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# Fourier transform of holomorphic discrete series
## — the case of tube domains —

by

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## §1. Preliminaries.

Let $G$ be a non-compact connected linear simple Lie group and $K$ a maximal compact subgroup of $G$. We assume throughout this note that $G/K$ carries a structure of hermitian symmetric space and that $G/K$ is holomorphically equivalent to a tube domain. The Lie algebras of $G$ and $K$ are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition with the associated Cartan involution $\theta$. Since $G/K$ is a hermitian symmetric space, there is a linear operator $J$ on $\mathfrak{p}$ such that $J$ commutes with $(\mathrm{Ad}\ k)|_{\mathfrak{p}}$ $(k \in K)$ and $J^2 = -1_{\mathfrak{p}}$. One knows that $J$ is written as $J = (\mathrm{ad}\ Z_0)|_{\mathfrak{p}}$ for some element $Z_0$ in the center $\mathfrak{c}$ of $\mathfrak{k}$. Note that since $G$ is assumed to be simple, $\mathfrak{c}$ is necessarily of one dimension.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then one can prove that $\mathfrak{t}$ is a (compact) Cartan subalgebra of $\mathfrak{g}$. Let $\Delta$ be the root system with respect to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and we denote by $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ $(\alpha \in \Delta)$ the root subspace corresponding to the root $\alpha \in \Delta$. Then,

$\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$. Let $\Delta_c$ (resp. $\Delta_n$) be the set of all roots $\alpha \in \Delta$ such that $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{k}_{\mathbb{C}}$ (resp. $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$). A root $\alpha$ in $\Delta_c$ (resp. in $\Delta_n$) is said to be compact (resp. non-compact). We introduce an order in $\Delta$ compatible with the complex structure of G/K so that the +i (resp. -i)-eigenspace $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) of the J extended to $\mathfrak{p}_{\mathbb{C}}$ by complex linearity coincides with the sum of all root subspaces corresponding to non-compact positive (resp. negative) roots. The set of all positive roots is denoted by $\Delta^+$ and $\Delta_c^+$ (resp. $\Delta_n^+$) stands for the set of all compact (resp. non-compact) positive roots. Both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$ normalized by $K_{\mathbb{C}}$.

Let $\gamma_1, \ldots, \gamma_\ell$ be a maximal system of strongly orthogonal non-compact positive roots constructed as follows: for each j, $\gamma_j$ is the largest positive non-compact root strongly orthogonal to $\gamma_{j+1}, \ldots, \gamma_\ell$. Let B be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. For every $\alpha \in \Delta$, we choose $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ and $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ so that

$$(1.1)\qquad \begin{aligned} &B(H_\alpha, H) = \alpha(H) \quad (\forall H \in \mathfrak{t}_{\mathbb{C}}), && X_\alpha - X_{-\alpha} \in \mathfrak{k} + i\mathfrak{p}, \\ &i(X_\alpha + X_{-\alpha}) \in \mathfrak{k} + i\mathfrak{p}, && [X_\alpha, X_{-\alpha}] = \frac{2H_\alpha}{\alpha(H_\alpha)} =: H'_\alpha. \end{aligned}$$

Then $H_\alpha \in i\mathfrak{t}$ and one can prove that $\mathfrak{a} := \sum_{1 \leq i \leq \ell} \mathbb{R}(X_{\gamma_i} + X_{-\gamma_i})$ is a maximal abelian subspace of $\mathfrak{p}$. Hence $\ell$ is equal to the real rank of G.

Let $G_{\mathbb{C}}$ be the complexification of G. We denote by $K_{\mathbb{C}}$ and $P_\pm$ the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_\pm$ respectively. Then, every element x in $P_+K_{\mathbb{C}}P_-$ can be expressed in a unique way as

$$x = \exp \xi_+(x)\cdot k(x)\cdot \exp \xi_-(x)$$

with $\xi_\pm(x) \in \mathfrak{p}_\pm$ amd $k(x) \in K_{\mathbb{C}}$. Furthermore $G \subset P_+K_{\mathbb{C}}P_-$. Since $\xi_+(xk) = \xi_+(x)$ $(k \in K_{\mathbb{C}})$, we have a mapping $\psi: G \to \mathfrak{p}_+$ defined by $\psi(gK) = \xi_+(g)$ $(g \in G)$. The $\psi$ is a holomorphic diffeomorphism onto a bounded symmetric domain $\mathscr{D}$ in $\mathfrak{p}_+$. The image $\mathscr{D} = \psi(G/K)$ $\subset \mathfrak{p}_+$ is called the Harish-Chandra realization of $G/K$. Let $q \in \mathscr{D}$. If $g \in G_{\mathbb{C}}$ satisfies $g \exp q \in P_+K_{\mathbb{C}}P_-$, then $g\cdot q \in \mathfrak{p}_+$ is well-defined and is given by $\xi_+(g \exp q)$. Moreover, for $g \in G$ and $x \in G/K$, $g\cdot\psi(x)$ is always well-defined and we have $g\cdot\psi(x) = \psi(gx)$. Let

(1.2) $$m_* = \exp \pi Z_0.$$

It is clear that $m_*$ lies in the center of $K$ and $\theta = \operatorname{Ad} m_*$. Thus $m_* \in N_K(A)$, the normalizer of $A := \exp \mathfrak{a}$ in $K$. Now it is easily seen that $m_*\cdot q = -q$ $(\forall q \in \mathscr{D})$, so $m_*$ gives the symmetry of $\mathscr{D}$ at the origin $0 \in \mathscr{D}$.

