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Well-posedness of the Cauchy problem for systems in a complex domain (as an application of the determinant theory)

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1. Introduction and results. C. Wagchal [11] showed that various formulations of the Cauchy problems are possible for a non-characteristic hypersurface for a given system of linear partial differential equations. We shall show that such Cauchy problems are all reduced to the one's for systems of normal type in the time derivative. Moreover, we shall give the necessity of the non-characteristicness of the initial hypersurface for the well-posed Cauchy problem. In the proof, we need some results on the invertibility of matrices of linear partial differential operators which are obtained by applying the determinant theory over the ring of linear partial differential operators due to Sato and Kashiwara [9].

We first give a list of notations.

Let $x = (x_1,\dots,x_n) = (x_1,x') \in \mathbb{C}^n$, $D = (D_1,\dots,D_n) = (D_1,D')$ $(D_j = \partial/\partial x_j)$ and Ω be a domain of $\mathbb{C}^n$.

$\mathcal{D}(\Omega)$; the ring of l.p.d.op. with holomorphic coefficients in Ω,

$M_N(\mathcal{D}(\Omega))$; the set of N×N-matrices with entries in $\mathcal{D}(\Omega)$,

$GL_N(\mathcal{D}(\Omega))$; the set of invertible matrices in $M_N(\mathcal{D}(\Omega))$.

<u>Cauchy problem</u> $(A,\mu)_p$: Let $A = (A_{ij}(x,D)) \in M_N(\mathcal{D}(\Omega))$, $\mu = (\mu_1,\ldots,\mu_N) \in \mathbb{Z}_+^N$ ($\mathbb{Z}_+ = \{0,1,2,3,\ldots\}$) and $p = (p_1,p') \in \Omega$. Then the Cauchy problem $(A,\mu)_p$ is defined by

$$(A,\mu)_p \begin{cases} \sum_{j=1}^{N} A_{ij}(x,D)u_j(x) = f_i(x) \in \mathcal{O}_p, & 1 \leqq i \leqq N, \\ D_1^k u_j \big|_{x_1 = p_1} = u_{jk}(x') \in \mathcal{O}_{p'}, & 0 \leqq k < \mu_j, \quad 1 \leqq j \leqq N, \end{cases}$$

where $\mathcal{O}_p$ denotes the germs of holomorphic functions at $x = p$.

<u>Well-posedness</u>: (1) $(A,\mu)_p$ is well-posed $\overset{\text{def.}}{\Longleftrightarrow}$ $(A,\mu)_p$ has a unique solution $(u_j(x)) \in \mathcal{O}_p^N$ for any given $(f_i(x)) \in \mathcal{O}_p^N$ and $(u_{jk}(x')) \in \mathcal{O}_{p'}^{|\mu|}$, where $|\mu| = \mu_1 + \ldots + \mu_N$.
(2) (A,μ) is well-posed in Ω $\overset{\text{def.}}{\Longleftrightarrow}$ $(A,\mu)_p$ is well-posed at every point $p \in \Omega$.

We begin with examples to make clear our problem.

<u>Example</u> 1. Let $A = \begin{pmatrix} D & -1 \\ 0 & D \end{pmatrix} \in M_2(\mathcal{D}(\mathbb{C}))$, where $D = d/dx$. Then the associated system to this matrix is

$$Du - v = f(x) \in \mathcal{O}_0, \quad Dv = g(x) \in \mathcal{O}_0.$$

We ask that what kind of the Cauchy data are possible.

(a) The case of $\mu = (1,1)$, i.e., $\{u(0), v(0)\}$ are given as a Cauchy data. In this case the Cauchy problem is trivially well-posed.

(b) The case of $\mu = (2,0)$, i.e., $\{u(0), u'(0)\}$. In this case also the Cauchy problem is well-posed. We interpret it as follows. Let $P = \begin{pmatrix} D & 1 \\ -1 & 0 \end{pmatrix} \in GL_2(\mathcal{D}(\mathbb{C}))$. Then we have $PA = \begin{pmatrix} D^2 & 0 \\ -D & 1 \end{pmatrix}$.

