+1)} \frac{d}{dt} \int_a^b |u_{\varepsilon X}|^{p+1} dx + \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx \\ & + \frac{\lambda}{p+1} |u_{\varepsilon X}(a, t)|^p u_{\varepsilon X}(a, t) - \frac{\lambda}{p+1} |u_{\varepsilon X}(b, t)|^p u_{\varepsilon X}(b, t) = 0 \quad (2.4) \end{aligned}$$

Here and from now on we abbreviate x and t variables in the integrand. Since u_{ε} is nonnegative, we easily see that

$$u_{\varepsilon X}(a, t) \geq 0 \quad \text{and} \quad u_{\varepsilon X}(b, t) \leq 0.$$

Hence integrating (2.4) from 0 to t , we obtain that, any $p \geq 1$

$$\begin{aligned} & \frac{1}{p(p+1)} \int_a^b |u_{\varepsilon X}|^{p+1} dx + \int_0^t \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx dt \\ & \leq \frac{1}{p(p+1)} \int_a^b |u_{0X}|^{p+1} dx \end{aligned}$$

from which it follows that

$$\|u_{\varepsilon X}(t)\|_{L^{p+1}(a, b)} \leq \|u_{0X}\|_{L^{p+1}(a, b)} \quad \text{for any } t > 0 \quad (2.5)$$

and

$$\int_0^t \int_a^b (u_{\varepsilon} + \varepsilon) |u_{\varepsilon X}|^{p-1} u_{\varepsilon XX}^2 dx dt \leq C \|u_{0X}\|_{L^{p+1}(a, b)}^{p+1} \quad (2.6)$$

From (2.5) we easily have (2.1).

Lemma 2. Let $0 < \varepsilon < \varepsilon_0$ where ε_0 is a fixed number. For any $T > 0$,

$$\|u_{\varepsilon t}\|_{L^2([a,b] \times [0,T])}^2 \leq C \quad (2.7)$$

where C is a positive constant independent of ε .

Proof. Using (1.1) _{ε} and integrating by parts, we get

$$\begin{aligned} \int_0^T \int_a^b u_{\varepsilon t}^2 dx dt &= \int_0^T \int_a^b (u_{\varepsilon} + \varepsilon)^2 u_{\varepsilon xx}^2 dx dt - \frac{2}{3} \varepsilon \lambda \int_0^T \{u_x(b,t)^2 - u_x(a,t)^2\} dt \\ &\quad + \left(\frac{2}{3} \lambda + \lambda^2\right) \int_0^T \int_a^b u_{\varepsilon x}^4 dx dt \\ &\leq \|u_{\varepsilon}\|_{L^\infty([a,b] \times [0,T])}^2 + \varepsilon_0 \int_0^T \int_a^b (u_{\varepsilon} + \varepsilon) u_{\varepsilon xx}^2 dx dt \\ &\quad + \frac{4}{3} \varepsilon \lambda T \|u_{\varepsilon x}\|_{L^\infty([a,b] \times [0,\infty))}^3 + \left(\frac{2}{3} \lambda + \lambda^2\right) (b-a) T \|u_{\varepsilon x}\|_{L^\infty([a,b] \times [0,\infty))}^4 \end{aligned}$$

From (2.1) and (2.2) with $p = 1$, we can easily obtain (2.7).

Proof of Theorem 1. From (2.1), (2.3) and (2.7), we see that there exists a nonnegative function $u \in L^\infty([0,\infty); C([a,b]) \cap W^{1,\infty}[a,b])$ with $u_t \in L^2([a,b] \times [0,T])$ (for any $T > 0$) and we can extract a subsequence of $\{u_{\varepsilon}\}$, which is denoted by $\{u_{\varepsilon_i}\}$, such that, as $\varepsilon_i \rightarrow 0$,

$$u_{\varepsilon_i} \longrightarrow u \quad \text{strongly in } C([a, b] \times [0, T])$$

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{weakly star in } L^\infty([a, b] \times [0, \infty))$$

and

$$u_{\varepsilon_i t} \longrightarrow u_t \quad \text{weakly in } L^2([a, b] \times [0, T]) .$$