Put

(1.3) $$c := \exp \frac{\pi}{4} \sum_{j=1}^{\ell} (X_{\gamma_i} - X_{-\gamma_i}) \in G_{\mathbb{C}}.$$

Then $c \in P_+K_{\mathbb{C}}P_-$. Setting

(1.4) $$X_0 := \sum_{i=1}^{\ell} X_{\gamma_i}, \qquad H_0' := \sum_{i=1}^{\ell} H_{\gamma_i}', \qquad Y_0 := \sum_{i=1}^{\ell} X_{-\gamma_i},$$

we have

(1.5) $$\xi_+(c) = X_0, \qquad k(c) = \exp(\log\sqrt{2})H_0', \qquad \xi_-(c) = -Y_0.$$

Let $\tau$ be the conjugation in $\mathfrak{g}_{\mathbb{C}}$ relative to the compact real form $\mathfrak{k} + i\mathfrak{p}$. One knows that $(x,y) := -B(x,\tau y)$ $(x,y \in \mathfrak{g}_{\mathbb{C}})$ defines a hermitian inner product on $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{t}^- := \sum_{j=1}^{\ell} \mathbb{R}H_{\gamma_j} \subset i\mathfrak{t}$ and $\mathfrak{t}^+$ be the orthogonal complement to $\mathfrak{t}^-$ in $i\mathfrak{t}$. Set $\nu = \mathrm{Ad}_{G_{\mathbb{C}}} c$, where $c$ is the element in $G_{\mathbb{C}}$ defined by (1.3). Then $\nu$ is an isometry of $\mathfrak{g}_{\mathbb{C}}$ and we have

$$\nu(X_{\gamma_j}+X_{-\gamma_j}) = H'_{\gamma_j}, \qquad \nu(X_{\gamma_j}-X_{-\gamma_j}) = X_{\gamma_j} - X_{-\gamma_j},$$

(1.6)

$$\nu(H'_{\gamma_j}) = -(X_{\gamma_j}+X_{-\gamma_j}).$$

Hence $\mathfrak{t}^- = \nu(\mathfrak{a})$, and $\mathfrak{t}^+_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} = \nu^{-1}(\mathfrak{t}_{\mathbb{C}})$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. For every $\alpha \in \Delta$, $\mathrm{res}_{\mathfrak{t}^-}\alpha$ will stand for the restriction of $\alpha$ to $\mathfrak{t}^-$. We denote still by $\gamma_j$ the restriction $\mathrm{res}_{\mathfrak{t}^-}\gamma_j$.

Let $\alpha_j := \gamma_j \circ \nu$ $(j = 1,2,\ldots\ell)$. Since we are assuming that G/K is holomorphically equivalent to a tube domain, the restricted root theorem due to Moore [6] can be stated as follows.

**Theorem** 1.1 (Moore). *Let* $\Delta(\mathfrak{a})$ *be the* $\mathfrak{a}$*-root system. Then, the positive system* $\Delta(\mathfrak{a})^+$ *of* $\Delta(\mathfrak{a})$ *is described as*

$$\Delta(\mathfrak{a})^+ = \{\tfrac{1}{2}(\alpha_m+\alpha_k);\ 1\leq k\leq m\leq \ell\} \cup \{\tfrac{1}{2}(\alpha_m-\alpha_k);\ 1\leq k< m\leq \ell\}.$$

For any $\alpha \in \Delta(\mathfrak{a})$, we denote by $\mathfrak{g}_{\alpha}$ the corresponding $\mathfrak{a}$-root subspace. Put

(1.7) $u_k := \frac{i}{2}\left(H'_{\gamma_k} - X_{\gamma_k} + X_{-\gamma_k}\right)$ $(k = 1,2,\ldots,\ell)$.

Since $H'_{\gamma_k} \in i\mathfrak{t}$ and $X_{\gamma_k} - X_{-\gamma_k} \in i\mathfrak{p}$ (cf. (1.1)), we see that $u_k \in \mathfrak{g}$. Moreover, (1.6) leads us to $\nu^{-1}(X_{\gamma_k}) = iu_k$, so that $u_k \in \mathfrak{g}_{\alpha_k}$.

Let

(1.8) $s := \sum_{k=1}^{\ell} u_k \in \mathfrak{g}(1)$, $a_0 := \sum_{k=1}^{\ell} \frac{1}{2}(X_{\gamma_k} + X_{-\gamma_k}) \in \mathfrak{a}$.

Then, ad $a_0$ is semisimple. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and put

$$\mathfrak{g}(0) = \mathfrak{m} + \mathfrak{a} + \sum_{k<m} (\mathfrak{g}_{(\alpha_m-\alpha_k)/2} + \mathfrak{g}_{-(\alpha_m-\alpha_k)/2}),$$

$$\mathfrak{g}(1) = \sum_{k\leq m} \mathfrak{g}_{(\alpha_m+\alpha_k)/2}, \qquad \mathfrak{g}(-1) = \sum_{k\leq m} \mathfrak{g}_{-(\alpha_m+\alpha_k)/2}.$$

Then, $\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1)$, an orthogonal direct sum of vector subspaces. It is easy to see that $\mathfrak{g}(k)$ is the k-eigenspace of ad $a_0$. Letting $\mathfrak{g}(k) = \{0\}$ for $|k| > 1$, we have

(1.9) $[\mathfrak{g}(k),\mathfrak{g}(m)] \subset \mathfrak{g}(k+m)$.