(c) The case of $\mu = (0,2)$, i.e., $\{v(0), v'(0)\}$. In this cae the Cauchy problem is not well-posed. In fact, the Cauchy data should satisfy a compatibility condition $v'(0) = g(0)$, and also $u(0)$ can not be determined from the Cauchy data.

This example shows that various types of the Cauchy data are possible for a system. The next example shows that we can not give the Cauchy data even if the initial hypersurface is not characteristic if we consider the Cauchy problem in our sense.

Example 2. Let $A(x,D) = \begin{pmatrix} x_1D_1 & x_1D_1 - 1 \\ D_1^2 + D_2 & D_1^2 \end{pmatrix} \in M_2(\mathcal{D}(\mathbb{C}^2))$.

Then $\det_\sigma A = \xi_1^2 - x_1\xi_1\xi_2$, and hence the hyperplane $x_1 = 0$ is not characteristic to this matrix, where $\det_\sigma A$ denotes the determinant of $A(x,D)$ in the sense of Sato and Kashiwara [9]. To show this, it is sufficient to see

$$\begin{pmatrix} D_1 & -x_1 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} D_1 - x_1D_2 & 0 \\ D_1^2 + D_2 & D_1^2 \end{pmatrix}.$$

We can easily see that the Cauchy problem for system associated to this matrix is well-posed only when we give the Cauchy data by $(u(0,x_2), (\partial v/\partial x_1)(0,x_2)) \in \mathcal{O}_o^2$, when we consider the Cauchy problem at the origin. We note that such a Cauchy problem is the one excluded. On the other hand, when we consider the Cauchy problem at $p = (p_1, 0)$ $(p_1 \neq 0)$, the Cauchy problem $(A,\mu)_p$ is well-posed if and only if $\mu = (1,1)$. It will be easily proved, so we omit it.

Now the following result due to C. Wagschal is the starting point of our research.

THEOREM (Wagschal [11]) Let $A(x,D) \in M_N(\mathcal{D}(\Omega))$ be a non-degenerate matrix with $\mathrm{order}_D A = m$. If the hyperplane $x_1 = p_1$ is not characteristic for A at $p = (p_1, p')$, then there is at least one μ with $|\mu| = m$ such that the Cauchy problem $(A,\mu)_p$ is well-posed.

We remark that Wagschal proved the existence of such μ that the Cauchy problem (A,μ) is well-posed in a neighbourhood of p. We have to note that the well-posedness of $(A,\mu)_p$ does not imply the well-posedness of (A,μ) in any neighbourhood of p, even if the assumptions of the theorem are satisfied (see an example in §4).

We give here definitions of terminology used in the above theorem.

<u>Definition</u>. Let $A = (A_{ij}(x,D)) \in M_N(\mathcal{D}(\Omega))$ and put $m_{ij} = \mathrm{order}_D A_{ij}$, where $\mathrm{order}_D A_{ij} = -\infty$ if $A_{ij} \equiv 0$. Then the <u>total order</u> of A which is denoted by $\mathrm{order}_D A$ is defined by

$$(1.1) \qquad \mathrm{order}_D A = \max_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{N} m_{i\sigma(i)} \in \mathbb{Z}_+ \cup \{-\infty\},$$

where we define $-\infty + \ell = -\infty$, and $\mathfrak{S}_N$ denotes the permutation group of $\{1,2,\ldots,N\}$. Let $\mathrm{order}_D A = m \geqq 0$. Then A is said to be <u>non-degenerate</u> if

$$(1.2) \qquad \deg_\xi \det A(x,\xi) = m \quad (\xi \in \mathbb{C}^n).$$

In this case, the <u>characteristic polynomial</u> $a(x,\xi)$ of A is defined by

$$(1.3) \qquad a(x,\xi) = \text{homogeneous part of degree } m \text{ in } \xi \text{ of } \det A(x,\xi),$$

and a hyperplane $x_1 = p_1$ is called to be <u>non-characteristic</u> at $p = (p_1, p')$ if $a(p,(1,0,\ldots,0)) \neq 0$.

Now our first purpose is to give a relation between A and μ for the well-posed Cauchy problem, which is stated as follows.