In order to show that u is a weak solution of (1.1)-(1.3), it suffices to show that, for any $T > 0$

$$|u_{\varepsilon_i x}|^2 \longrightarrow |u_x|^2 \quad \text{in } L^1([a, b] \times [0, T]) ,$$

and this implies

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{strongly in } L^2([a, b] \times [0, T]) .$$

From (2.1) and (2.2), we have

$$\|(u_\varepsilon^2)_{xx}\|_{L^2([a, b] \times [0, T])} \leq 2 \|u_\varepsilon u_{\varepsilon xx} + u_{\varepsilon x}^2\|_{L^2([a, b] \times [0, T])} \leq C .$$

We also have

$$\|(u_\varepsilon^2)_{xt}\|_{L^2(0, T; H^{-1}(a, b))} \leq C \|(u_\varepsilon^2)_t\|_{L^2([a, b] \times [0, T])} \leq C .$$

By virtue of Aubin's compactness theorem (see J.L.Lions [4]), we may assume that

$$(u_{\varepsilon_i}^2)_x = 2u_{\varepsilon_i} u_{\varepsilon_i x} \longrightarrow 2uu_x = (u^2)_x \quad \text{strongly in } L^2([a, b] \times [0, T]) .$$

Hence we may also assume that

$$u_{\varepsilon_i} u_{\varepsilon_i x} \longrightarrow uu_x \quad \text{a.e. in } [a, b] \times [0, \infty)$$

from which it follows that

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{a.e. in } [a, b] \times [0, \infty) \quad (2.8)$$

since $\frac{\partial u}{\partial x} = 0$ a.e. in $E = \{x \in [a, b]; u=0\}$ (see Kinderlehrer-Stampacchia [3], p53) and

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{a.e. in } {}^c E = \{x \in [a, b]; u > 0\}.$$

In view of Lebesgue's bounded convergence theorem we can easily obtain

$$\lim_{m \rightarrow \infty} \int_0^T \int_a^b |u_{mx}^2| \, dx dt = \int_0^T \int_a^b |u_x^2| \, dx dt. \quad (2.9)$$

On the other hand, from (2.1) we may assume that u_{mx} converges to u_x weakly in $L^2([a, b] \times [0, T])$. Hence

$$u_{mx} \longrightarrow u_x \quad \text{strongly in } L^2([a, b] \times [0, T]) \quad (2.10)$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2.

Lemma 3. Let u_0 satisfy (H1)_w and (H2). Then, for any $t > 0$

$$|u_\varepsilon(a, t)| \leq \sqrt{\varepsilon} \, C \quad (3.1)$$

and

$$|u_\varepsilon(b, t)| \leq \sqrt{\varepsilon} \, C \quad (3.2)$$

Proof. We only show that (3.1) hold valid. From (H2) we see that for some $\delta > 0$ and $C_1 > 0$

$$0 \leq u_0(x) \leq C_1 \{(x-a)^2 + \sqrt{\varepsilon} (x-a)\} \quad \text{for any } x \in (a, a+\delta) \quad (3.3)$$

Let $T > 0$ be fixed. For any $(x, t) \in [a, a+\delta] \times [0, T]$ set

$$\bar{u}(x, t) = A \{(x-a)^2 + \sqrt{\varepsilon} (x-a)\} \quad (3.4)$$

where A is chosen so large that

$$A \geq C_1 \quad (3.5)$$

and

$$A(\delta^2 + \sqrt{\varepsilon} \delta) \geq \max_{\substack{a \leq x \leq a+\delta \\ 0 \leq t \leq T}} u_\varepsilon(x, t). \quad (3.6)$$

