<table>
<thead>
<tr>
<th>Title</th>
<th>CFT on $\mathbb{P}^1$ and Monodromy Representations of the Braid Groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kanie, Yukihiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1989年08月号 700: 64-84</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1989-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/101479">http://hdl.handle.net/2433/101479</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
CFT on $\mathbb{P}^1$ and Monodromy Representations of the Braid Groups

(Talk at Research Institute of Mathematical Sciences,
Kyoto Univ., July 19, 1988)

Yukihiro Kanie

Mie University, Fac. of Education, Dept. of Math.

§0. From the differential equations of $N$-point functions of vertex operators in the conformal field theory on $\mathbb{P}^1$, arise the monodromy representations of the braid group $B_N$. In the meeting of last year, I reported that these monodromy representations give "all" irreducible representations of the Hecke algebra $H_N(q)$ of type $A_{N-1}$ (obtained by H. Wenzl [W]) associated with the affine Lie algebra of type $A_n^{(1)}$. In this meeting, I will report that associated with the affine Lie algebras of type $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$, the monodromy representations of the group $B_N$ give "all" irreducible representations of the Birman-Wenzl-Murakami algebra, the q-analogue of Brauer's centralizer algebras. Very important is Jimbo-Miwa-Okado's
calculations\cite{JMO}, and in the case of type $C_n^{(1)}$ the representations are equivalent to the ones obtained by J. Murakami\cite{M}.

§1. Let $g$ be the simple Lie algebra of type $X_n$, and $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}$ be the affine Lie algebra of type $X_n^{(1)}$. Fix an integer $\ell \geq 1$ and introduce the number $\kappa = \ell + g$, where $g$ is the dual Coxeter number of $\hat{g}$.

Denote by $P_+$ the set of dominant integral weights of $g$ and by $P_{\ell}$ the set of elements $\lambda \in P_+$ satisfying $(\theta, \lambda) \leq \ell$, where $\theta$ is the maximum root. For a weight $\lambda \in P_{\ell}$, we denote by $V_\lambda$ the irreducible representation of $g$ of highest weight $\lambda$, by $\mathcal{V}_\lambda$ the integrable representation of $\hat{g}$ of highest weight $\ell \Lambda_0 + \lambda$ and by $|\lambda\rangle$ the (fixed) highest weight cyclic vector of $V_\lambda$ and $\mathcal{V}_\lambda$.

The Virasoro algebra also acts on $\mathcal{V}_\lambda$ by the Sugawara forms $L(m), m \in \mathbb{Z}$, and the space $\mathcal{V}_\lambda$ is graded by means of the eigenspace decomposition w.r.t. the operator $L(0)$:

$$\mathcal{V}_\lambda = \sum_{d \in \mathbb{Z}_{\geq 0}} \mathcal{V}_\lambda(d), \mathcal{V}_\lambda(d) = \{v \in \mathcal{V}_\lambda; L(0)v = (\Delta_\lambda + d)v\},$$

where $\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa}$ and $\rho$ is the half sum of positive roots of $g$. Note that $\dim \mathcal{V}_\lambda(d) < \infty$ and $\mathcal{V}_\lambda(0) \cong V_\lambda$. 

- 2 -
There are dual right $g$ and $\hat{g}$-module $V^*_\lambda$ and $\mathfrak{h}^*_\lambda$, and the nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V^*_\lambda \otimes V^*_\lambda$ and $\mathfrak{h}^*_\lambda \otimes \mathfrak{h}^*_\lambda$ with the normalized condition $\langle \lambda | \lambda \rangle = 1$ where $\langle \lambda | \lambda \rangle$ is a fixed highest weight vector of $V^*_\lambda \otimes \mathfrak{h}^*_\lambda(0)$ and $\mathfrak{h}^*_\lambda$.

A triple $v = \begin{pmatrix} \lambda \\ \mu_2 \mu_1 \end{pmatrix}$ of weights in $P_\lambda$ is called a vertex and is drawn as

\[ v = \begin{array}{c} \lambda \\ \mu_2 \mu_1 \end{array} \]

A multi-valued, holomorphic function

\[ \Phi(z) : V_\lambda \otimes \mathfrak{h}^*_\mu_1 \rightarrow \mathfrak{h}^*_\mu_2 = \prod_{d \in \mathbb{Z}_{\geq 0}} \mathfrak{h}^*_\mu_2(d) \]

on $\mathbb{P}^1 \setminus \{0, \infty\}$ is called a vertex operator of type $\begin{pmatrix} \lambda \\ \mu_2 \mu_1 \end{pmatrix}$ (sometimes called of weight $\lambda$), if it satisfies the following:

(Gauge Condition) $[X(m), \Phi(z)(u \Theta \cdot)] = z^m \Phi(z)(Xu \Theta \cdot)$

\[ (X \in g, m \in \mathbb{Z}, u \in V_\lambda) ; \]

(Eq.of Motion) $[L(m), \Phi(z)] = z^m \{z \frac{d}{dz} + (m+1) \Delta_\lambda \} \Phi(z)$,
where $X(m)=X\otimes^m$ and the number $\Delta_{\lambda}$ is called the *conformal dimension* of the vertex operator $\Phi(z)$.