<u>THEOREM</u> 1. Let $A \in M_N(\mathcal{D}(\Omega))$ be a non-degenerate matrix with non-characteristic hyperplane $x_1 = p_1$ at $p = (p_1, p')$. Then the Cauchy problem (A, μ) is well-posed in a neighbourhood of p if and only if the following conditions are satisfied:

1) $|\mu| = \mathrm{order}_D A$,

2) There is a unique $P(x,D) \in GL_N(\mathcal{D}_p)$ such that PA is of μ-normal type in D_1, that is,

$$PA = (D_1^{\mu_j}\delta_{ij} + b_{ij}(x,D)), \quad \mathrm{order}_{D_1} b_{ij} < \mu_j,$$

where $\mathcal{D}_p$ denotes the germs of l.p.d.op. with holomorphic coefficients at p.

In the above theorem the initial hyperplane is assumed to be non-characteristic, but in some cases we can prove its necessity.

<u>THEOREM</u> 2. Let $A \in M_N(\mathcal{D}(\Omega))$ be a non-degenerate matrix with $\mathrm{order}_D A = m$. Then the Cauchy problem (A, μ) with $|\mu| = m$ is well-posed in a neighbourhood of $p = (p_1, p')$ only if the hyperplane $x_1 = p_1$ is not characteristic at p.

In section 2, we shall give a brief summary of the determinant theory and give characterizations of the invertible matrices which play important role in the proof of our theorems. Our theorems will be proved in section 3 following the fundamental idea due to Kitagawa and Sadamatsu [4].

In section 4, we shall give an example of non-degenerate matrix A with non-characteristic hyperplane $x_1 = p_1$ at $p = (p_1, p')$ such that the well-posedness of $(A,\mu)_p$ does not imply the well-posedness of (A,μ) in any neighbourhood of p.

In this note we shall study matrices of non-degenerate type, but my preprint [7] treats more general matrices and obtains the similar results under the assumption that $n = 2$.

At the end we note that Sadamatsu [8] proved similar result to Theorem 1 under more restricted situation.

2. Review of the determinant theory. The determinant theory over non-commutative field was first studied by Dieudonné [3], and then the determinant for matrices of differential operators was defined by Hufford [12] by embedding the ring of differential operators into the non-commutative field which is a field of quotients. After that Sato and Kashiwara [9] studied the determinant theory over the ring of micro-differential operators (the detailed proof of their results can be found in [1] by Andronikov).

Let $\mathcal{O}(\Omega)[\xi]$ be the set of polynomials in $\xi \in \mathbb{C}^n$ with coefficients in $\mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is the set of holomorphic functions in Ω. We denote by $\mathcal{O}(\Omega)[\xi]_m$ $(m \geqq 0)$ the set of polynomials of degree at most m in ξ. Then we define $\mathrm{gr}\,\mathcal{O}(\Omega)[\xi]$ by

$$(2.1) \qquad \mathrm{gr}\,\mathcal{O}(\Omega)[\xi] = \{0\} \bigoplus_{m=0}^{\infty} \mathcal{O}(\Omega)[\xi]_m \big/ \mathcal{O}(\Omega)[\xi]_{m-1} ,$$

where $\mathcal{O}(\Omega)[\xi]_{-1} = \{0\}$. Then the determinant for matrices in $M_N(\mathcal{D}(\Omega))$, which is denoted by $\det_\sigma \cdot$, is defined as a homomorphism,

$$\det_\sigma : M_N(\mathcal{D}(\Omega)) \longrightarrow \operatorname{gr} \mathcal{O}(\Omega)[\xi],$$

so that the following properties are satisfied:

1) For two matrices $A \in M_N(\mathcal{D}(\Omega))$ and $B \in M_{N'}(\mathcal{D}(\Omega))$, we have

$$\det_\sigma(A \oplus B) = \det_\sigma A \cdot \det_\sigma B .$$

2) For matrices A, B in $M_N(\mathcal{D}(\Omega))$, we have $\det_\sigma(AB) = \det_\sigma A \cdot \det_\sigma B$.

3) If $A \in M_N(\mathcal{D}(\Omega))$ is non-degenerate, then

$\det_\sigma A =$ the characteristic polynomial of A.