Note that $u \in C^{2,1}((a, b) \times [0, T])$. Direct calculation gives

$$\begin{aligned} & \bar{u}_t - (\bar{u} + \varepsilon) \bar{u}_{xx} + \lambda (\bar{u}_x)^2 \\ &= 2(2\lambda - 1)A^2(x-a)^2 + 2(2\lambda - 1)A^2\sqrt{\varepsilon}(x-a) + 2\varepsilon A(\lambda A - 2) \\ &\geq 0 \quad \text{in } (a, a+\delta) \times (0, T) \end{aligned} \quad (3.7)$$

provided that $\lambda \geq \frac{1}{2}$ and that A is so large that

$$\lambda A - 2 > 0.$$

By virtue of (3.3)-(3.7) we apply the maximum principle to obtain

$$0 \leq u_\varepsilon(x, t) \leq \bar{u}(x, t) \quad \text{in } [a, a+\delta] \times [0, T].$$

Hence

$$0 \leq u_\varepsilon(a, t) = \lim_{h \downarrow 0} \frac{u_\varepsilon(a+h, t) - u_\varepsilon(a, t)}{h} \leq \lim_{h \downarrow 0} \frac{u(a+h)}{h} = A\sqrt{\varepsilon}.$$

Thus we have (3.1).

Lemma 4. Under the same assumption

$$|u_{\varepsilon XX}| \leq \frac{C}{t} \quad \text{for all } t > 0. \quad (3.8)$$

Moreover, if $u_{0XX} \leq C_2$ then

$$u_{\varepsilon XX} \leq C_3 \quad (3.9)$$

where C_3 is a constant.

Proof. Putting $p = \frac{u_{\varepsilon t}}{u_{\varepsilon} + \varepsilon}$, we have

$$p_t = (u_{\varepsilon} + \varepsilon) p_{XX} + 2(1-\lambda) u_{\varepsilon X} p_X + p^2 \quad (x, t) \in (a, b) \times (0, \infty)$$

$$p(a, t) = p(b, t) = 0 \quad t \in (0, \infty)$$

$$p(x, 0) = u_{0XX} - \frac{\lambda u_{0X}^2}{u_0 + \varepsilon} \quad x \in (a, b).$$

The standard comparison theorem yields that

$$p \geq -\frac{1}{t}$$

Using (1.1) _{ε} , we easily see that

$$u_{\varepsilon XX} \geq -\frac{1}{t} \quad (3.10)$$

We put $q = u_{\varepsilon XX}$ to obtain that

$$q_t = (u_{\varepsilon} + \varepsilon) q_{XX} + 2(1-\lambda) u_{\varepsilon X} q_X + (1-2\lambda) q^2 \quad (3.11)$$

As for the boundary conditions, we utilize (1.1) _{ε} to get

$$q(a, t) = \frac{\lambda}{\varepsilon} |u_{\varepsilon X}(a, t)|^2, \quad q(b, t) = \frac{\lambda}{\varepsilon} |u_{\varepsilon X}(b, t)|^2 \quad (3.12)$$

for any $t > 0$. In view of Lemma 3, we see that

$$0 \leq q(a, t) \leq \lambda C^2, \quad 0 \leq q(b, t) \leq \lambda C^2 \quad (3.13)$$

Hence the comparison theorem yields that, if $\lambda > \frac{1}{2}$

$$q(x, t) = u_{\varepsilon XX}(x, t) \leq \frac{C}{(2\lambda - 1)}$$

for some constant $C > 0$.

If $u_{0XX} \leq C_2$, $\lambda \geq \frac{1}{2}$ and (3.11)-(3.13) yield that

$$u_{\varepsilon XX}(x, t) \leq C_3 \quad (3.14)$$

where $C_3 = \max(\lambda C^2, C_2)$ is independent of ε .

Proof of Theorem 2. Because of Lemma 4, we see that $\{u_{\varepsilon XX}\}$ is bounded in $L^\infty([a, b] \times [\delta, \infty))$ for every $\delta > 0$. Hence we can assume that

$$u_{\varepsilon_i XX} \longrightarrow u_{XX} \quad \text{weakly star in } L^\infty([a, b] \times [\delta, \infty))$$

and

$$|u_{XX}(x, t)| \leq \frac{C}{t} \quad \text{for any } (x, t) \in [a, b] \times [\delta, \infty).$$

If $u_{0XX} \leq C$, from (3.14) we have

$$u_{XX}(x, t) \leq C \quad \text{for any } (x, t) \in [a, b] \times [0, \infty).$$

This completes the proof of Theorem 2.

