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Note on Hirzebruch's Proportionality Principle

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Abstract. A θ-stable homogeneous space G/H is introduced with the associated Riemannian space of compact type G_U/H_U. The equation among the characteristic classes over $\Gamma\backslash G/H$ inherits from the corresponding one over G_U/H_U. As an application we also obtain a certain necessary condition for the existence of a uniform lattice.

§1. INTRODUCTION

In [Hi], Hirzebruch showed

FACT(HIRZEBRUCH'S PROPORTIONALITY PRINCIPLE). *Let D be a bounded Hermitian symmetric domain, Γ a torsionless discrete cocompact subgroup of the automorphism group $Aut(D)$ of D, and M the compact Hermitian symmetric space dual to D. Then there is a real number $A = A(\Gamma)$ such that $c^\alpha(\Gamma\backslash D)[\Gamma\backslash D] = Ac^\alpha(M)[M]$ for any c^α, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index and $c^\alpha = c_1^{\alpha_1} \cup \cdots \cup c_k^{\alpha_k}$ is a monomial of Chern classes.*

The purpose of this note is to clarify this principle by eliminating unnecessary conditions. Let us explain the idea briefly in the above case. In order to compare D and M, we shall take a common complexification of D and

M which we look upon as real manifolds by forgetting the original complex structures. This enables us to treat non-Riemannian case and non-complex case as well as Hermitian case.

In this paper, we shall deal with not only characteristic numbers but also characteristic classes. Furthermore, we can replace the tangent bundles over Hermitian symmetric spaces by homogeneous vector bundles over a wide class of homogeneous spaces, – which we call *$\theta-$stable homogeneous space* (see §3)–, containing the cases where the isotropy subgroup is the group of the fixed points of an automorphism of finite order (eg. semisimple symmetric spaces), compact (homogeneous Riemannian spaces), or a Levi part of a parabolic subgroup, etc. Formulation and our main theorem are stated in §4, asserting that equations among characteristic classes ($\mathbb{R}-$coefficient) of a homogeneous vector bundle over a $\theta-$stable homogeneous space inherit from those of an associated Riemannian space of compact type. Our approach is elementary alike Weyl's unitary trick or Flensted-Jensen duality in representation theory (see [FJ]), and the results lead to interesting corollaries:

COROLLARY 1. *Let X be a Riemannian manifold of constant curvature. Then all the Pontrjagin class vanishes in $H^*(X;\mathbb{R})$.*

COROLLARY 2. *Let G be a semisimple Lie group contained in a connected complexified Lie group $G_{\mathbb{C}}$, Γ be any discrete subgroup of $G_{\mathbb{C}}$ acting on $G_{\mathbb{C}}/G$ freely and properly discontinuously. Then all the Pontrjagin class of $\Gamma\backslash G_{\mathbb{C}}/G$ vanishes in $H^*(\Gamma\backslash G_{\mathbb{C}}/G;\mathbb{R})$.*

COROLLARY 3. *Let G/H be a (not necessarily Riemannian) semisimple symmetric space and G_U/H_U the associated Riemannian symmetric space* *of compact type. Let Γ be any discrete subgroup of G acting on G/H freely and properly discontinuosly. If $\sum a_\alpha p^\alpha(G_U/H_U) = 0$ in $H^*(G_U/H_U;\mathbb{R})$, then $\sum a_\alpha p^\alpha(\Gamma\backslash G/H) = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{R})$. Here p^α denotes a monomial of Pontrjagin classes. Furthermore, if H is connected, the above result holds when we replace p^α by a monomial of Pontrjagin classes and the Euler class.*

COROLLARY 4. *Let H be the centralizer of a toral subgroup of a connected semisimple Lie group G, and G_U/H_U an associated Riemannian space of compact type (generalized flag variety). Then there is an embedding $G/H \hookrightarrow G_U/H_U$, through which G/H carries a $G-$invariant complex structure induced from a G_U-invariant complex structure on G_U/H_U. Let Γ be any discrete subgroup of G acting on G/H freely and properly discontinuosly. If $\sum a_\alpha c^\alpha(G_U/H_U) = 0$ in $H^*(G_U/H_U;\mathbb{R})$, then $\sum a_\alpha c^\alpha(\Gamma\backslash G/H) = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{R})$. Here c^α denotes a monomial of Chern classes.*

Corollary 1 can also be deduced from the following

FACT(1.1)([S]). *If X is a Riemannian manifold of constant curvature, then $X \times S^1$ admits a flat affine connection.*

The proof of this fact is not given in [S], so we show it for the sake of completeness (see Appendix B).

Note that H is noncompact in general. G/H is a bounded Hermitian symmetric domain in Corollary 4 if and only if $\mathrm{Ad}_G(H)$ is a maximal compact subgroup of the adjoint group $\mathrm{Int}(\mathfrak{g}) \equiv \mathrm{Ad}_G(G)$.

If Γ is a uniform lattice in G/H and H is connected, the converse statement of Corollary 3 and Corollary 4 also holds. It is well-known that there exists a uniform lattice in G/H when H is a compact and G is linear ([Bo]). On the other hand, when H is noncompact, a discrete subgroup of G does not necessarily act on G/H properly discontinuously. Various aspects arise about the discrete subgroup which can act properly discontinuously on G/H: some admit uniform lattices, some admit only finite groups. (see [Ko] and the references there). Applying the results to Euler class, we have

COROLLARY 5. *Let (G,H) be a linear θ-stable pair. If* $\operatorname{rank} G = \operatorname{rank} H$ *and* $\dim_{\mathbb{R}} \mathfrak{q} \cap \mathfrak{k}$ *is odd, then G/H admits no uniform lattice, that is, there exists no discrete subgroup Γ of G such that $\Gamma\backslash G/H$ is a compact smooth manifold.*

For example, let

$$G/H = SO(i+k, j+l)/SO(i,j) \times SO(k,l).$$

Then there is no uniform lattice of G/H when when three elements among i, j, k, l are odd and the other is even.

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§2. Preliminaries

In this section, we review the notion of invariant connection of reductive homogeneous space and the reduction of connections to real forms (cf. [N], [K-N]).