4) $A \in GL_N(\mathcal{D}(\Omega))$ if and only if $\det_\sigma A \equiv a(x) \neq 0$ in Ω, i.e. $\det_\sigma A$ is a unit element in $\operatorname{gr} \mathcal{O}(\Omega)[\xi]$.

In a sense, the determinant is nothing but the characteristc polynomial of a matrix.

In the foundation of the determinant theory, the following property of $\mathcal{D}_p$ called Ore property plays an essential role.

Ore property. For any non-zero elements A and B in $\mathcal{D}_p$, we have $\mathcal{D}_p A \cap \mathcal{D}_p B \neq \{0\}$ ($A\mathcal{D}_p \cap B\mathcal{D}_p \neq \{0\}$), where $\mathcal{D}_p A$ is a left ideal of $\mathcal{D}_p$ generated by A. That is, A and B have commom left (right) multiples. (see Schapira [10, Remark 1.3.8]).

We note that the Ore property asserts that we can construct left (right) quotient field of $\mathcal{D}_p$, and hence the determinant theory due to Dieudonné can be applied in our case. By this property, the determinant is calculated as follows.

We introduce the following three matrices.

$$P = (\vec{e}_1, \ldots, \overset{i}{\vec{e}_j}, \ldots, \overset{j}{-\vec{e}_i}, \ldots, \vec{e}_N), \quad \det_\sigma P = 1,$$

where $\vec{e}_i = {}^t(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ and $(\vec{a}_1, \ldots, \vec{a}_N) \in M_N(\mathcal{D}(\Omega))$ denotes a matrix with the j-th column vector a_j.

$$Q = (\vec{e}_1, \dots, \overset{j}{\vec{e}_j + \vec{e}_i}, \dots, \vec{e}_N), \quad \det_\sigma Q = 1,$$

$$R = \mathrm{diag}(1, \dots, 1, \overset{j}{a(x,D)}, 1, \dots, 1), \quad \det_\sigma R = \sigma(a)(x,\xi),$$

where $\mathrm{diag}(a_1,\dots,a_N)$ denotes a diagonal matrix with the j-th diagonal component a_j and $\sigma(a)$ denotes the principal symbol of a.

Let $A \in M_N(\mathcal{D}(\Omega))$. Then by using the Ore property for $\mathcal{D}_p$, we obtain a matrix $S \in M_N(\mathcal{D}_p)$ which is obtained by multiplying matrices of above types such that

$$SA = \mathrm{triang}(\alpha_1(x,D), \dots, \alpha_N(x,D)),$$

(triangular matrix with the j-th diagonal component α_j). Then

$$\det_\sigma A = \left(\prod_{j=1}^{N} \sigma(\alpha_j)(x,\xi) \right) \cdot (\det_\sigma S)^{-1} \in \mathrm{gr}\, \mathcal{O}_p[\xi].$$

Next, we give results on characterization of invertible matrices, which play crucial role in the proof of our results.

PROPOSITION 2.1 (Andronikov[2]) Let $A(x,D)$ be a matrix of $N' \times N$ type with entries in $\mathcal{D}(\Omega)$. Then the mapping

$$A(x,D) : \mathcal{O}_p^N \longrightarrow \mathcal{O}_p^{N'}$$

is bijective at every point p in Ω if and only if $N' = N$ and $A(x,D) \in GL_N(\mathcal{D}(\Omega))$.

PROPOSITION 2.2. Let $\Omega = \Omega_1 \times \Omega'$, where $\Omega_1 \subset \mathbb{C}_{x_1}$ and $\Omega' \subset \mathbb{C}^{n-1}_{x'}$. Then $A(x,D') \in GL_N(\mathcal{D}(\Omega))$ if and only if $A(p_1,x',D') \in GL_N(\mathcal{D}(\Omega'))$ at every $p_1 \in \Omega_1$.

For the proof, the following lemma is essential.

LEMMA 2.3. Let Ω be as above. Let $A(x,D') \in M_N(\mathcal{D}(\Omega))$ and put $\det_\sigma A(x,D') = a(x,\xi')$. Then we have

1) If $a(x,\xi') \equiv 0$, then $\det_{\sigma'} A(p_1,x',D') \equiv 0$ for every $p_1 \in \Omega_1$, where $\det_{\sigma'} \cdot$ denotes the determinant for matrices in $M_N(\mathcal{D}(\Omega'))$.