Denote by $\mathcal{F}(\nu)$ the space of all vertex operators of type $\nu$, and introduce the space

$$\mathcal{F}(\lambda) = \sum_{\mu_1, \mu_2 \in P_\lambda} \mathcal{F}(\left[ \begin{array}{c} \lambda \\ \mu_2, \mu_1 \end{array} \right])$$

of all vertex operators of weight $\lambda$.

Introduce the subalgebra $\mathfrak{sl}(2;\mathbb{C}) = \mathbb{C}\langle X_0, [X_0, X_{-1}], X_{-1} \rangle \cong \mathfrak{sl}(2;\mathbb{C})$ of $g$ and the subspace $\mathcal{F}(\nu)$ of $\text{Hom}_g (V_{\lambda, \mu_1}^\nu, V_{\mu_2}^\nu)$ defined by

$$\mathcal{F}(\nu) = \cap \ker \pi_{\theta} (j, j_1^\nu, j_2^\nu)$$

where the intersection is taken over the set $\{j, j_1, j_2 \in \frac{1}{2} \mathbb{Z}_{\geq 0}; j+ j_1 + j_2 \in \mathbb{Z}\}$, and $\pi_{\theta} (j, j_1^\nu, j_2^\nu)(\varphi) \in \text{Hom}_g (W_j \otimes W_{j_1} \otimes W_{j_2})$ is defined as

$$\pi_{\theta} (j, j_1^\nu, j_2^\nu)(\varphi) = \text{proj}_{W_{j_2}} \varphi |_{W_j \otimes W_{j_1}} \ (\varphi \in \text{Hom}_g (V_{\lambda, \mu_1}^\nu, V_{\mu_2}^\nu))$$

where $W_j, W_{j_1}, W_{j_2}$ are $\theta$-simple submodules of $V_{\lambda, \mu_1}^\nu, V_{\mu_2}^\nu$ with spin $j, j_1, j_2$ respectively.

By Equation of Motion, $\Phi$ is expressed as a formal
Laurent series

\[ \Phi(z) = \sum_{m \in \mathbb{Z}} \phi(m) z^{-m - \hat{\Delta}(\nu)}, \]

where \( \hat{\Delta}(\nu) = \Delta_{\lambda} + \Delta_{\mu_1} - \Delta_{\mu_2} \) and \( \phi(m) \) is homogeneous of degree \( m \), i.e.

\[ \phi(m): V_{\lambda} \otimes_{\mu_1} \rightarrow V_{\mu_2}^{(d-m)} \quad \text{for any } d. \]

The principal branch of \( \Phi(z) \) is taken such as the value of \( z^{-\hat{\Delta}(\nu)} \) is positive for \( z \in \mathbb{R}_+ = \{ z \in \mathbb{R}; z > 0 \} \) and uniquely continued to the region \( \mathbb{C}_+ = \{ z \in \mathbb{C}; \operatorname{Im} z > 0 \} \), and we refer this for the value of \( \Phi(z) \) on \( \mathbb{C}_+ \).

For any vertex operator \( \Phi \in \mathcal{F}_\ell_1(\nu) \), its initial term \( \varphi = \Phi(0) \big|_{V_{\lambda} \otimes_{\mu_1}^{\nu_2}(0)} = \operatorname{proj}_{V_{\lambda} \otimes_{\mu_1}^{\nu_2}} z^{\hat{\Delta}(\nu)} \Phi(z) \big|_{V_{\lambda} \otimes_{\mu_1}^{\nu_2}} \) belongs to \( \mathcal{F}(\nu) \). Under this correspondence,

**Theorem 1.** The space \( \mathcal{F}_\ell_1(\nu) \) of \( N \)-point functions of type \( \nu \) is isomorphic with the space \( \mathcal{F}(\nu) \) of initial terms of type \( \nu \).

Call \( \nu \) a \( \mathcal{CG}(\ell \text{-constrained Clebsch-Gordan}) \) vertex, if \( \mathcal{F}(\nu) \neq 0 \), and denote by \( \mathcal{CG} \) the set of all \( \mathcal{CG} \) vertices.
For each $\phi \in \mathcal{F}(v)$, denote by $\Phi_\phi$ the vertex operator with
the initial term $\phi$.

**Notes.**

i) Even if we assume that $\lambda \in P_+$ and $\mu_1 \in P_-$,
$\mathcal{F}(\begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array}) \neq 0$ implies that $\lambda \in P_-$.

ii) Operator product expansions of currents $X(z) = \sum_{m \in \mathbb{Z}} X(m) z^{-m-1}$ (with $X \in \mathcal{F}$) and the energy-momentum tensor $T(z) = \sum_{m \in \mathbb{Z}} L(m) z^{-m-2}$ with vertex operators allow the extension of
the vertex operators $\Phi(z)$ of type $\begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array}$ to the operators
$\Phi(z) : \begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array} \rightarrow \begin{array}{c} \hat{\lambda} \\
\hat{\mu}_2 \\
\hat{\mu}_1 \end{array}$ by means of contour integrals.