Let $\pi : P \to X$ be a smooth principal $H-$bundle. A connection on $P \to X$ is a splitting of the tangent bundle $TP \to P$ into an $H-$equivariant

Whitney sum $TP = Ver(P) \oplus Hor(P)$, where $Ver(P) = Ker(d\pi : TP \to TX)$ is the tangent bundle along fibers, and $Hor(P)$ is so called a horizontal subbundle. The connection form $\alpha \in \mathcal{E}^1(P, \mathfrak{h})$ is defined by the composition of $TP \to Ver(P)$, the first projection of the splitting $TP = Ver(P) \oplus Hor(P)$, and $Ver(P) \to \mathfrak{h}$, the inverse of $\mathfrak{h} \ni X \mapsto X_p^* \in Ver(P)_p$, where X^* denotes the fundamental vector field on P. The curvature form $\Omega \equiv D\alpha$ is the horizontal $\mathfrak{h}-$valued 2-form on P given by $\Omega(X,Y) \equiv D\alpha(X,Y) \stackrel{\text{def}}{=} d\alpha(\text{pr}X, \text{pr}Y)$ $(X, Y \in TP)$, where pr: $TP \to Hor(P)$ stands for the second projection of $TP = Ver(P) \oplus Hor(P)$.

Let H' be a subgroup of H, $P' \to X$ a smooth principal $H'-$bundle. $P' \to X$ is called a reduction of $P \to X$ if $P = P' \underset{H'}{\times} H$. If Y is a submanifold of X and a smooth principal $H'-$bundle $\pi' : Q \to Y$ is a reduction of $\pi_{|Y} : P_{|Y} \to Y$ satisfying,

$$(TQ)_p \subset Ker(d\pi'_p) \oplus Hor(P)_p, \tag{2.1}$$

for any $p \in Q$, we have a connection on Q induced from the one on P. Namely, let $Hor(Q)_p \stackrel{\text{def}}{=} Hor(P)_p \cap (TQ)_p$, then the subbundle $Hor(Q)$ of TQ determines a connection on $Q \to Y$.

For $E = P \underset{\rho}{\times} V$, the vector bundle associated to a representation $\rho : H \to GL(V)$, we have a connection induced from a connection on P. The curvature form Ω^E of this connection is a $End(E)$ valued 2-form described as follows:

$$\Omega^E(u,v) \stackrel{\text{def}}{=} [p, d\rho(\Omega(\widetilde{u}, \widetilde{v}))],$$

via the identification $P \underset{Ad(\rho)}{\times} End(V) = End(E)$. Here for $x \in X, p \in P$

with $\pi(p) = x$, $\widetilde{u}, \widetilde{v} \in (TP)_p$ are lifts of $u, v \in (TX)_x$ respectively, and $d\rho$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ induced from ρ.

We call a homogeneous space G/H is *reductive* when there exists an $Ad(H)-$stable vector subspace $\mathfrak{q}$ complementary to $\mathfrak{h}$ in $\mathfrak{g}$. For a reductive homogeneous space G/H, a connection on a principal $H-$bundle $G \to G/H$ is defined as follows: for $g \in G$,

$$Hor(G)_g \stackrel{\text{def}}{=} L_{g*}\mathfrak{q}.$$

This connection is called the canonical connection of the second kind on G/H in [N]. The curvature form is given by $\Omega_o(X,Y) = -[X,Y]_{|\mathfrak{h}}$ where $Z_{|\mathfrak{h}}$ denotes the $\mathfrak{h}$ component of $Z \in \mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, o is the origin corresponding to the identity element of G and $X, Y \in \mathfrak{q}$.

For any reductive homogeneous space G/H contained in its complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$, the canonical connection of the second kind on $G_{\mathbb{C}}/H_{\mathbb{C}}$ induces the one on G/H. In fact, $\mathfrak{g} \subset \mathfrak{h} \oplus \mathfrak{q}_{\mathbb{C}}$ implies the condition (2.1). Thus the principal bundle $G \to G/H$ inherits the connection from $G_{\mathbb{C}|G/H} \to G/H$.

§3. θ-STABLE PAIR

In this section we introduce a notion of a *θ-stable pair* (G, H) and construct an algebra homomorphism between the cohomology rings of $\Gamma\backslash G/H$ and of *the associated Riemannian space of compact type* G_U/H_U.

Let $\mathfrak{g}$ be a semisimple Lie algebra defined over $\mathbb{R}$. We call a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is *θ-stable* when there exists a Cartan involution θ of $\mathfrak{g}$ such that $\theta\mathfrak{h} = \mathfrak{h}$. Then the following lemma is proved by standard arguments (see [War] Ch.1 §1).

LEMMA(3.1). *Let $\mathfrak{h}$ be a θ-stable subalgebra in $\mathfrak{g}$, $\mathfrak{q}$ the orthogonal subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ gives a direct decomposition as a $\mathfrak{h}$-module. Furthermore, the adjoint representation $\mathrm{ad}_{|\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is semisimple. Especially, $\mathfrak{h}$ is a reductive Lie algebra, that is, $\mathfrak{h}$ is decomposed into a direct sum of the center and the semisimple ideal* $[\mathfrak{h}, \mathfrak{h}]$.

EXAMPLE(3.2). Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. The following subalgebras are θ-stable in $\mathfrak{g}$.

1) The centralizer (or normalizer) of a θ-stable subalgebra in $\mathfrak{g}$.
2) The fixed point subalgebra of a linear automorphism of $\mathfrak{g}$ of finite order ([He] p.277).
3) A semisimple subalgebra ([M]).

Now we introduce a notion of a '*θ-stable pair*'.

DEFINITION(3.3). Let G be a connected semisimple Lie group, H a closed subgroup of G. We call (G, H) a *θ-stable pair* when the following two conditions are satisfied:

a) There is a Cartan involution θ of G such that H has a polar decomposition $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding Cartan decomposition of $\mathfrak{g}$ and K is the connected subgroup of G with Lie algebra $\mathfrak{k}$.
b) The connected Lie subgroup corresponding to the Lie algebra $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes \mathbb{C}$ is closed in the adjoint group $Int(\mathfrak{g}_{\mathbb{C}})$.