2) If $a(p_1,x',\xi') \not\equiv 0$, then $\det_{\sigma'} A(p_1,x',D') = a(p_1,x',\xi')$.

3) If $a(x,\xi') \not\equiv 0$ and $a(p_1,x',\xi') \equiv 0$, then we have

$$\mathrm{order}_\sigma A > \mathrm{order}_{\sigma'} A(p_1,x',D'),$$

where $\mathrm{order}_\sigma A$ denotes the degree of $\det_\sigma A$ as a polynomial in ξ'.

For the proof of above lemma and proposition, see [7].

3. Proof of theorems. The proof is done by following the argument in Kitagawa and Sadamatsu [4].

Let $A = (A_{ij}) \in M_N(\mathcal{D}(\Omega))$ and put $\ell_{ij} = \mathrm{order}_{D_1} A_{ij}$. Then we define

$$(3.1) \qquad \mathrm{order}_{D_1} A := \max_{\sigma \in \mathfrak{S}_N} \sum_{i=1}^{N} \ell_{i\sigma(i)} \in \mathbb{Z}_+ \cup \{-\infty\}.$$

Let assume $\mathrm{order}_{D_1} A = \ell \geqq 0$. Then by Volevič's lemma, there is a system of integers $\{s_i,\ t_j\}$ satisfying

$$(3.2) \qquad \ell_{ij} \leqq t_j - s_i \text{ and } \ell = |t| - |s|,$$

where $|t| = t_1 + \dots + t_N$, etc. We set

$$A_{ij} = \sum_{k=0}^{t_j - s_i} a_{ijk}(x,D') D_1^{t_j - s_i - k},$$

and set

$$(3.3) \qquad A_0(x,D') = (a_{ij0}(x,D')).$$

We remark that if the assumptions in Theorem 1 are satisfied, then $\mathrm{order}_{D_1} A = \mathrm{order}_D A$ and we can choose $\{s_i, t_j\}$ so that

$$\mathrm{order}_D A_{ij} \leqq t_j - s_i \quad \text{and} \quad \mathrm{order}_D A = |t| - |s| .$$

Hence in this case we may put $a_{ij0} \equiv a_{ij0}(x)$ and $\det A_0(p) \neq 0$.

We introduce two matrices $\Delta^{(\nu)}$ and $\mathcal{L}^{(\nu)}$ for $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{Z}_+^N$.

$$(3.4) \qquad \Delta^{(\nu)} = \mathrm{diag}(D_1^{\nu_1}, \dots, D_1^{\nu_N}) \qquad (N\times N\text{-matrix}),$$

$$(3.5) \qquad \mathcal{L}^{(\nu)} = \begin{pmatrix} 1 & & 0 \\ D_1 & & \vdots \\ \vdots & & \vdots \\ D_1^{\nu_1-1} & \ddots & \vdots \\ 0 & & 0 \\ \vdots & & 1 \\ \vdots & & D_1 \\ \vdots & & \vdots \\ 0 & & D_1^{\nu_N-1} \end{pmatrix} \qquad (|\nu|\times N\text{-matrix}).$$

We note that we may choose $\{s_i, t_j\}$ so that

$$(3.6) \qquad t_j > \mu_j \quad \text{and} \quad s_i > 0 \quad (i,j=1,\dots,N).$$

Now applying $\mathcal{L}^{(s)}$ $(s = (s_1, \dots, s_N))$ to the system $Au = f$ from the left, we get $\mathcal{L}^{(s)} Au = \mathcal{L}^{(s)} f$. By the above choice of $\{s_i, t_j\}$, in $\mathcal{L}^{(s)} Au$, only $(D_1^k u_j;\ 0 \leqq k < t_j,\ 1 \leqq j \leqq N)$ appear. By this reason, this relation can be rewritten as follows.

$$(3.7) \quad \mathfrak{A}(x,D') \begin{pmatrix} \Delta^{(\mu)} \\ \mathcal{L}^{(t-\mu-1)} \Delta^{(\mu+1)} \end{pmatrix} u + \mathfrak{B}(x,D')\, \mathcal{L}^{(\mu)} u = \mathcal{L}^{(s)} f,$$

where $\mathfrak{A}(x,D')$ is an $|s|\times(|t|-|\mu|)$-matrix and $\mathfrak{B}(x,D')$ is an $|s|\times|\mu|$-matrix. Here $t - \mu - 1 = (t_1-\mu_1-1, \dots, t_N-\mu_N-1)$, etc.