(*Nuclear Democracy*)

iii) By the same arguments as in §3, the analytic
continuation of a vertex operator $\Phi$ of type $\begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array}$ along
the path $\gamma_0$ gives a vertex operator of type $\begin{array}{c} \lambda \\
\mu_1 \\
\mu_2 \end{array}$, where

$$\gamma_0(t) = z e^{\pi i T t}, \quad t \in [0,1], \quad z \in \mathbb{R}_+.$$

This gives an isomorphism $C_{\gamma_0}$ of $\mathcal{F}(\begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array})$ to $\mathcal{F}(\begin{array}{c} \lambda \\
\mu_1 \\
\mu_2 \end{array})$
and the corresponding isomorphism

$$C_{\gamma_0} : \mathcal{F}(\begin{array}{c} \lambda \\
\mu_2 \\
\mu_1 \end{array}) \rightarrow \mathcal{F}(\begin{array}{c} \lambda \\
\mu_1 \\
\mu_2 \end{array})$$

- 6 -
is given by

$$C_{\nu_0} = e^{\pi \sqrt{-1}} \hat{\Delta}(\nu) T,$$

where $T$ is the transposition:

$$T : \text{Hom}(V \otimes_{\lambda} \mu_1, V \mu_2) \longrightarrow \text{Hom}(V \otimes_{\lambda} \mu_1, V \mu_2)$$

$$(T\phi)(u \otimes v) = \phi(v \otimes u).$$

A vertex operator $\Phi(z)$ of type $\nu$ is also considered as an operator from $\mathfrak{g}_{\mu_1}$ to $\mathfrak{h}_{\mu_2}$ parametrized by $V_{\lambda}$, i.e.

$$\Phi(u;z)(v) = \Phi(z)(u \otimes v), \quad (u \in V_{\lambda}, v \in \mathfrak{g}_{\mu_1}).$$

§2. It is convenient to introduce the spaces $\mathfrak{g} = \sum_{\lambda \in P_L} \mathfrak{g}_{\lambda}$
and $\hat{\mathfrak{g}} = \sum_{\lambda \in P_L} \hat{\mathfrak{g}}_{\lambda}$ and consider vertex operators as linear operators of $\mathfrak{g}$ to $\hat{\mathfrak{g}}$. The vacuum $|0\rangle$ of $\mathfrak{g}_0$ is called a Virasoro vacuum, since $L(m)|0\rangle = 0$ for $m \geq -1$. Note that $V_0 = \mathbb{C}|0\rangle$.

For an $N$-ple $\Lambda = (\lambda_N, \ldots, \lambda_1)$ of weights in $P_L$, denote

$$V_{\Lambda} = V_{\lambda_N} \otimes \cdots \otimes V_{\lambda_1} \quad \text{and} \quad V_{\Lambda}^\vee = \text{Hom}_\mathbb{C}(V_{\Lambda}, \mathbb{C}).$$

For any vertex operators $\Phi^i(z_i)$ of weight $\lambda_i (1 \leq i \leq N)$,
\[ \langle 0| \Phi^N(z_N) \cdots \Phi^1(z_1)|0 \rangle \]

is the coefficient of $|0\rangle$ the iterated application
\[ \Phi^N(z_N) \cdots \Phi^1(z_1)|0 \rangle \] to the vector $|0\rangle$, and this is a
\[ V^\Lambda_g(A) \] -valued formal Laurent series in $z_N, \ldots, z_1$ called the
$N$-point function of weight $\Lambda$ and is denoted by
\[ \langle \Phi^N(z_N) \cdots \Phi^1(z_1) \rangle \]. Denote by $\mathcal{P}_\Lambda(A)$ the space of all
$N$-point functions of weight $\Lambda$.

The space $V^\Lambda_g(A)$ is decomposed as
\begin{align*}
V^\Lambda_g(A) &= \sum_{\mu} V^\Lambda_g(A)_{\mu}, \\
&\quad \quad \mu = (\mu_{N-1}, \ldots, \mu_1) \in (P_+)^{N-1}; \\
&\quad \quad \quad \quad C_A \overset{\sim}{\simeq} \Hom^g(V, \otimes V_{\mu_{N-1}}, \otimes V_0) \otimes \cdots \otimes \Hom^g(V, \otimes V_{\mu_{i-1}}, \otimes V_{\mu_i}) \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \cdots \otimes \Hom^g(V, \otimes V_0, \otimes V_{\mu_1}),
\end{align*}

where the identification $C_A$ is given by
\begin{align*}
C_A(\varphi_N \otimes \cdots \otimes \varphi_1)(u_N \otimes \cdots \otimes u_1) \\
&= \langle 0| \varphi_N(u_N \otimes \varphi_{N-1}(\cdots \varphi_2(u_2 \otimes \varphi_1(u_1 \otimes 0)\cdots)) \\
&= \langle 0| \varphi_N(u_N) \otimes \cdots \otimes \varphi_1(u_1)(|0\rangle),
\end{align*}

for $\varphi_i \in \Hom^g(V, \otimes V_{\lambda_i}, \otimes V_{\mu_i}) \cong \Hom^g(V, \otimes \Hom(V, \otimes V_{\mu_{i-1}}, \otimes V_{\mu_i}))$,

(1 \leq i \leq N; $\mu_N = \mu_0 = 0$), and $u_N \otimes \cdots \otimes u_1 \in \mathcal{V}_A$. 