When (G, H) is a θ-stable pair, we call G/H a *$\theta-$stable homogeneous* *space.*

When G has a faithful finite dimensional representation, we call (G, H) is a *linear θ-stable pair.* In this case, the connected components of H are finite because K is compact.

The condition $a)$ in the above definition implies $\theta\mathfrak{h} = \mathfrak{h}$, and so $\mathfrak{h}$ is a θ-stable subalgebra in $\mathfrak{g}$. Conversely if H is connected, the condition $a)$ can be replaced by the condition that $\mathfrak{h}$ is a θ-stable subalgebra in $\mathfrak{g}$.

Let (G, H) be a θ-stable pair. Then there is a closed subgroup $H_{\mathbb{C}}$ of a connected Lie group $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes \mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ respectively such that the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathbb{C}$ induces the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\iota} & G_{\mathbb{C}} \\ \cup & & \cup \\ H & \xrightarrow{\iota} & H_{\mathbb{C}}, \end{array}$$

and that

$$H_{\mathbb{C}} = \iota(H) \cdot (H_{\mathbb{C}})_o \tag{3.4}$$

(Say, choose $G_{\mathbb{C}}$ the adjoint group $Int(\mathfrak{g}_{\mathbb{C}})$ and put $H_{\mathbb{C}}$ by (3.4).)

Let (G, H) be a θ-stable pair, θ a Cartan involution of $\mathfrak{g}$ which makes $\mathfrak{h}$ stable, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Then we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{p} + \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p},$$

as a vector space. Let G_U be a connected Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_U = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Set $H_U = H_{\mathbb{C}} \cap G_U$. Then H_U, G_U are compact real forms of $H_{\mathbb{C}}, G_{\mathbb{C}}$ respectively, and we have a natural map

$$G/H \underset{\text{covering}}{\longrightarrow} \iota(G)/H_{\mathbb{C}} \cap \iota(G) \underset{\text{complexification}}{\hookrightarrow} G_{\mathbb{C}}/H_{\mathbb{C}} \underset{\text{complexification}}{\hookleftarrow} G_U/H_U.$$

We call G_U/H_U (resp. $G_{\mathbb{C}}/H_{\mathbb{C}}$) *an associated Riemannian space of compact type* (resp. *a complexification*) for a given θ–stable homogeneous space G/H.

REMARK(3.5). Each connected component of H_U meets $H_{\mathbb{C}}$. Moreover the cohomology ring $H^*(G_U/H_U;\mathbb{R})$ is independent of the choice of the above complex Lie group $G_{\mathbb{C}}$. This notice is sometimes convenient for actual calculation.

EXAMPLE(3.6). Let G be a connected semisimple Lie group. (G,H) is a θ-stable pair in either of the following cases:

1) H is the centralizer (or normalizer) in G of a θ-stable subalgebra $\mathfrak{t}$. When $\mathfrak{t}$ is a θ-stable abelian subspace, an associated Riemannian space of compact type G_U/H_U is called a (generalized) flag variety (cf. lemma(6.1)).
2) H is an open subgroup in the group of the fixed points of an automorphism σ of finite order of G. When σ is involutive, the homogeneous space G/H is called *a semisimple symmetric space.*
3) H is a semisimple connected subgroup in G ([Y] guarantees that $H_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$).
4) H is compact.

Let (G,H) be a θ-stable pair. Then G/H, G_U/H_U and $G_{\mathbb{C}}/H_{\mathbb{C}}$ are reductive homogeneous spaces in the sense of §2 with complementary subspaces $\mathfrak{q}$, $\mathfrak{q}_U = \mathfrak{q}\cap\mathfrak{k}+\sqrt{-1}\mathfrak{q}\cap\mathfrak{p}$ and $\mathfrak{q}\otimes\mathbb{C}$ respectively. Therefore invariant forms are identified with the invariant elements in the exterior algebra of the cotangent space at the origins. Namely,

$$\mathcal{E}^*(G/H;\mathbb{R})^G \simeq (\bigwedge \mathfrak{q}^*)^H,$$

$$\mathcal{E}^*(G_U/H_U;\mathbb{R})^{G_U} \simeq (\bigwedge \mathfrak{q}_U{}^*)^{H_U},$$

and

$$\mathcal{E}^*(G_{\mathbb{C}}/H_{\mathbb{C}};\mathbb{C})^{G_{\mathbb{C}}} \simeq (\bigwedge \mathfrak{q}_{\mathbb{C}}{}^*)^{H_{\mathbb{C}}}.$$

Define a linear map $d : \bigwedge(\mathfrak{q}^*)^H \to \bigwedge(\mathfrak{q}^*)^H$ by

$$(dh)(X_1,\ldots,X_n) = \sum_{i<j}(-1)^{i+j}h([X_i,X_j]_{|\mathfrak{q}},X_1,\overset{i\ j}{\check{\ }\check{\ }}\ldots,X_n),\ (X_i \in \mathfrak{q}),$$

$d : \bigwedge(\mathfrak{q}_U^*)^{H_U} \to \bigwedge(\mathfrak{q}_U^*)^{H_U}$ by

$$(dh)(X_1,\ldots,X_n) = \sum_{i<j}(-1)^{i+j}h([X_i,X_j]_{|\mathfrak{q}_U},X_1,\overset{i\ j}{\check{\ }\check{\ }}\ldots,X_n),\ (X_i \in \mathfrak{q}_U),$$

and $d : \bigwedge(\mathfrak{q}_{\mathbb{C}}{}^*)^{H_{\mathbb{C}}} \to \bigwedge(\mathfrak{q}_{\mathbb{C}}{}^*)^{H_{\mathbb{C}}}$ by

$$(dh)(X_1,\ldots,X_n) = \sum_{i<j}(-1)^{i+j}h([X_i,X_j]_{|\mathfrak{q}_{\mathbb{C}}},X_1,\overset{i\ j}{\check{\ }\check{\ }}\ldots,X_n),\ (X_i \in \mathfrak{q}_{\mathbb{C}}).$$