We also have

(i) $\dim \mathfrak{g}_{\alpha_k} = 1$ for all $1\leq k\leq \ell$,

(1.10) (ii) $a := \dim \mathfrak{g}_{(\alpha_m-\alpha_k)/2} = \dim \mathfrak{g}_{(\alpha_m+\alpha_k)/2}$ is independent of $m$, $k$ $(m > k)$.

## §2. **Realization of** $G/K$ **as a tube domain.**

2.1. **Basic facts about Jordan algebras.** We begin this section with the definition of Jordan algebra. Our reference is the book [1]. Let $\mathfrak{U}$ be a finite dimensional vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A *product* $x,y \mapsto xy$ in $\mathfrak{U}$ is, by definition, a bilinear mapping $\mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$. The associative law is not assumed here. The vector space $\mathfrak{U}$, equipped with a product, is called a *Jordan algebra* if

(J-1) $$xy = yx,$$

(J-2) $$x^2(xy) = x(x^2y)$$

hold for all $x,y \in \mathfrak{U}$. Now let $\mathfrak{U}$ be a Jordan algebra with the unit element $e$. For $x \in \mathfrak{U}$, we define a linear operator $L(x)$ on $\mathfrak{U}$ by

(2.1) $$L(x)y = xy.$$

Then we have $L(x)y = L(y)x$ and the assignment $x \mapsto L(x)$ is clearly linear. In terms of these operators, (J-2) is rewritten as

$$[L(x),L(x^2)] = 0.$$

We know that any Jordan algebra is *power-associative*, that is, defining the power $x^n$ of an element $x \in \mathfrak{U}$ by $x^n = xx^{n-1}$ inductively, we have $x^m x^n = x^{m+n}$. Therefore the subalgebra $\mathbb{K}[x]$

generated by $e$ and $x$ is associative. Set

$$P(x) = 2L(x)^2 - L(x^2) \qquad (x \in \mathfrak{A}). \tag{2.2}$$

The mapping $x \mapsto P(x)$ is called the *quadratic representation* of $\mathfrak{A}$. It is well-known that $P(x^n) = P(x)^n$ $(n = 1, 2, \ldots)$. Furthermore we have the following formula named as the *fundamental formula*:

$$P(P(x)y) = P(x)P(y)P(x) \qquad (\forall x, y \in \mathfrak{A}). \tag{2.3}$$

An element $x \in \mathfrak{A}$ is said to be *invertible* if one of the following three mutually equivalent conditions holds:

(i) The operator $P(x)$ is invertible, that is, $\det P(x) \neq 0$.

(ii) There is $y \in \mathbb{K}[x]$ such that $xy = e$.

(iii) There is $y \in \mathfrak{A}$ such that $[L(x), L(y)] = 0$ and $xy = e$.

Then, if $x$ is invertible, the $y$ in (ii) or (iii) is uniquely given by $y = P(x)^{-1}x$, and will be written as $x^{-1}$. The set of all invertible elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}^\times$. Moreover $P(x^{-1}) = P(x)^{-1}$ holds for any $x \in \mathfrak{A}^\times$.

Now let $\mathfrak{A}$ be a real Jordan algebra. $\mathfrak{A}$ is said to be *formally real* if

(FR-1) $x^2 + y^2 = 0$ implies $x = y = 0$.

It is known that (FR-1) is equivalent to the following (FR-2):

(FR-2) the symmetric bilinear form $x, y \mapsto \operatorname{tr} L(xy)$ is positive definite.

We remark here that the linear form $\mathfrak{A} \ni x \mapsto \operatorname{tr} L(x)$ is associative in the sense that

$$\text{(2.4)} \qquad \operatorname{tr} L((xy)z) = \operatorname{tr} L(x(yz)) \qquad (\forall x,y,z \in \mathfrak{A}).$$

In particular, the operators $L(x)$ (hence $P(x)$, too) are symmetric with respect to the bilinear form $\operatorname{tr} L(xy)$.

We assume now that $\mathfrak{A}$ is a formally real Jordan algebra. Then $\mathfrak{A}$ has the unit element $e$. The *positive cone* $\Omega$ is, by definition, the interior of the squares, *i.e.*, $\Omega = \operatorname{Int}\{x^2;\ x \in \mathfrak{A}\}$. $\Omega$ is an open convex cone in $\mathfrak{A}$ and selfdual with respect to the inner product $\operatorname{tr} L(xy)$:

$$\Omega = \{y \in \mathfrak{A};\ \operatorname{tr} L(xy) > 0 \ \text{ for all } \ x \in (\mathrm{C}\ell\ \Omega)\setminus\{0\}\}.$$

We note:

(i) $\Omega$ coincides with the connected component of $\mathfrak{A}^{\times}$ containing $e$.

(ii) $x \in \Omega$ if and only if $L(x)$ is positive definite.

(iii) If $x \in \Omega$, then $P(x)$ is positive definite.

Finally, since the mapping $\Omega \ni x \mapsto x^2 \in \Omega$ is a diffeomorphism (its tangent mapping at $x_0 \in \Omega$ is $2L(x_0)$), its inverse mapping will be denoted by $\Omega \ni y \mapsto y^{1/2} \in \Omega$.

2.2. **Jordan algebra structure on** $\mathfrak{g}(1)$. We retain the notation of §1 and recall the element $s$ defined by (1.8).