We note

$$\mathcal{L}^{(\mu)} u \Big|_{x_1=p_1} = (u_{jk}(x');\ 0 \leqq k < \mu_j,\ 1 \leqq j \leqq N) \in \mathcal{O}_{p'}^{|\mu|}$$

is a known vector from the Cauchy data.

Next, by applying $\Delta^{(s)}$ to the system $Au = f$ from the left, we get

$$(3.8) \qquad A_0(x,D')\Delta^{(t)}u + \mathcal{C}(x,D')\mathcal{L}^{(t)}u = \Delta^{(s)}f,$$

where $\mathcal{C}(x,D')$ is an $N \times |t|$-matrix.

We note that the well-posedness of $(A,\mu)_p$ implies the existence of the formal solution of $(A,\mu)_p$,

$$(F)_p \qquad u(x) = \sum_{k=0}^{\infty} u^{(k)}(x')(x_1-p_1)^k/k!, \qquad u^{(k)}(x') \in \mathcal{O}_{p'}^N.$$

This implies the surjectivity of the mapping,

$$(\#)_p \qquad \mathfrak{A}(p_1,x',D') : \mathcal{O}_{p'}^{|t|-|\mu|} \longrightarrow \mathcal{O}_{p'}^{|s|}.$$

Now the following proposition is easily obtained from the Ore property of $\mathcal{D}_p$ and the definition of the determinant.

PROPOSITION 3.1. The surjectivity of the mapping $(\#)_p$ implies the following properties.

1) $|s| \leqq |t| - |\mu|$, that is, $|\mu| \leqq \text{order}_{D_1} A$,

2) $\text{rank}\, \mathfrak{A}(p_1,x',D') = |s|$, that is, $\mathfrak{A}(p_1,x',D')$ has at least one minor of degree $|s|$ with non-vanishing determinant.

The following proposition plays a crucial role in the proof of Theorem 1.

PROPOSITION 3.2. Let $\Omega = \Omega_1 \times \Omega'$, and assume $A_0(x,D') \in GL_N(\mathcal{D}(\Omega))$. Then the Cauchy problem $(A,\mu)_q$ has a unique formal solution $(F)_q$ at every $q \in \Omega$ if and only if the following conditions are satisfied: 1) $|s| = |t| - |\mu|$, 2) $\mathfrak{A}(x,D') \in GL_{|s|}(\mathcal{D}(\Omega))$.

Proof. The if part is obvious, since $\mathfrak{A}(x,D') \in GL_{|s|}(\mathcal{D}(\Omega))$ implies $\mathfrak{A}(q_1,x',D') \in GL_{|s|}(\mathcal{D}(\Omega'))$ for every $q_1 \in \Omega_1$ (Proposition 2.2). The assumption that $A_0(x,D') \in GL_N(\mathcal{D}(\Omega))$ and the unique existence of the formal solution $(F)_q$ $(q \in \Omega)$ implies that the mapping $(\#)_q$ is bijective, which implies

$$|s| = |t| - |\mu| \text{ and } \mathfrak{A}(q_1,x',D') \in GL_{|s|}(\mathcal{D}(\Omega'))\,(q_1 \in \Omega_1)$$

by Proposition 2.1. Hence by Proposition2.2, we have the assertion.

Proof of Theorem 1. First of all, we note that the Cauchy problem $(A,\mu)_p$ is well-posed if and only if $(A,\mu)_p$ has a unique formal solution $(F)_p$ under the assumptions in the theorem. To see this, it is sufficient to show that every formal solution $(F)_p$ of $(A,\mu)_p$ converges, which will be easily proved.