Then it is easy to see that these d's correspond to the exterior derivatives under the above isomorphisms. Finally, define a linear isomorphism $\phi : \mathfrak{q} \to \mathfrak{q}_U$ by

$$\phi(X+Y) = X + \sqrt{-1}Y \quad (X \in \mathfrak{q}\cap\mathfrak{k}, Y \in \mathfrak{q}\cap\mathfrak{p}).$$

Then we have the following

LEMMA(3.7). *With notation as above, let ξ and ξ_U be invariant differential forms on G/H and G_U/H_U respectively. Assume ξ and ξ_U satisfy the following condition:*

$$\xi_U(\phi(X_1),\dots,\phi(X_a),\phi(Y_1),\dots\phi(Y_b)) = (\sqrt{-1})^b\xi(X_1,\dots,X_a,Y_1,\dots Y_b), \tag{3.8}$$

for any $X_i \in \mathfrak{q}\cap\mathfrak{k}, Y_j \in \mathfrak{q}\cap\mathfrak{p}$.

Then if ξ_U is an exact form, there is a $G-$invariant form η on G/H such that $\xi = d\eta$. If ξ_U is a closed form, ξ is also closed.

PROOF: The natural isomorphism (see Remark(3.5))

$$(\bigwedge \mathfrak{q}^*)^H \otimes \mathbb{C} \simeq (\bigwedge \mathfrak{q_C}^*)^{H_\mathbb{C}} \simeq (\bigwedge \mathfrak{q}_U{}^*)^{H_U} \otimes \mathbb{C}$$

induces

$$\mathcal{E}^*(G/H;\mathbb{R})^G \otimes \mathbb{C} \simeq \mathcal{E}^*(G_\mathbb{C}/H_\mathbb{C};\mathbb{C})^{G_\mathbb{C}} \simeq \mathcal{E}^*(G_U/H_U;\mathbb{R})^{G_U} \otimes \mathbb{C}.$$

The assumption (3.8) imply that ξ and ξ_U are the same images in the middle term. Suppose ξ_U is an exact form. Then there exists a G_U-invariant form η_U on G_U/H_U such that $d\eta_U = \xi_U$, because ξ_U is G_U-invariant and G_U is compact. Let $\eta \in \mathcal{E}^*(G/H;\mathbb{R})^G \otimes \mathbb{C}$ be the corresponding element of η_U under the above isomorphism. Then we have $\xi = d\eta$. The second claim is similar and easy. Thus the lemma is proved. ∎

PROPOSITION(3.9). *With notation as above, let Γ be a discrete subgroup of G acting on G/H freely and properly discontinuously. Then*

$$\mathcal{E}^*(G_U/H_U;\mathbb{R})^{G_U} \otimes \mathbb{C} \xrightarrow{\sim} \mathcal{E}^*(G/H;\mathbb{R})^G \otimes \mathbb{C} \hookrightarrow \mathcal{E}^*(\Gamma\backslash G/H;\mathbb{R}) \otimes \mathbb{C}$$

induces a $\mathbb{C}-$*algebra homomorphism*

$$\Upsilon : H^*(G_U/H_U;\mathbb{C}) \to H^*(\Gamma\backslash G/H;\mathbb{C}).$$

If $\Gamma\backslash G/H$ *is compact and* H *is connected, then* Υ *is injective.*

PROOF: The first claim is an immediate consequence of the preceding lemma. If $\Gamma\backslash G/H$ is compact and H is connected, G_U/H_U and G/H have G and G_U invariant orientation respectively. Therefore Υ is injective from the Poincaré duality. ∎

REMARK(3.10). For the injectivity of Υ, the assumption of connectedness of H can be replaced by $Ad(H)_{|\mathfrak{q}} \subset SL(\mathfrak{q})$, which means that G/H admits a $G-$invariant orientation. But in general Υ is neither injective nor surjective. As the proof of our theorem in §5 shows, Υ transfers the characteristic classes on G_U/H_U to the corresponding ones on $\Gamma\backslash G/H$.

REMARK(3.11). It is easy to see that G_U/H_U is a compact symmetric space if and only if (G,H) is a semisimple symmetric pair. It is a well known fact due to É.Cartan that $H^*(G_U/H_U;\mathbb{C}) \simeq \mathcal{E}^*(G_U/H_U;\mathbb{R})^{G_U} \otimes \mathbb{C}$ if G_U/H_U is a symmetric space.

§4. STATEMENT OF RESULTS

Let (G,H) be a θ-stable pair. Retain notations in §3. Let $\rho : H \to GL(V,\mathbb{R})$, $\rho_U : H_U \to GL(V_U,\mathbb{R})$ be finite dimensional representations. We call ρ and ρ_U has *the same complexification* when there are a complex vector space $V_{\mathbb{C}}$, a representation $\rho_{\mathbb{C}} : H_{\mathbb{C}} \to GL(V_{\mathbb{C}},\mathbb{C})$ and isomorphisms

$\psi : V \otimes \mathbb{C} \xrightarrow{\sim} V_{\mathbb{C}}$ and $\psi_U : V_U \otimes \mathbb{C} \xrightarrow{\sim} V_{\mathbb{C}}$ such that the following diagram commutes.

$$\begin{array}{ccccc} H & \xrightarrow{\iota} & H_{\mathbb{C}} & \hookleftarrow & H_U \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho_{\mathbb{C}}} & & \downarrow{\scriptstyle\rho_U} \\ GL(V,\mathbb{R}) & \underset{\psi_\sharp}{\hookrightarrow} & GL(V_{\mathbb{C}},\mathbb{C}) & \underset{\psi_{U\sharp}}{\hookleftarrow} & GL(V_U,\mathbb{R}). \end{array}$$

Now we are ready to state our main theorem.