**Lemma** 2.1. (i) *The real vector space* $\mathfrak{g}(1)$ *has a structure of Jordan algebra by* $x\cdot y = -\frac{1}{2}[[x,\theta s],y]$ $(x,y \in \mathfrak{g}(1))$. *The unit element is* $s$.
(ii) *Let* $L(x)$ *be the operator defined by* $L(x)y = x\cdot y$ $(x,y \in \mathfrak{g}(1))$. *Then,* $\mathrm{tr}_{\mathfrak{g}(1)}L(x\cdot y) = -2B(x,\theta y)$, *so that* $\mathfrak{g}(1)$ *with the product in* (i) *is a formally real Jordan algebra.*

Henceforth we denote by $\mathfrak{A}$ the formally real Jordan algebra described in Lemma 2.1. Now consider the complexification $\mathfrak{g}(1)_{\mathbb{C}}$. The product $x\cdot y$ in $\mathfrak{g}(1)$, which is a real bilinear mapping, is naturally extended to a complex bilinear mapping $\mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1)_{\mathbb{C}} \to \mathfrak{g}(1)_{\mathbb{C}}$. It is easy to see that the complex vector space $\mathfrak{g}(1)_{\mathbb{C}}$ with this complex bilinear product becomes a Jordan algebra. We denote by $\mathfrak{A}_{\mathbb{C}}$ the complex Jordan algebra thus obtained. The multiplication operators, the quadratic representation of $\mathfrak{A}_{\mathbb{C}}$ are still denoted by $L(x)$, $P(x)$ respectively.

Consider the tube domain $T_\Omega := \mathfrak{A} + i\Omega \subset \mathfrak{A}_{\mathbb{C}}$.

**Lemma** 2.2. (i) *One has* $T_\Omega \subset (\mathfrak{A}_{\mathbb{C}})^{\times}$, *that is, every* $z \in T_\Omega$ *is invertible in the Jordan algebra* $\mathfrak{A}_{\mathbb{C}}$.
(ii) *If* $z \in T_\Omega$, *then* $-z^{-1} \in T_\Omega$. *Moreover the mapping* $T_\Omega \ni z \mapsto -z^{-1} = -P(z)^{-1}z \in T_\Omega$ *is holomorphic and has the unique fixed point* is, *where* $s$ *is the unit element of* $\mathfrak{A}_{\mathbb{C}}$ *defined by* (1.8).

**Sketch**. Let $z \in T_\Omega$ and put $z = x + iy$ with $x \in \mathfrak{A}$ and $y \in \Omega$. (i) Set $u = y^{1/2} \in \Omega$. Then,

$$x + iy = P(u)(P(u)^{-1}x + is). \tag{2.5}$$

Thus it suffices to consider the elements of the form $x + is$

$(x \in \mathfrak{U})$. But the following formula shows that $x + is$ is invertible:

$$P(x+is)P(x-is) = P(x^2+s),$$

because $x^2 + s \in \Omega$. (ii) Since $-(x+is)^{-1} = -(x^2+s)^{-1}(x-is)$ (this computation is done in the associative algebra $\mathbb{C}[x]$), we see immediately $-(x+is)^{-1} \in T_\Omega$. Thus by (2.5), $-z^{-1} \in T_\Omega$ for any $z \in T_\Omega$. For the rest, it suffices to solve the equations $x^2 - y^2 + s = 0$, $x \cdot y = 0$. Q.E.D.

On the other hand, one knows that $c \cdot \mathcal{D} \subset \mathfrak{p}_+$ and that $\nu^{-1} \circ c(\mathcal{D}) = T_\Omega$ (note $\nu(\mathfrak{g}(1)_\mathbb{C}) = \mathfrak{p}_+$). Thus $T_\Omega$ realizes $G/K$ and $G$ acts on $T_\Omega$ by

$$(2.6) \qquad g \cdot z = \nu^{-1}(c \cdot (g \cdot q)) \qquad (g \in G,\ z \in T_\Omega),$$

where $q = c^{-1} \cdot (\nu(z)) \in \mathcal{D}$. We will make (2.6) more explicit for some elements of $G$.

Let $G(0) = Z_G(a_0)$, the centralizer in $G$ of the $a_0 \in \mathfrak{a}$ defined by (1.8). Then, $\mathfrak{g}(0) = \mathrm{Lie}\ G(0)$ and $G(0)$ is reductive. Let $G(1) = \exp \mathfrak{g}(1)$ and $P_0 := G(1)G(0)$. Then, $P_0$ is a maximal parabolic subgroup of $G$ with $G(0)$ a Levi part, $G(1)$ the unipotent radical. We have $cP_0c^{-1} \subset P_+K_\mathbb{C}$, so that

$$(2.7) \qquad g \cdot z = \nu^{-1}\zeta_+(cgc^{-1} \exp \nu(z)) \qquad (g \in P_0,\ z \in T_\Omega).$$

Now let $g_0 \in G(0)$. Then $cg_0c^{-1} \in K_\mathbb{C}$, and so

$$(2.8) \qquad g_0 \cdot z = (\mathrm{Ad}\ g_0)z \qquad (g_0 \in G(0),\ z \in T_\Omega).$$

For $a \in \mathfrak{g}(1)$, recalling $\nu(\mathfrak{g}(1)_{\mathbb{C}}) = \mathfrak{p}_+$, we see easily that

$$(2.9) \qquad (\exp a)\cdot z = z + a \qquad (a \in \mathfrak{g}(1),\ z \in T_\Omega).$$

Finally for the element $m_*$ defined by (1.2), we get

$$(2.10) \qquad m_*\cdot z = -z^{-1}.$$

Since $P_0$ and the element $m_*$ generate $G$, the formulas (2.8) ~ (2.10) describe the $G$-action on $T_\Omega$.