The if part is now obvious, since hyperplanes $x_1 = q_1$ are non-characteristic at $q = (q_1,q')$ when q varies in a neighbourhood of p. The only if part is also obvious except the existence of $P \in GL_N(\mathcal{D}_p)$ such that PA is of μ-normal type in D_1, by Proposition 3.2. By Proposition 3.2, we know that $\mathfrak{A}(x,D') \in GL_{|s|}(\mathcal{D}(\omega))$, where ω is a small neighbourhood of p. We put

$$P(x,D) = Q\,\mathfrak{A}^{-1}\,\mathcal{L}^{(s)}, \quad Q = \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right) \quad (N \times |s|\text{-matrix}) .$$

Then obviously we see that PA is of μ-normal type in D_1. Therefore we have only to prove that $P \in GL_N(\mathcal{D}_p)$. We note that Kitagawa and Sadamatsu [4, Prop. 4] proved this fact, but we give here our proof. Since PA is of μ-normal type, we put

$$(3.9) \qquad PA = \Delta^{(\mu)} + (C_{ij}(x,D)), \quad \text{order}_{D_1} C_{ij} < \mu_j .$$

By the above observations, we see that the Cauchy problems for two systems

$$A u = f \in \mathcal{O}_q^N \quad \text{and} \quad PA u = P f \qquad (q \in \omega)$$

with the same μ-Cauchy data have the same formal solution $(F)_q$. We note that $\mathcal{L}^{(t-\mu)} PA u = \mathcal{L}^{(t-\mu)} P f$ is written in the similar form to (3.7),

$$\text{(3.10)} \qquad \tilde{\mathfrak{A}}(x,D') \begin{bmatrix} \Delta^{(\mu)} \\ \mathcal{L}^{(t-\mu-1)} \Delta^{(\mu+1)} \end{bmatrix} u + \tilde{\mathfrak{B}}(x,D') \mathcal{L}^{(\mu)} u = \mathcal{L}^{(t-\mu)} P f ,$$

where $\tilde{\mathfrak{A}} \in M_{|s|}(\mathcal{D}(\omega))$ and $\tilde{\mathfrak{B}}$ is an $|s| \times |\mu|$-matrix. Since PA is of μ-normal type in D_1, we can easily see that the mapping

$$\tilde{\mathfrak{A}}(q_1, x'D') : \mathcal{O}_{q'}^{|s|} \longrightarrow \mathcal{O}_{q'}^{|s|}$$

is bijective at every point $q \in \omega$. This implies that $\tilde{\mathfrak{A}}(x,D') \in GL_{|s|}(\mathcal{D}(\omega))$. Thus we have following two systems,

$$\begin{bmatrix} \Delta^{(\mu)} \\ \mathcal{L}^{(t-\mu-1)} \ \Delta^{(\mu+1)} \end{bmatrix} u + \mathfrak{A}^{-1} \mathfrak{B} \mathcal{L}^{(\mu)} u = \mathfrak{A}^{-1} \mathcal{L}^{(s)} f,$$

$$\begin{bmatrix} \Delta^{(\mu)} \\ \mathcal{L}^{(t-\mu-1)} \ \Delta^{(\mu+1)} \end{bmatrix} u + \tilde{\mathfrak{A}}^{-1} \tilde{\mathfrak{B}} \mathcal{L}^{(\mu)} u = \tilde{\mathfrak{A}}^{-1} \mathcal{L}^{(t-\mu)} P f .$$

These two systems determine the same coefficients $\{u_{jk}(x');\ \mu_j \leqq k < t_j,\ 1 \leqq j \leqq N\}$ in the formal solution $(F)_q$. By choosing $f = 0$, we have $(\mathfrak{A}^{-1}\mathfrak{B})\big|_{x_1=q_1} = (\tilde{\mathfrak{A}}^{-1}\tilde{\mathfrak{B}})\big|_{x_1=q_1}$ for any q_1. Hence we have $\mathfrak{A}^{-1}\mathfrak{B} = \tilde{\mathfrak{A}}^{-1}\tilde{\mathfrak{B}}$. Therefore we have