THEOREM. *Let (G,H) be a θ-stable pair, G_U/H_U an associated Riemannian space of compact type, $G_{\mathbb{C}}/H_{\mathbb{C}}$ a complexification. Let Γ be any discrete subgroup of G acting on G/H freely and properly discontinuously from the left.*

1) Let $\rho : H \to GL(V,\mathbb{R})$, $\rho_U : H_U \to GL(V_U,\mathbb{R})$ be finite dimensional representations with the same complexification. Set ${}^{\Gamma}E \overset{\text{def}}{=} \Gamma\backslash G \underset{\rho}{\times} V$, $E_U \overset{\text{def}}{=} G_U \underset{\rho_U}{\times} V_U$.

If there is a relation $\sum a_\alpha p^\alpha(E_U) = 0$ in $H^*(G_U/H_U;\mathbb{R})$, then the equation $\sum a_\alpha p^\alpha({}^{\Gamma}E) = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{R})$ holds. Here $\alpha = (\alpha_1,\dots,\alpha_k)$ is a multi-index and $p^\alpha = p_1^{\alpha_1} \cup \cdots \cup p_k^{\alpha_k}$ is a monomial of Pontrjagin classes.

2) Let V be a finite dimensional vector space over $\mathbb{C}$, $\rho : H_{\mathbb{C}} \to GL(V,\mathbb{C})$ a representation of $H_{\mathbb{C}}$. Set ${}^{\Gamma}E \overset{\text{def}}{=} \Gamma\backslash G \underset{\rho|H}{\times} V$, $E_U \overset{\text{def}}{=} G_U \underset{\rho|H_U}{\times} V$.

If there is a relation $\sum a_\alpha c^\alpha(E_U) = 0$ in $H^*(G_U/H_U;\mathbb{R})$, then the equation $\sum a_\alpha c^\alpha({}^{\Gamma}E) = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{R})$ holds.

3)If $\Gamma\backslash G/H$ is compact and H is connected, the converse statement of 1) and 2) also holds.

EXAMPLE(4.1). Let $D = SO_o(n,2)/SO(n) \times SO(2)$ be a complex quadric, Γ a discrete cocompact subgroup of the automorphism group $Aut(D)$ of D, and M the compact Hermitian symmetric space dual to D. Then $c_j(\Gamma\backslash D) \neq 0$ for any j with $1 \leq j \leq n = \dim_{\mathbb{C}} D$ because we know that the corresponding result for M holds.

EXAMPLE(4.2). The total Chern class $c(\mathbb{C}P^n) = 1 + c_1(\mathbb{C}P^n) + \cdots + c_n(\mathbb{C}P^n)$ of a complex projective space $\mathbb{C}P^n$ is given by

$$c(\mathbb{C}P^n) \equiv (1+x)^{n+1} \mod x^{n+1},$$

where x is the first Chern class of the hyperplane section bundle. Let $X(p,q) = U(p+1,q)/U(1) \times U(p,q) \quad (p+q=n)$. Then $X(n,0) = \mathbb{C}P^n$ and $X(0,n)$ be the dual Hermitian symmetric domain of noncompact type (ref. [He] for the terminology). Let Γ be a discrete subgroup of $U(p+1,q)$ acting on $X(p,q)$ freely and properly discontinuously. Then we have

$$c_j(\Gamma\backslash X(p,q)) = \left(\prod_{l=0}^{j-1} \frac{n+1-l}{n+1}\right) c_1(\Gamma\backslash X(p,q))^j \quad (1 \leq j \leq n).$$

If Γ is a uniform lattice, $c_j(\Gamma\backslash X(p,q)) \neq 0$ for any j with $1 \leq j \leq n$. It can be proved that there exists a uniform lattice for $X(0,n), X(n,0)$ (Riemannian case) and $X(1,2r)$, whereas any discrete subgroup acting properly discontinuously on $X(p,q)$ with $p \geq q$ is finite (see [Ko]).

REMARK(4.3). We do not require that Γ is cocompact, so Theorem holds even when $\Gamma = 1$.

§5. PROOF OF THEOREM

Let (G,H) be a θ-stable pair. We retain notations in §2 and §3. As we prepared in §2, the curvature forms Ω and Ω_U of $G \to G/H$ and $G_U \to G_U/H_U$ are given by

$$\Omega_o(X,Y) = -[X,Y]_{|\mathfrak{h}} \quad (X,Y \in \mathfrak{q}),$$

$$\Omega_{Uo}(X_U,Y_U) = -[X_U,Y_U]_{|\mathfrak{h}_U} \quad (X_U,Y_U \in \mathfrak{q}_U),$$

where $Z_{|\mathfrak{h}}, Z_{|\mathfrak{h}_U}$ and $Z_{|\mathfrak{h}_{\mathbf{C}}}$ denote the second projections with respect to the decompositions $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}, \mathfrak{g}_U = \mathfrak{h}_U + \mathfrak{q}_U$ and $\mathfrak{g}_{\mathbf{C}} = \mathfrak{h}_{\mathbf{C}} + \mathfrak{q}_{\mathbf{C}}$ respectively. So the curvature forms Ω_o^E, $\Omega_o^{E_U}$ of homogeneous vector bundles $E \to G/H$ and $E_U \to G_U/H_U$ are given by

$$\Omega_o^E(X,Y) = -\rho([X,Y]_{|\mathfrak{h}}) \in \mathfrak{gl}(V),$$

$$\Omega_o^{E_U}(X_U,Y_U) = -\rho_U([X_U,Y_U]_{|\mathfrak{h}_U}) \in \mathfrak{gl}(V_U),$$

where we identify $\mathfrak{gl}(V)$ and $\mathfrak{gl}(V_U)$ with $End(E)_o$ and $End(E_U)_o$ respectively.