4. Proof of Theorem 3.

Let u and v be two weak solutions of (1.1)-(1.3) with initial data u_0 and v_0 , respectively. Let $T > 0$ be fixed and put $w(x, t) = u(x, t) - v(x, t)$ and $w_0(x) = u_0(x) - v_0(x)$.

Then we have

$$\begin{aligned} \int_a^b w(x, T) \psi(x, T) dx &= \int_a^b w_0(x) \psi(x, 0) dx \\ &+ \int_0^T \int_a^b \{w \psi_t - (u u_x - v v_x) \psi_x - (\lambda + 1) (|u_x|^2 - |v_x|^2) \psi\} dx dt \end{aligned} \quad (4.1)$$

for any $\psi \in C^{2,1}([a, b] \times [0, \infty))$ with compact support in (a, b) .

For each $n \in \mathbb{N}$ define

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < s \\ ns & \text{if } |s| \leq \frac{1}{n} \\ -1 & \text{if } s < -\frac{1}{n} \end{cases}$$

and

$$\Psi = \{g_n((u^2 - v^2) \theta_k \theta_m \star \rho_\nu \star \sigma_\mu) \star \rho_\nu \star \sigma_\mu\} \theta_k \theta_m$$

where ρ_ν and σ_μ are the standard mollifiers with respect to x

and t , respectively; $\theta_k(\frac{x}{k})$ where $\theta \in C_0^\infty((a, b))$ with $0 \leq \theta \leq 1$

and $\theta(x) = 1$ in a neighborhood of 0 (we may assume $0 \in (a, b)$) and

$\theta_m(t) \in C_0^\infty((0, \infty))$ such that $0 \leq \theta_m \leq 1$ and $\theta_m(t)$ tends to the indicator function of $[s_1, s_2]$ ($0 < s_1 < s_2$) as $m \rightarrow \infty$. Then

$\Psi \in C_0^\infty((a, b) \times (0, \infty))$ and $\Psi(x, t) \geq 0$ for any $(x, t) \in (a, b) \times (0, \infty)$.

Substituting Ψ for a test function ψ in (3.1), we have

$$\int_0^T \int_a^b \{w\Psi_t - (uu_x - vv_x)\Psi_x - (\lambda+1)(|u_x|^2 - |v_x|^2)\Psi\} dx dt. \quad (4.2)$$

From $w_t \in L^2([a, b] \times [0, T])$ for any $T > 0$ and $\Psi \in C_0^\infty((a, b) \times (0, \infty))$ we get

$$\int_0^T \int_a^b w\Psi_t dx dt = - \int_0^T \int_a^b w_t \Psi dx dt.$$

Letting ν and μ tend to infinity, we can easily see that

$$I_1(k, m, n) - I_2(k, m, n) - I_3(k, m, n)$$

$$= \int_0^T \int_a^b w_t \theta_k \theta_m g_n((u^2 - v^2) \theta_k \theta_m) dx dt$$

$$- [- \int_0^T \int_a^b (uu_x - vv_x) \{g_n((u^2 - v^2) \theta_k \theta_m)\}_x dx dt]$$

$$- [-(\lambda+1) \int_0^T \int_a^b (|u_x|^2 - |v_x|^2) \theta_k \theta_m g_n((u^2 - v^2) \theta_k \theta_m) dx dt] = 0 \quad (4.3)$$

As n tends to infinity, we find $I_1(k, m, n)$ tends to

$$\tilde{I}_1(k, m) = \int_0^T \int_a^b w_t \theta_k \theta_m \operatorname{sgn}((u-v) \theta_k \theta_m) dx dt \quad (4.4)$$

since $\operatorname{sgn}((u^2 - v^2) \theta_k \theta_m) = \operatorname{sgn}((u-v) \theta_k \theta_m)$.