- 8 -
Introduce the subspace $\mathcal{V}(A)$ of $V^\vee_g(A)$ defined, through $\mathcal{C}_A$, by

$$\mathcal{V}(A) = \sum_{\mu} \mathcal{V}(A)_{\mu}, \quad \mu = (\mu_{N-1}, \ldots, \mu_1) \in (P_\mu)^{N-1};$$

where

$$\mathcal{V}(A)_{\mu} = \mathcal{V}(v_N(\mu)) \otimes \cdots \otimes \mathcal{V}(v_1(\mu)) \otimes \cdots \otimes \mathcal{V}(v_{N-2}(\mu)) \subset V^\vee_g(A)$$

and

$$v_N(\mu) = \begin{pmatrix} \lambda_N \\ \mu_{N-1} \end{pmatrix}, \ldots, v_i(\mu) = \begin{pmatrix} \lambda_i \\ \mu_{i-1} \end{pmatrix}, \ldots, v_1(\mu) = \begin{pmatrix} \lambda_1 \\ \mu_0 \end{pmatrix}.$$

Then the space $\mathcal{V}(A)$ is isomorphic to $\mathcal{V}_\ell(A)$ of $N$-point functions of weight $A$ as follows: to each $\varphi = C(\varphi_N \otimes \cdots \otimes \varphi_1) \in \mathcal{V}(A)$, assign the $N$-point function

$$\Phi_{\varphi_N \otimes \cdots \otimes \varphi_1}(z) = \langle \varphi_N(z_N) \cdots \varphi_1(z_1) \rangle \in \mathcal{V}_\ell(A).$$

Now introduce a system $KZ(A)$ of differential equations on $\text{Hom}_g(V_A, \mathbb{C})$-valued functions $\Phi(z)$ on $X_N = \{z = (z_N, \ldots, z_1) \in \mathbb{C}^N; \ z_i \neq z_j \ (i \neq j)\}$.
$$KZ(\Lambda) \left[ \frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \atop k \neq i}}^N \frac{\Omega_{ik}}{z_i - z_k} \right] \phi(z) = 0 \quad (1 \leq i \leq N)$$

due to Knizhnik-Zamolodchikov [KZ], where

$$\Omega_{ik} = \sum_{a=1}^{\text{dim} g} \rho_i(x^a) \rho_k(x_a),$$

$\rho_i$ denotes the $g$-action on the $i$-th component of $\text{Hom}(V_{\Lambda}, \mathbb{C})$ and $\{x^a\}$ and $\{x_a\}$ are dual bases of $g$.

Further introduce an additional $g$-constraint condition, i.e. a system $\mathcal{C}(A)$ of algebraic equations

$$\mathcal{C}(A) \sum_{|m_i|=L_i}^{L_i} \left[ \begin{array}{c} L_i \\ m_i \end{array} \right] \prod_{k \neq i} (z_k - z_i)^{-m_k} \phi(z)(x^N_1 u_N, \ldots, x^L_i u_L, \ldots, x^1_1 u_1) = 0,$$

$$(1 \leq i \leq N)$$

for any $u_k \in V_{\lambda_k}(k \neq i)$, where $m_i = (m^N_i, \ldots, \hat{m_i}, \ldots, m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$, $|m_i| = \sum_{k \neq i} m_k$, $L_i = \ell - (\lambda_i, \theta) + 1$ and $\left[ \begin{array}{c} L_i \\ m_i \end{array} \right]$ is the multinomial coefficient.

Remark. The system $KZ(\Lambda)$ of differential equations is completely integrable because of the infinitesimal pure braid relations among the operators $\Omega_{ik}$ (see [A]). The system $\mathcal{C}(A)$ is compatible with the system $KZ(\Lambda)$.

Any $N$-point function of weight $\Lambda$ satisfies the systems
Theorem 2.

i) For any N-ple $\Lambda$ of weights in $P_\Lambda$, any N-point function $\langle \Phi^N(z_N) \cdots \Phi^1(z_1) \rangle$ of weight $\Lambda$ is absolutely convergent in the region $\mathcal{R}_N$, and is analytically continued to a multivalued holomorphic function on $X_N$, where $\mathcal{R}_N$ is defined by

$$\mathcal{R}_N = \{ z = (z_N, \cdots, z_1) \in \mathbb{C}_+^N; |z_N| > \cdots > |z_1| \} \subset X_N.$$

ii) The solution space of the joint system $KZ(\Lambda)$ and $\mathcal{L}C(\Lambda)$ is isomorphic with $\mathcal{R}e\mathcal{L}(\Lambda)$, hence with $\mathcal{R}(\Lambda)$.