## §3. **Holomorphic discrete series.**

3.1. **Realization on** $D$. We retain the notation in the preceding sections. Let $\Lambda$ be a $K$-dominant $K$-integral form on $\mathfrak{t}_{\mathbb{C}}$: thus

(i) $\Lambda(H_\alpha) \geq 0$ for all $\alpha \in \Delta_c^+$,
(ii) $\xi_\Lambda(h) := \exp \Lambda(\log h)$ is a character of $T := \exp \mathfrak{t}$.

We denote by $\tau_\Lambda$ the irreducible unitary representation of $K$ on a finite dimensional Hilbert space $E_\Lambda$ with highest weight $\Lambda$. The inner product on $E_\Lambda$ is written as $(\cdot,\cdot)_\Lambda$. We describe here a realization of holomorphic discrete series of $G$ following Vergne-Rossi [9]. Let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Every element of $U(\mathfrak{g}_{\mathbb{C}})$ is canonically considered as a left invariant differential operator on $G$. Let $\mathcal{O}(\Lambda)$ be the space of all $E_\Lambda$-valued $C^\infty$-functions on $G$ such that

(i) $\varphi(xk) = \tau_\Lambda(k)^{-1}\varphi(x) \qquad (x \in G,\ k \in K)$,

(ii) $X\varphi = 0$ for all $X \in \mathfrak{p}_-$,

(iii) $\int_G |\varphi(x)|^2 \, dx < \infty$,

where $dx$ is the Haar measure on $G$. We define an inner product on $\mathcal{O}(\Lambda)$ by

$$(\varphi_1, \varphi_2) := \int_G (\varphi_1(x), \varphi_2(x))_\Lambda \, dx.$$

Then, one knows that $\mathcal{O}(\Lambda)$ with this inner product is a Hilbert space. $G$ acts on $\mathcal{O}(\Lambda)$ by left translations: $L_\Lambda(g)\varphi(x) = \varphi(g^{-1}x)$. The representation $(U_\Lambda, \mathcal{O}_\Lambda)$ of $G$ is irreducible and belongs to holomorphic discrete series of $G$ if $\dim \mathcal{O}(\Lambda) > 0$. By Harish-Chandra [4], the condition $\dim \mathcal{O}(\Lambda) > 0$ is equivalent to

$$(3.1) \qquad (\Lambda+\rho)(H_\beta) < 0 \quad \text{for all} \quad \beta \in \Delta_n^+,$$

where $2\rho = \sum_{\alpha \in \Delta^+} \alpha$.

We will assume from now on that $\Lambda$ satisfies (3.1), so that $\mathcal{O}(\Lambda) \neq \{0\}$. In order to get a relization of holomorphic discrete series on a function space on $D$, one needs a map $\Phi\colon G \to GL(E_\Lambda)$ such that

$$\Phi(gk) = \Phi(g)\tau_\Lambda(k) \qquad (g \in G \quad k \in K),$$
$$X\Phi = 0 \qquad \text{for all} \quad X \in \mathfrak{p}_-.$$

We note that since $P_0 = G(0)G(1)$ is a parabolic subgroup of $G$, one has $G = P_0K$. Thus recalling $cP_0c^{-1} \subset P_+K_{\mathbb{C}}$, we get

$$(3.2)\qquad cG \subset cP_0K \subset (cP_0c^{-1})cK \subset P_+K_{\mathbb{C}}P_+K_{\mathbb{C}}P_-K \subset P_+K_{\mathbb{C}}P_-,$$

so that $k(cg) \in K_{\mathbb{C}}$ is well-defined for any $g \in G$. Extending $\tau_\Lambda$ to a holomorphic representation of $K_{\mathbb{C}}$, we now set, after Vergne-Rossi [9, p.18],

$$(3.3)\qquad \Phi_\Lambda(g) = \tau_\Lambda(k(c))^{-1}\tau_\Lambda(k(cg)).$$

Then we have immediately that $\Phi_\Lambda(e) = 1_{E_\Lambda}$ and

$$(3.4)\qquad \begin{aligned} &\Phi_\Lambda(gk) = \Phi_\Lambda(g)\tau_\Lambda(k) \qquad (g \in G,\ \ k \in K), \\ &X\Phi_\Lambda = 0 \qquad \text{for all } X \in \mathfrak{p}_-. \end{aligned}$$

We further extend $\tau_\Lambda$ to a representation of the semidirect product $P_+K_{\mathbb{C}}$ by defining $\tau_\Lambda(p) = 1_{E_\Lambda}$ for all $p \in P_+$. Noting that for $g \in P_0$, we have $k(cg) = k(cgc^{-1})k(c)$ by (3.2), we see

$$\Phi_\Lambda(g) = \tau_\Lambda(k(c)^{-1})\tau_\Lambda(cgc^{-1})\tau_\Lambda(k(c)) \qquad (g \in P_0).$$

Thus $\Phi_\Lambda|P_0$ is a representation of the parabolic subgroup $P_0$. Moreover by (1.5) and (1.6), we have

$$(3.5)\qquad \begin{aligned} &\Phi_\Lambda(g_0) = \tau_\Lambda(cg_0c^{-1}) && (g_0 \in G(0)), \\ &\Phi_\Lambda(\exp x) = 1_{E_\Lambda} && (x \in \mathfrak{g}(1)). \end{aligned}$$

We remark here that $[\det P(t)]^{-1/2}dt$, $dt$ being the Lebesgue measure on $\mathfrak{g}(1)$, is a $P_0$-invariant measure on $\Omega$, where $t \mapsto P(t)$ is the quadratic representation of the Jordan algebra $\mathfrak{A}$ described in Lemma 2.1 (recall that $P(t)$ is positive definite for $t \in \Omega$).