As Pontrjagin classes of a real vector bundle F are determined by Chern classes of its complexification $F_{\mathbf{C}} = F \otimes \underline{\mathbf{C}}$, we shall compare the curvatures of $E \otimes \underline{\mathbf{C}}$ and $E_U \otimes \underline{\mathbf{C}}$;

$$\Omega_o^{E\otimes\underline{\mathbf{C}}}(X,Y) = -\rho_{\mathbf{C}}([X,Y]_{|\mathfrak{h}}) \in \mathfrak{gl}(V_{\mathbf{C}}),$$

$$\Omega_o^{E_U\otimes\underline{\mathbf{C}}}(X_U,Y_U) = -\rho_{\mathbf{C}}([X_U,Y_U]_{|\mathfrak{h}_U}) \in \mathfrak{gl}(V_{\mathbf{C}}),$$

where in the first equality $\mathfrak{gl}(V_{\mathbf{C}})$ is identified with $End(E \otimes \underline{\mathbf{C}})_o$ and in the second equality $\mathfrak{gl}(V_{\mathbf{C}})$ is identified with $End(E_U \otimes \underline{\mathbf{C}})_o$ under ψ and ψ_U respectively (notation §4). Define a linear isomorphism $\phi : \mathfrak{q} \to \mathfrak{q}_U$ by

$$\phi(X+Y) = X + \sqrt{-1}Y \quad (X \in \mathfrak{q} \cap \mathfrak{k}, Y \in \mathfrak{q} \cap \mathfrak{p}).$$

Since $[X,Y]_{|\mathfrak{h}} = [X,Y]_{|\mathfrak{h}_{\mathbf{C}}}$ and $[\phi(X),\phi(Y)]_{|\mathfrak{h}_U} = [\phi(X),\phi(Y)]_{|\mathfrak{h}_{\mathbf{C}}}$, we have

$$[\phi(X),\phi(Y)]_{|\mathfrak{h}_U} = (\sqrt{-1})^{\delta(X)+\delta(Y)}[X,Y]_{|\mathfrak{h}},$$

and so

$$\Omega_o^{E_U\otimes\underline{\mathbf{C}}}(\phi(X),\phi(Y)) = (\sqrt{-1})^{\delta(X)+\delta(Y)}\Omega_o^{E\otimes\underline{\mathbf{C}}}(X,Y),$$

where X and Y are elements of $\mathfrak{q}\cap\mathfrak{k}$ or $\mathfrak{q}\cap\mathfrak{p}$ and we set $\delta(W) = 0$ if $W \in \mathfrak{q}\cap\mathfrak{k}$, $\delta(W) = 1$ if $W \in \mathfrak{q}\cap\mathfrak{p}$.

By Chern-Weil theory ([D],[K-N]), characteristic classes are represented by using curvatures. Namely, there is an $\mathbb{C}$−algebra homomorphism

$$w : Inv(L) \longrightarrow H^*(BL;\mathbb{C}),$$

for a Lie group L, where $Inv(L)$ denotes the ring of $\mathbb{C}$−valued invariant polynomials of the Lie algebra $\mathfrak{l}$ of L, and BL denotes the classifying space of a Lie group L. When L is a complex Lie group, we denote by $Inv_{\mathbb{C}}(L)$ the subring of $Inv(L)$ consisting of holomorphic polynoimials. The Chern classes are considered as elements of $H^*(BGL(n,\mathbb{C});\mathbb{R})$. For $f \in Inv(L)$ and a principal L bundle $P \to X$, the Chern class is represented by the differential form on X corresponding to the tensorial (i.e. L−invariant and horizontal) form $f(\Omega,\ldots,\Omega)$ on P where f is identified with its polarized multilinear form. If L is compact and connected, the homomorphism w is an isomorphism.

For a complex vector bundle $F \to X$ of rank n, the k-th Chern form c_k of F is represented by $f_k(\Omega^F,\ldots,\Omega^F)$ on X, where f_k is the homogeneous part of degree k in t of the real valued polynomial

$$\widetilde{f}(A)(t) = \det(I - \frac{t}{2\pi\sqrt{-1}}A) \quad (A \in \mathfrak{u}(n)).$$

This formula is also applicable for $GL(n,\mathbb{C})$ vector bundles and gives a representative of the total Chern class via the identification:

$$Inv_{\mathbb{C}}(GL(n,\mathbb{C})) \simeq Inv(U(n)) \simeq H^*(BU(n);\mathbb{C}) \simeq H^*(BGL(n,\mathbb{C});\mathbb{C}).$$

Considering f as a multilinear form as before, we have,

$$\begin{aligned}(5.1)\qquad & f(\Omega_o^{E_U\otimes\underline{\mathbb{C}}},\dots,\Omega_o^{E_U\otimes\underline{\mathbb{C}}})(\phi(X_1),\dots,\phi(X_a),\phi(Y_1),\dots\phi(Y_b))\\ &= (\sqrt{-1})^b f(\Omega_o^{E\otimes\underline{\mathbb{C}}},\dots,\Omega_o^{E\otimes\underline{\mathbb{C}}})(X_1,\dots,X_a,Y_1,\dots Y_b),\end{aligned}$$

where $X_i \in \mathfrak{q}\cap\mathfrak{k}, Y_j \in \mathfrak{q}\cap\mathfrak{p}$, and $f \in Inv(GL(n,\mathbb{C}))$.

If $[(wf)(E_U \otimes \underline{\mathbb{C}})] = 0$ in $H^*(G_U/H_U;\mathbb{R})$, there exists a $G-$invariant form η on G/H such that $d\eta = (wf)(E\otimes\underline{\mathbb{C}})$ owing to lemma(3.3). Since $f(\Omega_o^{E\otimes\underline{\mathbb{C}}},\dots,\Omega_o^{E\otimes\underline{\mathbb{C}}})$ and η are locally invariant (i.e. its pullback is $G-$invariant on G/H), the characteristic class $[(wf)(^\Gamma E\otimes\underline{\mathbb{C}})] = 0$ in $H^*(\Gamma\backslash G/H;\mathbb{C})$.

Applying this to the case that $[wf]$ is the image of $\sum a_\alpha c^\alpha$ under the homomorphism

$$H^*(BGL(n,\mathbb{C});\mathbb{R}) \to H^*(BGL(n,\mathbb{R});\mathbb{R}),$$

we get 1) in Theorem. The proof of 2) in Theorem is similar and 3) is derived from the last statement of Proposition(3.7).

§6. Proof of Corollaries

Proof of Corollary 1.