Note. If $\nu = \begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix} \in (CG)$, then $\mu = \lambda$, $\hat{\Delta}(\nu) = 0$, and $\mathcal{R}(\nu) \cong \text{Hom}_g(V^\lambda, V^\mu) = \mathbb{C} \text{id}$.

If $\nu = \begin{pmatrix} \lambda \\ \mu \\ 0 \end{pmatrix} \in (CG)$, then $\mu = \lambda^\ast$, $\hat{\Delta}(\nu) = 2\Delta_\lambda$, and $\mathcal{R}(\nu) \cong \text{Hom}_g(V^\lambda, \otimes V^\lambda, \mathbb{C}) = \mathbb{C} \nu$, where the anti-weight $\lambda^\ast$ of $\lambda$ is defined as $-\lambda^\ast (=\omega_0 \lambda)$ is the lowest weight of $V^\lambda$ and $\nu$ is normalised as $\nu(|\lambda> \otimes \omega_0 |\lambda^\ast>) = 1$, where $\omega_0$ is the longest element of the Weyl group of $g$.

3-point functions are essentially nothing but vertex
operators. The assignation to $\varphi \in FC\left( \begin{array}{c} \lambda \\ \mu_2 \end{array} \mu_1 \right)$ the element

$$\nu \otimes \varphi \otimes \text{id} \in FC(\mu_2, \lambda, \mu_1),$$

$$\nu \otimes \varphi \otimes \text{id}(\mid u \rangle \otimes \mid v \rangle \otimes \mid w \rangle)$$

$$= \nu(\mid u \rangle \otimes \varphi(\mid v \rangle \otimes \mid w \rangle) \otimes \varepsilon_{\mu_2} \otimes \lambda \otimes \mu_1),$$

gives the isomorphism between them. Hence the space

$$\mathcal{P} FC\left( \begin{array}{c} \lambda \\ \mu_2 \end{array} \mu_1 \right)$$

of vertex operators is isomorphic with the space $\mathcal{P} FC(\mu_2, \lambda, \mu_1)$ of 3-point functions. More precisely, the classical sector $\text{proj}_{\mu_2} \Phi(z) |_{\lambda \otimes \mu_1}$ of the vertex operator $\Phi(z)$ is given by

$$\lim_{z_t \to 0} \lim_{z_s \to 0} 2^{\Delta_{\mu_2}} \left\langle \Phi(z_t) \Phi(z) \Phi \text{id}(z_s) \right\rangle.$$

§3. Denote by $\mathcal{P} FC(\nu_2), \mathcal{P} FC(\nu_1)$ the space of compositions $\Phi^2(\nu_2) \Phi^1(\nu_1)$ of vertex operators $\Phi^1$ of type $\nu_1$. Then

$$\sum_{\mu \in P_L} \mathcal{P} FC\left( \begin{array}{c} \lambda_2 \\ \mu_t \end{array} \mu \right) \cdot \mathcal{P} FC\left( \begin{array}{c} \lambda_1 \\ \mu_s \end{array} \right) \cong \mathcal{P} FC(\Lambda) \cong FC(\Lambda),$$

where $\Lambda = (\mu_t, \mu_2, \lambda_1, \mu_s)$.

The composition $\Phi^2(\nu_2) \Phi^1(\nu_1)$ is determined by the classical sector $\text{proj}_{\lambda_2} \Phi^2(\nu_2) \Phi^1(\nu_1) |_{\lambda \otimes \mu_1} \in FC(\nu_2, \lambda_2, \lambda_1, \mu_s) \otimes FC(\nu_1)$ and it is given by

- 12 -
\[
\lim_{z_t \rightarrow z_s}^{2\Delta_{\mu_t}} <\Phi_{\nu}(z_t)\Phi^2(z_2)\Phi^1(z_1)\Phi_{id}(z_s)> .
\]

Hence by Theorem 2, the composition $\Phi^2(z_2)\Phi^1(z_1)$ is absolutely convergent in the range $\mathcal{R}_2 = \{(z_2, z_1) \in \mathbb{C}^2; |z_2| > |z_1| > 0\}$, so by the analytic continuation it defines the holomorphic (multivalued) function valued in

\[
\text{Hom}(V_{\lambda_2}, \Theta V_{\lambda_1}, \text{Hom}(\mathcal{R}_1, \mathcal{R}_2))
\]
on the complex manifold $M_2 = \{(z_2, z_1) \in (\mathbb{C}\setminus\{0\})^2; z_1 \neq z_2\}$.