Let $S = (\exp \sum_{\alpha\in\Delta(a)^+} \mathfrak{g}_\alpha)A$, the Iwasawa solvable subgroup of $G$. Put $S(0) = G(0) \cap S$. We denote by $\eta_0$ the diffeomorphism of $\Omega$ onto $S(0)$ such that $(\mathrm{Ad}\,\eta_0(t))s = t$ $(t \in \Omega)$.

With these preparations, we now introduce a Hilbert space $H(\Lambda)$ of $E_\Lambda$-valued holomorphic functions $F$ on $T_\Omega$ such that

$$\|F\|^2 := \int_{T_\Omega} \|\Phi_\Lambda(\eta_0(y))^{-1}F(x+iy)\|_\Lambda^2 \frac{dxdy}{\det P(y)} < \infty.$$

Letting $\alpha(g) = g\cdot is \in T_\Omega$ $(g \in G)$, we define

$$T_\Lambda F(g) := \Phi_\Lambda(g)^{-1}F(\alpha(g)) \qquad (F \in H(\Lambda),\ g \in G).$$

Then $T_\Lambda$ is a unitary mapping from $H(\Lambda)$ onto $\mathcal{O}(\Lambda)$. Let $\pi_\Lambda(g) := T_\Lambda^{-1}L_\Lambda(g)T_\Lambda$ $(g \in G)$. To describe $\pi_\Lambda(g)$, we set

(3.6) $$J_\Lambda(g,\alpha(h)) := \Phi_\Lambda(gh)\Phi_\Lambda(h)^{-1} \qquad (g \in G,\ h \in S).$$

Then, one has

$$J_\Lambda(g_1g_2,z) = J_\Lambda(g_1,g_2\cdot z)J_\Lambda(g_2,z) \qquad (g_1,g_2 \in G,\ z \in T_\Omega).$$

Now, a simple computation yields

(3.7) $$\pi_\Lambda(g)F(z) = J_\Lambda(g^{-1},z)^{-1}F(g^{-1}\cdot z) \qquad (g \in G,\ z \in T_\Omega).$$

We note that since $\Phi_\Lambda|P_0$ is a representation, we have $J_\Lambda(g,z) = \Phi_\Lambda(g)$ for all $g \in P_0$ and $z \in T_\Omega$. We also note that by (3.3),

(3.8) $$J_\Lambda(k,is) = \tau_\Lambda(k) \qquad \text{for all } k \in K.$$

3.2. **Some integrals over** $\Omega$. Let us set $\langle x,y\rangle := -B(x,\theta y)$ for $x,y \in \mathfrak{g}(1)$. Then, $\langle\cdot,\cdot\rangle$ is an inner product of $\mathfrak{g}(1)$ relative to which $\Omega$ is selfdual (cf. Lemma 2.1 (ii)). For $\lambda \in \mathfrak{g}(1)$ and $u \in E_\Lambda$, define

$$\Gamma_\Lambda(\lambda;v) := \int_\Omega e^{-2\langle\lambda,t\rangle} \|\Phi_\Lambda(\eta_0(t))^{-1}v\|_\Lambda^2 \frac{dt}{\det P(t)}$$

(3.9)

$$E_\Lambda(\lambda) := \{v \in E_\Lambda;\ \Gamma_\Lambda(\lambda;v) < \infty\}.$$

It is an immediate consequence of the Minkowski's inequality that $E_\Lambda(\lambda)$ is a subspace of $E_\Lambda$. Moreover, we have

$$\Gamma_\Lambda(\lambda;v) = \infty \quad \text{for all} \quad \lambda \notin C\ell\ \Omega \quad \text{and non-zero} \quad v \in E_\Lambda,$$

(3.10)

$$E_\Lambda(\lambda) = E_\Lambda \quad \text{for all} \quad \lambda \in \Omega$$

(for a proof, see Rossi-Vergne [8, Lemmas 5.13 ~ 5.16]). Next we set for $\lambda \in \Omega$

(3.11) $$\Gamma_\Lambda(\lambda) := \int_\Omega e^{-2\langle\lambda,t\rangle} \Phi_\Lambda(\eta_0(t)^{-1})^*\Phi_\Lambda(\eta_0(t)^{-1}) \frac{dt}{\det P(t)},$$

where $\Phi_\Lambda(\eta_0(t)^{-1})^*$ denotes the adjoint operator of $\Phi_\Lambda(\eta_0(t)^{-1})$.

**Lemma** 3.1. *The integral in* (3.11) *is absolutely convergent for any* $\lambda \in \Omega$, *so that* $\Gamma_\Lambda(\lambda)$ *is a positive definite hermitian operator.*

The following estimate of $\|\Gamma_\Lambda(\lambda)\|$ plays an important role in the last part of Theorem 4.1 below.