A Riemannian manifold of constant negative (otherwise the statement is obvious) curvature is a quotient of the n-dimensional hyperbolic space form $\mathbb{H}^n = SO_o(n,1)/SO(n)$ by a torsion free discrete group of isometries. Thus from the knowledge of $G_U/H_U = S^n$, we obtain Corollary 1.

Proof of Corollary 2.

The associated Riemannian space of compact type for the θ-stable pair $(G_{\mathbf{C}}, G)$ is $G_U \times G_U/\Delta G_U$, where $G_U = \{(g,g) \in G_U \times G_U; g \in G_U\}$. Since this space is diffeomorphic to a group manifold G_U, all the Pontrjagin class vanishes. Now, Corollary 2 is deduced from Theorem.

Proof of Corollary 3.

Corollary 3 holds when (G, H) is a θ-stable pair in general. Corollary 3 is almost proved by applying 1) in Theorem to the adjoint representations $Ad_{|H} : H \to GL(\mathfrak{q})$ and $Ad_{|H_U} : H_U \to GL(\mathfrak{q}_U)$. We only have to take account of Euler classes.

As G/H admits an indefinite metric by the Killing form restricted to $\mathfrak{q}$, the structure group of the tangent bundle can be reduced to $SO_o(p,q)$ for some $p, q \in \mathbb{N}$, where $p + q = \dim \mathfrak{q}$. To deal with Euler classes, we treat the complexified vector bundles again. From the fact that the rings of invariant polynomials of $SO_o(p,q)$, $SO(p+q)$ and the ring of the invariant holomorphic polynomials of $SO(p+q, \mathbb{C})$ are isomorphic, there is $P \in Inv_{\mathbf{C}}(SO(p+q, \mathbb{C}))$ such that

$$P_{|\mathfrak{so}(p+q)} = P^e,$$

where $P^e \in Inv(SO(p+q))$ is the invariant polynomial corresponding to the Euler class. Therefore in this case, we can calculate the Euler class by using P and the $SO_o(p,q)-$connection on G/H.

Proof of Corollary 4.

Let G be a connected semisimple Lie group and $G_{\mathbf{C}}$ a complex Lie group with complexified Lie algebra of G. Let θ be a Cartan involution of $\mathfrak{g}$, and

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. Let G_U be the compact real form of $G_{\mathbb{C}}$ whose Lie algebra is given by $\mathfrak{g}_U = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$. Fix an abelian subspace $\mathfrak{t}(\neq 0)$ in $\mathfrak{k}$. Let H, H_U, and $H_{\mathbb{C}}$ be the centralizers of $\mathfrak{t}$ in G, G_U, and $G_{\mathbb{C}}$ respectively. Fix a parabolic subgroup R of $G_{\mathbb{C}}$ with Levi part $H_{\mathbb{C}}$.
Then we have a generalized Borel embedding[1] :

LEMMA(6.1) (FOLKLORE). *With notation as above, both G_U/H_U and G/H are simply connected, and especially H_U and H are connected. Furthermore, there exists a G_U−invariant complex structure on a compact manifold $G_U/H_U = G_{\mathbb{C}}/R$, and G/H is realized in an open G−orbit of the identity coset of $G_{\mathbb{C}}/R$.*

We shall give a proof of this fact in Appendix A for the reader's convenience.

It is known that there is a Levi decomposition $R = H_{\mathbb{C}} \cdot N$, where N is the unipotent radical of R. As N is a normal subgroup in R, any representation $\rho_{\mathbb{C}} : H_{\mathbb{C}} \to GL(V, \mathbb{C})$ is extended to R by letting N act on V trivially. We also denote this extension by ρ for brevity.

As we define a complex structure on G_U/H_U by the isomorphism $G_U/H_U \simeq G_{\mathbb{C}}/R$, the holomorphic tangent bundle of G_U/H_U is given by $G_{\mathbb{C}} \underset{\mathrm{Ad}_{|R}}{\times} \mathfrak{g}_{\mathbb{C}}/\mathfrak{r} \simeq G_U \underset{\mathrm{Ad}_{|H_U}}{\times} \mathfrak{g}_{\mathbb{C}}/\mathfrak{r}$, and the holomorphic tangent bundle of

[1] [Griffiths-Schmid] (Acta. Math. 1969) treated when H is compact and called G/H dual manifolds of Kähler C-space. [Shapiro] (Comment. Math. Helv. 1971) treated when G/H is a semisimple symmetric space which was classified on the Lie algebra level in [Be].

G/H is given by $(G_{\mathbb{C}} \underset{\mathrm{Ad}_{|R}}{\times} \mathfrak{g}_{\mathbb{C}}/\mathfrak{r})_{|G/H} \simeq G \underset{\mathrm{Ad}_{|H}}{\times} \mathfrak{g}_{\mathbb{C}}/\mathfrak{r}$.

On the other hand, we have the following commutative diagrams.

$$\begin{array}{ccc} \mathfrak{r} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{r}) \\ \cup & & \uparrow \simeq \\ \mathfrak{h} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{q}) \end{array}$$

via the isomorphism $\mathfrak{q} \simeq \mathfrak{g}_{\mathbb{C}}/\mathfrak{r}$ induced from the inclusion $\mathfrak{q} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$, and

$$\begin{array}{ccc} \mathfrak{r} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{r}) \\ \cup & & \uparrow \simeq \\ \mathfrak{h}_U & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{q}_U) \end{array}$$

via the isomorphism $\mathfrak{q}_U \simeq \mathfrak{g}_{\mathbb{C}}/\mathfrak{r}$ induced from the inclusion $\mathfrak{q}_U \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. Therefore Corollary 3 is reduced to 2) in Theorem.

Proof of Corollary 5.

As G has a faithful finite dimensional representation, the connected components of H is finite. Therefore the non-existence of a uniform lattice in G/H is derived from the case where H is connected. When H is connected, H_U is also connected from (3.4), and the Euler number $\chi(G_U/H_U)$ does not vanish owing to Hirsch's formula[2] of the Poincaré polynomial of the maximal rank compact reductive pair (G_U, H_U). On the other hand, $\chi(\Gamma\backslash G/H) = 0$ because the tangent bundle $T(\Gamma\backslash G/H)$ splits according to the $H \cap K$ module decomposition $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$ and because $\dim_{\mathbb{R}} \mathfrak{q} \cap \mathfrak{k}$ is odd.