Denote by $\Phi^2(u_2; z_1)\Phi^1(u_1, z_2) = C_{\gamma} (\Phi^2(u_2; z_2)\Phi^1(u_1, z_1))$ its analytic continuation along the path $\gamma$:

\[
\gamma(t) = \left[\frac{z_2 + z_1}{2} + e^{\pi\sqrt{-1}t} \frac{z_2 - z_1}{2}, \frac{z_2 + z_1}{2} - e^{\pi\sqrt{-1}t} \frac{z_2 - z_1}{2}\right], \quad t \in [0, 1]
\]

for $(w, z) \in \mathcal{R}$, then the corresponding analytic continuation

\[
T <\Phi_{\nu}(z_t)\Phi^2(z_2)\Phi^1(z_1)\Phi_{id}(z_s)>
\]
satisfies the systems KZ(TA) and $\mathcal{L}(TA)$ as a $\text{Hom}(V_{\lambda_2}, \Theta V_{\lambda_1}, \lambda_2)$-valued function, where $T$ is the transposition operator: $\text{Hom}(V_{\lambda_2}, \Theta V_{\lambda_1}, A) \longrightarrow \text{Hom}(V_{\lambda_1}, \Theta V_{\lambda_2}, A),$

\[
(T\phi)(u_1 \Theta u_2) = \phi(u_2 \Theta u_1) \quad (u_2 \Theta u_1 \in V_{\lambda_1} \Theta V_{\lambda_2}),
\]

and $TA = (\mu^*_t, \lambda_1, \lambda_2, \mu_s)$. Hence the analytic continuation
along $\gamma$ gives an isomorphism between the spaces of compositions of vertex operators:

**Theorem 3. (Commutation Relations)**

For $A = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, $C(A) = C_{\gamma}(A)$ is an isomorphism:

$$C_{\gamma}(A) : \mathcal{V}(A) \longrightarrow \mathcal{V}(TA)$$

$$\sum_{\mu \in P} \mathcal{V}(\mu_2) \otimes \mathcal{V}(\mu_1) \longrightarrow \sum_{\mu \in P} \mathcal{V}(\mu_1) \otimes \mathcal{V}(\mu_2).$$

**Remark.** The isomorphisms $C_{\gamma}(A)$, $A \in P^4$ enjoy the braid relations: For any $N \geq 1$, $\mu_t, \mu_s \in P$, introduce the space

$$\mathcal{V}(N; \mu_t, \mu_s) = \sum_{\lambda_1, \cdots, \lambda_N \in P} \mathcal{V}(\mu_t^*, \lambda_N, \cdots, \lambda_1, \mu_s).$$

Define the operators $C_i$ (1 $\leq$ i $\leq$ N) on $\mathcal{V}(N; \mu_t, \mu_s)$ such that

$$C_i \mathcal{V}(\mu_t^*, \lambda_N, \cdots, \lambda_1, \mu_s) \subset \mathcal{V}(\mu_t^*, \lambda_N, \cdots, \lambda_i, \lambda_{i+1}, \cdots, \lambda_1, \mu_s)$$

and

$$C_i(\varphi_1 \otimes \cdots \otimes \varphi_i)$$

$$= \varphi_N \otimes \cdots \otimes \varphi_{i+2} \otimes C(\mu_{i+1}, \lambda_{i+1}, \lambda_1, \mu_{i-1})(\varphi_{i+1} \otimes \varphi_i) \otimes \varphi_{i-1} \otimes \cdots \otimes \varphi_1.$$
for \( \varphi_N \theta \cdots \theta \varphi_1 \in \mathcal{P}(\mu^*_t, \lambda_N, \cdots, \lambda_1, \mu_1) \)
\[ = \mathcal{P}\left( \begin{array}{c} \lambda_N \\ \mu_t \\ \mu_{N-1} \end{array} \right) \theta \cdots \theta \mathcal{P}\left( \begin{array}{c} \lambda_1 \\ \mu_i \\ \mu_{i-1} \end{array} \right) \theta \cdots \theta \mathcal{P}\left( \begin{array}{c} \lambda_1 \\ \mu_1 \\ \mu_s \end{array} \right). \]

Then
\[ C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1} \]
as isomorphisms of \( \mathcal{P}(N; \mu_t, \mu_s) \) to itself.

\[ \text{§4. The composition } \Phi^2(u_2; w) \Phi^1(u_1; z) \text{ is singular at } w = z \]
and its behaviour near \( w = z \) is described as follows.

For \( A = (\mu^*_t, \lambda_2, \lambda_1, \mu_s) \), the space \( V^\vee(A) \) has another decomposition
\[ V^\vee(A) \leftarrow \sum_{F} \frac{\Sigma}{\theta} \text{Hom}_{g} (V_{\lambda_2} \Theta V_{\lambda_1}, V_{\nu}) \Theta \text{Hom}_{g} (V_{\nu} \Theta V_{\mu_s}, V_{\mu_t}), \]
where the identification \( F \) is given by
\[ F(\varphi_2 \theta \varphi_1)(u_2 \theta u_1 \theta u_s) = \varphi_1(\varphi_2(u_2 \theta u_1) \theta u_s) \quad (u_i \in V_{\lambda_i}, u_s \in V_{\mu_s}), \]