**Proposition** 3.2. *There is a positive constant* $c_\Lambda$ *such that*

$$\|\Gamma_\Lambda(\lambda)\| \geq c_\Lambda \|\lambda\|^{(\Lambda+\rho)(H_0')} \qquad \textit{for all} \quad \lambda \in \Omega.$$

## §4. **Fourier transform of holomorphic discrete series.**

4.1. **Paley-Wiener theorem.** First of all, we note that if $F \in H(\Lambda)$, then for almost every $y \in \Omega$, the function

$$\mathfrak{U} \ni x \mapsto \Phi_\Lambda(\eta_0(y))^{-1} F(x+iy) \in E_\Lambda$$

is square integrable by Fubini's theorem. Hence we can consider its Fourier transform $\phi_y$: letting $\mathfrak{U}_t := \{x \in \mathfrak{U};\ \|x\| < t\}$ $(t = 1,2,\dots)$, we set

$$(4.1) \qquad \phi_y(\lambda) := \frac{1}{(2\pi)^{m/2}} \operatorname*{l.i.m.}_{t\to\infty} \int_{\mathfrak{U}_t} \Phi_\Lambda(\eta_0(y))^{-1} F(x+iy) e^{-i\langle\lambda,x\rangle}\, dx,$$

where $m = \dim \mathfrak{U} = \dim \mathfrak{g}(1)$.

On the other hand, recall the operator $\Gamma_\Lambda(\lambda)$ $(\lambda \in \Omega)$ defined by (3.11). We know by Lemma 3.1 that $\Gamma_\Lambda(\lambda)$ is positive definite hermitian. So, the positive definite square root $\Gamma_\Lambda(\lambda)^{1/2}$ is well-defined. We now introduce a Hilbert space $\hat{H}(\Lambda)$ of $E_\Lambda$-valued measurable functions $\phi$ on $\Omega$ such that

$$(4.2) \qquad \|\phi\|^2 := \int_\Omega \|\Gamma_\Lambda(\lambda)^{1/2}\phi(\lambda)\|_\Lambda^2\, d\lambda < \infty.$$

**Theorem** 4.1. *Let* $F \in H(\Lambda)$ *and define* $\phi_y$ *by* (4.1). *Then, there is a measurable* $E_\Lambda$*-valued function* $\phi$ *on* $\mathfrak{U}$ *with* $\mathrm{supp}\,\phi \subset \mathrm{C}\ell\,\Omega$ *such that*

$$\phi_y(\lambda) = e^{-\langle\lambda,y\rangle}\Theta_\Lambda(iy)^{-1}\phi(\lambda) \qquad (\lambda \in \mathfrak{U}).$$

*Moreover, one has* $\phi \in \hat{H}(\Lambda)$ *and the correspondence* $\mathcal{F}_\Lambda: H(\Lambda) \ni F \mapsto \phi \in \hat{H}(\Lambda)$ *is a unitary mapping. The inverse* $\mathcal{F}_\Lambda^{-1}: \hat{H}(\Lambda) \ni \phi \mapsto F \in H(\Lambda)$ *is given by the absolutely convergent integral*

$$(4.3) \qquad F(z) = \frac{1}{(2\pi)^{m/2}} \int_\Omega \phi(\lambda)e^{i\langle\lambda,z\rangle}\,d\lambda.$$

The absolute convergence of (4.3) is a consequence of the Schwarz inequality and of Proposition 3.2 together with the fact $\|\Gamma_\Lambda(\lambda)^{-1/2}\| = \|\Gamma_\Lambda(\lambda)\|^{-1/2}$.

4.2. **Holomorphic discrete series realized on** $\hat{H}(\Lambda)$. With the unitary mapping $\mathcal{F}_\Lambda$ in Theorem 4.1 at hand let us set $\hat{\pi}_\Lambda(g) := \mathcal{F}_\Lambda\pi_\Lambda(g)\mathcal{F}_\Lambda^{-1}$ $(g \in G)$. Then we get a holomorphic discrete series representation $\hat{\pi}$ of $G$ on $\hat{H}(\Lambda)$. We will describe representation operators $\hat{\pi}_\Lambda(g)$ $(g \in G)$. Recall the element $m_* \in N_K(A)$ defined by (1.2). Since $G$ is generated by $m_*$ and $P_0$, it suffices to describe $\hat{\pi}_\Lambda(g)$ $(g \in P)$ and $\hat{\pi}_\Lambda(m_*)$.

If $g \in P_0$, then since $J_\Lambda(g,z) = \Phi_\Lambda(g)$ for all $z \in T_\Omega$, we have

$$\pi_\Lambda(g)F(z) = \Phi_\Lambda(g)F(g^{-1}\cdot z) \qquad (g \in P_0,\ F \in H(\Lambda)).$$

Suppose further $g = g_0 \in G(0)$. Then by (2.8), we have $g_0^{-1}\cdot z =$

$(\mathrm{Ad}\ g_0)^{-1}z$. Therefore

$$\hat{\mathfrak{R}}_\Lambda(g_0)\phi(\lambda) = (\det_{\mathfrak{g}(1)}\mathrm{Ad}\ g_0)\Phi_\Lambda(g_0)\phi((\mathrm{Ad}\ g_0)^*\lambda) \qquad (\phi \in \hat{H}(\lambda)),$$

where $(\mathrm{Ad}\ g_0)^*$ is the adjoint to $(\mathrm{Ad}\ g_0)$ with respect to the inner product $\langle\cdot,\cdot\rangle$. Next let $g = \exp a$ $(a \in \mathfrak{g}(1))$. Then by (2.9), $g^{-1}\cdot z = z - a$, so that by virtue of (3.5)

$$\hat{\mathfrak{R}}_\Lambda(\exp a)\phi(\lambda) = e^{-i\langle\lambda,a\rangle}\phi(\lambda).$$

To describe $\hat{\mathfrak{R}}_\Lambda(m_*)$ we need the following lemma.