Moreover, $\theta_m(t) = 0$ near 0 and T , then we have

$$\begin{aligned} \tilde{I}_1(k, m) &= \int_0^T \int_a^b (|w \theta_k \theta_m|)_t dx dt - \int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \\ &= - \int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \end{aligned} \quad (4.5)$$

As for $I_2(k, m, n)$, using chain rule, we get

$$\begin{aligned} I_2(k, m, n) &= -2 \int_0^T \int_a^b (u u_x - v v_x)^2 g'_n((u^2 - v^2) \theta_k \theta_m) \theta_k \theta_m dx dt \\ &\quad - \int_0^T \int_a^b (u u_x - v v_x) g'_n((u^2 - v^2) \theta_k \theta_m) (u^2 - v^2) \theta_m^2 \theta_k (\theta_k)_x dx dt \\ &\quad - \int_0^T \int_a^b (u u_x - v v_x) g_n((u^2 - v^2) \theta_k \theta_m) \theta_m^2 \theta_k (\theta_k)_x dx dt. \end{aligned}$$

Since the first term on the right hand side is nonpositive and

$$\|(\theta_k)_x\| \leq \frac{C}{k}, \text{ we have}$$

$$I_2(k, m, n) \leq \frac{C}{k} (\|u\|_{L^\infty}^3 + \|v\|_{L^\infty}^3 + \|u\|_{L^\infty} + \|v\|_{L^\infty}) (\|u_x\|_{L^2} + \|v_x\|_{L^2})$$

where $L^p = L^p([a, b] \times [0, T])$ ($p=2, \infty$). Since $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$

and $\|v_x\|_{L^2}$ are bounded, we get

$$I_2(k, m, n) \leq \frac{C}{k} \quad (4.6)$$

where C depends on $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$ and $\|v_x\|_{L^2}$.

Since $\operatorname{sgn}((u^2 - v^2)\theta_k \theta_m) = \operatorname{sgn}(w \theta_k \theta_m)$, letting $n \rightarrow \infty$

we see that $I_3(k, m, n)$ tends to

$$\tilde{I}_3(k, m) = -(\lambda+1) \int_0^T \int_a^b (|u_x|^2 - |v_x|^2) \theta_k \theta_m \operatorname{sgn}(w \theta_k \theta_m) dx dt.$$

Recalling that u_{xx} and v_{xx} are semiconcave, we have

$$\begin{aligned} \tilde{I}_3(k, m) &= -(\lambda+1) \int_0^T \int_a^b (|w \theta_k \theta_m|)_x (u_x + v_x) dx dt \\ &\quad - (\lambda+1) \int_0^T \int_a^b (u-v)(u_x - v_x) (\theta_k)_x \theta_m \operatorname{sgn}(w \theta_k \theta_m) dx dt \end{aligned}$$

$$\begin{aligned}
& \leq (\lambda+1) \int_0^T \int_a^b |w \theta_k \theta_m| (u_{xx} + v_{xx}) dx dt \\
& + (\lambda+1) \int_0^T \int_a^b (|u| + |v|) (|u_x| + |v_x|) |(\theta_k)_x| dx dt \\
& \leq C \int_0^T \int_a^b |w \theta_k \theta_m| dx dt + \frac{C}{k} \quad (4.7)
\end{aligned}$$

Hence eq. (4.3) with (4.5), (4.6) and (4.7) implies that

$$\int_0^T \int_a^b |w \theta_k| (\theta_m)_t dx dt \leq C \int_0^T \int_a^b |w \theta_k \theta_m| dx dt + \frac{C}{k} \quad (4.8)$$

In (4.8) letting $k, m \rightarrow \infty$, we find that

$$\int_a^b |w(x, s_2)| dx - \int_a^b |w(x, s_1)| dx \leq C \int_{s_1}^{s_2} \int_a^b |w(x, s)| dx ds$$

for any s_1 and s_2 ($0 < s_1 < s_2$).

As $s_2 = t$ and s_1 tends to 0, we have

$$\int_a^b |w(x, t)| dx - \int_a^b |w_0(x)| dx \leq C \int_0^t \int_a^b |w(x, s)| dx ds$$

from which it follows that, for any $t \geq 0$

$$\int_a^b |w(x, t)| dx \leq e^{ct} \int_a^b |w_0(x)| dx \quad (4.9)$$

This completes the proof of Theorem 3. Corollary 2 is easily obtained from (4.9).

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