For \( \varphi_2 \in F\left( \begin{array}{c} \nu \\ \mu_t \end{array} \right) \) and \( \varphi_1 \in F\left( \begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \right), \) a "vertex operator"
\[ \Phi^f_{\varphi_2 \theta \varphi_1}(z) \text{ of } \hat{g}_{\mu_s} \text{ to } \hat{g}_{\mu_t} \text{ parametrized by } V_{\lambda_2} \Theta V_{\lambda_1} \text{ defined by} \]
\[ \Phi^f_{\varphi_2 \theta \varphi_1}(u_2 \theta u_1; z) = \Phi^f_{\varphi_1}(\varphi_2(u_2 \theta u_1); z) \quad (u_i \in V_{\lambda_i}). \]
Theorem 4. (Short range expansion or Fusion rule)

i) Near \( w=z ((w,z) \in \mathbb{R}_2^2) \),

\[
\Phi^2(u_2;w)\Phi^1(u_1;z) = \sum_{\nu \in P_{\ell}} \hat{\Delta}(w_1) \left[ \Phi^f_{\nu}(u_2 \otimes u_1;z) + O(w-z) \right]
\]

\[
- (\Delta_{\lambda_1} + \Delta_{\lambda_2}) \sum_{\nu \in P_{\ell}} (w-z) \Delta_{\nu} \Phi^f_{\nu}(u_2 \otimes u_1;z),
\]

where \( \nu \in \Phi\left(\begin{bmatrix} \lambda_2 \\ \mu_t \mu_s \end{bmatrix}\right) \theta \Phi\left(\begin{bmatrix} \nu \\ \mu_t \mu_s \end{bmatrix}\right) \), and \( O(w-z) \) is holomorphic near \( w=z \) and vanishes at \( w=z \):

\[
\begin{array}{ccc}
\lambda_2 & \lambda_1 \\
\mu_t & \mu & \mu_s
\end{array}
\]

The value of \( (w-z) \) is chosen as it is positive for \( (w,z) \in \mathbb{R}^2_2 \).

ii) For \( \Lambda = (\mu_t^1, \lambda_2, \lambda_1, \mu_s) \), the fusion gives an isomorphism

\[
F(\Lambda): \Phi(\Lambda) \cong \sum_{\nu \in P_{\ell}} \nu(\begin{bmatrix} \lambda_2 \\ \mu_t \mu_s \end{bmatrix}) \theta \nu(\begin{bmatrix} \lambda_1 \\ \mu_t \mu_s \end{bmatrix}) \rightarrow \sum_{\nu \in P_{\ell}} \nu(\begin{bmatrix} \lambda_2 \\ \nu \lambda_1 \\ \mu_t \mu_s \end{bmatrix}) \theta \nu(\begin{bmatrix} \nu \\ \lambda_1 \\ \mu_t \mu_s \end{bmatrix})
\]

defined by

\[
F(\Lambda)(\Phi^2 \Phi^1) = \sum_{\nu \in P_{\ell}} \psi_{\nu} (\Phi^2 \Phi^1) \in \Phi(\begin{bmatrix} \lambda_2 \\ \mu_t \mu_s \end{bmatrix}) \theta \Phi(\begin{bmatrix} \lambda_1 \\ \mu_t \mu_s \end{bmatrix}),
\]

- 16 -
where $\psi_i$ are the ones obtained in i) for $\phi^i = \phi^i_{\varphi}$.

Theorem 5.

For $A = (\mu^*_t, \lambda_2, \lambda_1, \mu_s)$, the following diagram commutes:

$$
\begin{array}{ccc}
\gamma(A) & \xrightarrow{F(A)} & \sum_{\nu \in P} \gamma\left(\begin{bmatrix} \nu \\ \mu_t \mu_s \end{bmatrix}\right) \gamma\left(\begin{bmatrix} \lambda_2 \\ \lambda_1 \end{bmatrix}\right) \\
\downarrow C_\gamma & & \downarrow \text{id} \otimes C_0 \\
\gamma(TA) & \xrightarrow{F(TA)} & \sum_{\nu \in P} \gamma\left(\begin{bmatrix} \nu \\ \mu_t \mu_s \end{bmatrix}\right) \gamma\left(\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}\right)
\end{array}
$$

**Remark.** The equation KZ($A$) in the limit $z_4 \to \infty$, $z_1 \to 0$ is reduced to a differential equation (reduced KZ-system) RKZ($A$) on $V_g(A)$—functions of one variable $\xi = z_3/z_2$. The equation RKZ($A$) has only regular singularities at $\xi = 0, 1, \infty$. The isomorphisms $C_{\gamma}(A)$ and $F(A)$ are essentially nothing but the connection matrices from the space of its solutions regularized at $\xi = 0$ to the spaces of solutions regularized at $\xi = \infty$ and $\xi = 1$ respectively.

§5. Naturally arises a problem to determine the
isomorphisms $C_\gamma(A)$ and $F(A)$, but it is very difficult to carry out for all cases. We succeeded (last year) in the case where $\hat{g}$ is an affine Lie algebra of type $A_n^{(1)}$ and $A = (\mu_\ell, \varnothing, \varnothing, \mu_s)$, where $\varnothing$ means a Young diagram consisting of one node and represent the vector representation of $g = sl(n+1, \mathbb{C})$.