**Lemma** 4.2. (i) *Let* $r > 0$. *Then*

$$J_\Lambda(m_*^{-1}, rz)^{-1} = r^{\Lambda(H_0')} J_\Lambda(m_*^{-1}, z)^{-1} \qquad (z \in T_\Omega).$$

(ii) *One has*

$$\int_{\mathfrak{U}} \| J_\Lambda(m_*^{-1}, z)^{-1} \| \, dx < \infty \qquad (z = x+iy \ \textit{with} \ y \in \Omega).$$

We now define an operator valued function $\mathcal{I}_\Lambda$ on $\Omega \times \Omega$ by

$$(4.4) \qquad \mathcal{I}_\Lambda(t,\lambda) := \frac{1}{(2\pi)^m}\int_{\mathfrak{U}} J_\Lambda(m_*^{-1}, z)^{-1}\exp -i(\langle\lambda, z^{-1}\rangle + \langle t, z\rangle)\, dx$$
$$(z = x+iy, \quad y \in \Omega),$$

where $m = \dim \mathfrak{U}$. We note that since $-z^{-1} \in T_\Omega$ if $z \in T_\Omega$, we have $\mathrm{Im}\ \langle\lambda, z^{-1}\rangle < 0$ for $\lambda \in \Omega$. Therefore

$$|\exp -i(\langle\lambda, z^{-1}\rangle + \langle t, z\rangle)| = \exp \mathrm{Im}(\langle\lambda, z^{-1}\rangle + \langle t, z\rangle) \le e^{t\ \mathrm{Im}\ z}.$$

Thus the integral in (4.4) is absolutely convergent by Lemma 4.2. We also note that since the integrand in (4.4) is holomorphic, $\mathcal{J}_\Lambda(t,\lambda)$ is indeed independent of $y \in \Omega$. We call the function $\mathcal{J}_\Lambda(t,\lambda)$ the *Bessel kernel* associated to the holomorphic discrete series $\pi_\Lambda$.

**Theorem** 4.3. *One has a realization* $\hat{\pi}_\Lambda$ *of holomorphic discrete series of* G *on* $\hat{H}(\Lambda)$. *The representation operators are given by*

(i) $\hat{\pi}_\Lambda(g_0)\phi(t) = (\det_{\mathfrak{g}(1)}\mathrm{Ad}\ g_0)\Phi_\Lambda(g_0)\phi((\mathrm{Ad}\ g_0)^*t) \qquad (g_0 \in G(0)),$

(ii) $\hat{\pi}_\Lambda(\exp a)\phi(t) = e^{-i\langle t,a\rangle}\phi(t) \qquad (a \in \mathfrak{g}(1)),$

(iii) $\hat{\pi}_\Lambda(m_*)\phi(t) = \int_\Omega \mathcal{J}_\Lambda(t,\lambda)\phi(\lambda)\ d\lambda \qquad (\phi \in C_c^\infty(\Omega,E_\Lambda) \subset \hat{H}(\Lambda)).$

We close this note by showing that $\mathcal{J}_\Lambda(t,\lambda)$ is determined by $\mathcal{J}_\Lambda(t) := \mathcal{J}_\Lambda(t,s)$, where $s$ is the unit element of the Jordan algebra $\mathfrak{A}$, that is, the element given by (1.8).

Since $\Omega$ is diffeomorphic to $G(0) \cap \exp \mathfrak{p}$, there is, for each $t \in \Omega$, a unique element $p_0(t) \in G(0) \cap \exp \mathfrak{p}$ such that $(\mathrm{Ad}_{\mathfrak{g}(1)}p_0(t))s = t$. Recall here the quadratic representation $P(\cdot)$ of the Jordan algebra $\mathfrak{A}$. We have $P(t^{1/2})s = t$ for every $t \in \Omega$.

**Lemma** 4.4. $\mathrm{Ad}_{\mathfrak{g}(1)}p_0(t) = P(t^{1/2})$ *for all* $t \in \Omega$.

**Proposition** 4.5. *One has, for all* $t,\lambda \in \Omega$

$$\mathcal{J}_\Lambda(t,\lambda) = (\det_{\mathfrak{g}(1)}\mathrm{Ad}\ p_0(\lambda))\Phi_\Lambda(p_0(\lambda))\mathcal{J}_\Lambda(P(\lambda^{1/2})t)\Phi_\Lambda(p_0(\lambda)).$$

It would be interesting to study the operator valued function

$\mathcal{J}_\Lambda$ in detail. For $G = Sp(\ell,\mathbb{R})$, $SU(\ell,\ell)$ and $SO^*(4\ell)$, $\mathcal{J}_\Lambda$ is essentially the reduced Bessel function investigated by Gross-Kunze [3]. For $G$ equal to one of the above three groups or $SO_0(\ell,2)$ but with $\tau_\Lambda$ one dimensional, $\mathcal{J}_\Lambda$ is essentially the Bessel function studied by Faraut-Travaglini [2].

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