[2] H.Cartan, J.-L.Koszul, and J.Leray, Colloque de Topologie, Bruxelles, 1950

REMARK (6.2). When $\operatorname{rank} G = \operatorname{rank} H$, it is easy to see that $\dim_{\mathbb{R}} \mathfrak{q}$ is even. Therefore $\dim_{\mathbb{R}} \mathfrak{q} \cap \mathfrak{k}$ is odd if and only if $\dim_{\mathbb{R}} \mathfrak{q} \cap \mathfrak{p}$ is odd.

§7. APPENDIX

A. Proof of Lemma (6.1).

From definition we have

(A.1) $$G \cap H_{\mathbb{C}} = H,\ G_U \cap H_{\mathbb{C}} = H_U.$$

Since $\mathfrak{h}_{\mathbb{C}} = (\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}) + (\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}})$, both H_U and H are real forms of $H_{\mathbb{C}}$. Then H and H_U are connected because $K \cap H_{\mathbb{C}} = Z_K(\mathfrak{t})$ and $G_U \cap H_{\mathbb{C}} = Z_{G_U}(\mathfrak{t})$ are connected (see [He] Ch.7 Corollary 2.8.). As H and H_U contain the center of G and G_U respectively, G/H and G_U/H_U do not depend on the choice of coverings of G and $G_{\mathbb{C}}$. Thus both G/H and G_U/H_U are simply connected, and from now on we may assume that G is contained in its simply-connected complexification $G_{\mathbb{C}}$.

Fix a general element Z in $\sqrt{-1}\mathfrak{t}$ so that $\mathfrak{h}_{\mathbb{C}} = \{X \in \mathfrak{g}_{\mathbb{C}}; [Z, X] = 0\}$. Then $\mathfrak{g}_{\mathbb{C}}$ is decomposed into the negative, 0, and the positive eigenspaces of $\operatorname{ad}(Z)$, namely, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- + \mathfrak{h}_{\mathbb{C}} + \mathfrak{n}$. Let R (resp. R^-) be a parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}$ (resp. $\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}^-$). The natural inclusions $G \subset G_{\mathbb{C}} \supset G_U$ induce

$$G/G \cap R \subset G_{\mathbb{C}}/R \supset G_U/G_U \cap R.$$

We will show that

(A.2) $$\mathfrak{g} + (\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}) = \mathfrak{g}_U + (\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}) = \mathfrak{g}_{\mathbb{C}}.$$

(A.3) $$G \cap R = G \cap H_{\mathbb{C}}, G_U \cap R = G_U \cap H_{\mathbb{C}}.$$

Then (A.2) implies $G/G\cap R$ and $G_U/G_U\cap R$ are open sets in $G_{\mathbf{C}}/R$, and since G_U is compact we have $G_{\mathbf{C}}/R = G_U/G_U\cap R$. Using (A.3), we have $G/H \subset G_{\mathbf{C}}/R = G_U/H_U$ which will complete the proof of the lemma.

Now let us show (A.2), (A.3). let τ be a conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to a real form $\mathfrak{g}$ (or $\mathfrak{g}_U$). Since $Z \in \mathfrak{k} = \mathfrak{g}\cap\mathfrak{g}_U$, we have

$$\tau(\mathfrak{h}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}, \tau(\mathfrak{n}) = \mathfrak{n}^-,$$

and

$$\tau(\mathfrak{n}^-) = \mathfrak{n}.$$

We also denote by τ its lifting to an automorphism of a simply connected Lie group $G_{\mathbf{C}}$. Let X be any element of $\mathfrak{n}^-$. Then $X = -\tau(X) + (X + \tau(X)) \in \mathfrak{n} + \mathfrak{g}$ (or $\in \mathfrak{n} + \mathfrak{g}_U$), which shows (A.2). Let g be any element of $G\cap R$ (or $G_U \cap R$). Acting τ to the equation $g\,R\,g^{-1} = R$, we get $g\,R^-\,g^{-1} = R^-$. Because R and R^- are self-normalizing, g $\in R\cap R^- = H_{\mathbf{C}}$, proving (A.3).

REMARK (A.4). With notation as above, G/H is a semisimple symmetric space if and only if the nilradical $\mathfrak{n}_{\mathbf{C}}$ is abelian, and a bounded Hermitian symmetric domain if and only if H is maximal compact in G. These symmetric spaces are called '$\frac{1}{2}$ – *Kähler*' and '*Kähler*' respectively in Berger's classification ([Be]).

B. Proof of Fact (1.1).

The simply connected hyperbolic space form $\mathbb{H}^n$ can be embedded into $\mathbb{R}^{n,1}$, which is $\mathbb{R}^{n+1} = \{(x_0, \ldots, x_n); x_i \in \mathbb{R}\}$ equipped with the indefinite metric $dx_0^2 + \cdots + dx_{n-1}^2 - dx_n^2$. As the isometry group of $\mathbb{H}^n$ is a subgroup of index 2 in $O(n,1)$, M can be written as $\Gamma\backslash\mathbb{H}^n$ where Γ is a discrete subgroup of $O(n,1)$. For any fixed $r > 0\,(r \neq 1)$, we define $\varphi = \varphi_r : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ by the scalar multiplication of r. $M \times S^1$ is diffeomorphic to $\Gamma\times <\varphi>\backslash\mathbb{R}^{n,1}_+$, where $\mathbb{R}^{n,1}_+$ is $\{(x_0,\ldots,x_n); x_n > 0\}$ and $<\varphi>$ is the group generated by φ in $GL(n+1,\mathbb{R})$. As $\Gamma\times <\varphi>$ is a subgroup of $GL(n+1,\mathbb{R})$, the standard flat affine connection on $\mathbb{R}^{n+1}$ is preserved under the action of $\Gamma\times <\varphi>$. Therefore $M \times S^1$ admits a flat affine connection.

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