Now let $\hat{g}$ be an affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $P_+$ be the set of weights $\lambda \in P_+$ such that the simple $g$-module $V_\lambda$ can appear in some tensor products of the vector representations $V_\varnothing$ of $g = sl(n+1, \mathbb{C})$, $\sigma(2n; \mathbb{C})$, $sp(2n; \mathbb{C})$, $o(2n; \mathbb{C})$, $o(2n+1; \mathbb{C})$ respectively.

For each $\tau \in P_\varnothing$, introduce the space

$$\mathcal{V}_N(\tau) = \sum_{\mu} \mathcal{V}_N(\tau)_\mu,$$

$$\mathcal{V}_N(\tau)_\mu = \mathcal{V}(\left[ \begin{array}{c} \varnothing \\ \tau \end{array} \right], \mu_{N-1}) \otimes \cdots \otimes \mathcal{V}(\left[ \begin{array}{c} \varnothing \\ \mu_i \end{array} \right], \mu_{i-1}) \otimes \cdots \otimes \mathcal{V}(\left[ \begin{array}{c} \varnothing \\ \mu_1 \end{array} \right], \mu_0),$$

where the summation is taken over the set $\mu \in (\mu_1, \cdots, \mu_{N-1})$. Then $\mathcal{V}_N(\tau)$ is the subspace of $\mathcal{V}(N; \tau, 0)$ which is invariant under the operators $C_i$ ($1 \leq i \leq N-1$).

The braid group $B_N$ with N-strings of $\mathbb{C}$ has a system 

$$\{ b_i; 1 \leq i \leq N-1 \}$$

of generators with the fundamental relations:

\[
\begin{cases}
  b_i b_j = b_j b_i \ (|i-j| \geq 2) \\
  b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \ (1 \leq i \leq N-2).
\end{cases}
\]

(BR)
These generators $b_i$ are represented by the curves on $\mathbb{C}$ defined by
\[
b_i(t) = \left\{ N, N-1, \ldots, i+ \frac{1}{2}(1+e^{\pi\sqrt{-1}t}), i+ \frac{1}{2}(1-e^{\pi\sqrt{-1}t}), \ldots, 2, 1 \right\}
\quad \text{for } t \in [0, 1],
\]

We now define a monodromy representation $\pi^\tau_N$ of $B_N$ on the space $\mathcal{F}_N(\tau)$ as $\pi^\tau_N(b_i) = C_i$ ($1 \leq i \leq N-1$). The we get the main theorems.

**Theorem 6.**

If $g$ is of type $A_n$, then the monodromy representation $r/\pi_{N}^{\lambda}$ in $\mathcal{F}_N(\tau)$ factors through the Iwahori-Hecke algebra $H_N(r)$, where $r = \exp(\frac{\pi\sqrt{-1}}{2+n+1})$.

**Note.** The algebra $H_N(r)$ is defined by generators $\{\tau_i, \tau_i^{-1} | 1 \leq i \leq N-1\}$ with the defining relations:
\[
\tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1, \quad \tau_i - \tau_i^{-1} = (r - r^{-1}) \quad (1 \leq i \leq N-1),
\]
\[
\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad \text{and} \quad \tau_i \tau_{j} = \tau_{j} \tau_{i} \quad (|i-j| \geq 2).
\]

**Theorem 7.** If $g$ is the simple Lie algebra of type $B_n$, $C_n$ or $D_n$. Then the monodromy representation $\pi^\lambda_N$ in $\mathcal{F}_N(\tau)$ factors through the Birman-Wenzl-Murakami algebra $C_N(g;r)$.
where \( r = \exp\left(\frac{\pi\sqrt{-1}}{2 + g}\right) \), \( g(B_n) = 2n-1 \), \( g(C_n) = n+1 \), \( g(D_n) = 2n-2 \); \( C_N(B_n; r) = C_N(r^{-n-1/2}, r) \), \( C_N(C_n; r) = C_N(r^n, r) \)
and \( C_N(D_n; r) = C_N(r^{-n}, r) \).

Note. The algebra \( C_N(a, r) \) is defined by generators
\[ \{\tau_i, \tau_i^{-1}, \epsilon_i \mid 1 \leq i \leq N-1\} \]
with the defining relations:
\[ \tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1, \quad \tau_i \epsilon_i = \epsilon_i \tau_i = -(a^2 - r)^{-1} \epsilon_i, \]
\[ \tau_i - \tau_i^{-1} = (r - r^{-1})(1 - \epsilon_i) \quad (1 \leq i \leq N-1), \]
\[ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_i + 1, \quad \epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_{i+1} \epsilon_i \epsilon_i + 1 \]
\[ \epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_{i+1} \epsilon_i \epsilon_i = \tau_i \epsilon_{i+1} \quad (1 \leq i \leq N-2), \]
\[ \tau_i \tau_j = \tau_j \tau_i, \quad \epsilon_i \tau_j = \tau_j \epsilon_i, \quad \epsilon_i \epsilon_j = \epsilon_j \epsilon_i \quad (|i-j| \geq 2), \]
The proof is carried out by the explicit calculation of a differential equation of 4-point function in a very special case and the algebraic arguments for the algebras \( H_N(r) \) and \( C_N(a, r) \).

References


- 20 -
191-208.