$$(\mathfrak{A}^{-1} \mathcal{L}^{(s)} f)\Big|_{x_1=q_1} = (\tilde{\mathfrak{A}}^{-1} \mathcal{L}^{(t-\mu)} Pf)\Big|_{x_1=q_1}$$

for any $f \in \mathcal{O}_q^N$ $(q \in \omega)$. This implies $\mathfrak{A}^{-1} \mathcal{L}^{(s)} = \tilde{\mathfrak{A}}^{-1} \mathcal{L}^{(t-\mu)} P$, i.e. $\mathcal{L}^{(s)} = \mathfrak{A}\, \tilde{\mathfrak{A}}^{-1} \mathcal{L}^{(t-\mu)} P$. Hence we have

$$I_N = R\mathcal{L}^{(s)} = R\,\mathfrak{A}\,\tilde{\mathfrak{A}}^{-1}\,\mathcal{L}^{(t-\mu)}\,P\,,$$

where

$$R = \begin{array}{c} \overbrace{\quad}^{s_1}\ \overbrace{\quad\quad}^{s_2}\ \cdots\ \overbrace{\quad}^{s_N} \\ \begin{bmatrix} 10\cdots00\cdots\cdots\cdots\cdots0 \\ 0\cdots010\cdots0\cdots\cdots0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0\cdots\cdots\cdots\cdots010\cdots0 \end{bmatrix} \end{array} \qquad (N \times |s|\text{-matrix}).$$

This proves that $P \in GL_N(\mathcal{D}(\omega))$ and $P^{-1} = R\,\mathfrak{A}\,\tilde{\mathfrak{A}}^{-1}\,\mathcal{L}^{(t-\mu)}$.
At the end, we note that we can prove the unique existence of P by the same way. The proof of Theorem 1 is now completed.

Proof of Theorem 2. First we note that from Proposition 3.1- 1) and the assumption that $|\mu| = \text{order}_D A$ imply that

$$|\mu| = \text{order}_{D_1} A\,,\ \text{i.e.,}\ \mathfrak{A}(x,D') \in M_{|s|}(\mathcal{D}(\Omega)).$$

Hence we can take $\{s_i, t_j\}$ so that

$$(3.11)\qquad \begin{cases} A_0 \equiv A_0(x), \\ \det_{\sigma'}\mathfrak{A}(q_1,x',D') \not\equiv 0 \quad \text{for any } q_1 \text{ near } p_1, \end{cases}$$

(see Proposition 3.1- 2)). Again by Proposition 3.1- 1), we can prove that

$$(3.12)\qquad \det A_0(x) \not\equiv 0 \quad \text{in } \Omega.$$

In fact, let assume $\det A_0(x) \equiv 0$ and take a left null-vector $(q_1(x),\ldots,q_N(x)) \in \mathcal{O}_p^N \setminus \{0\}$ of $A_0(x)$. Without loss of generality we may assume that

$$s_1 \leqq s_2 \leqq \cdots \leqq s_N \quad \text{and} \quad q_1(x) \not\equiv 0.$$

We define a matrix $Q(x,D_1)$ by

$$Q(x,D_1) = \begin{bmatrix} q_1 & q_2 D_1^{s_2-s_1} & \cdots\cdots & q_N D_1^{s_N-s_1} \\ 0 & 1 & 0\cdots\cdots & 0 \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}.$$

Then $Q(x,D_1)$ is invertible at a point where $q_1 \neq 0$. By this construction of Q, we see that

$$\text{order}_{D_1} QA < \text{order}_{D_1} A = |\mu|,$$

which is a contradiction.

At a point q_1 such that $\det A_0(q_1,x') \not\equiv 0$, the hyperplane $x_1 = q_1$ is not characteristic almost every where. Therefore the well-posedness of the Cauchy problem implies that the mapping $(\#)_q$ is bijective at a dense point q' on $x_1 = q_1$, which proves

$$(3.13) \qquad \det_{\sigma'} \mathfrak{A}(q_1,x',D') \equiv \alpha_{q_1}(x') \not\equiv 0.$$

Hence by (3.11) and (3.13) we can prove

$$(3.14) \qquad \det_\sigma \mathfrak{A}(x,D') \equiv \alpha(x) \not\equiv 0 \quad \text{and} \quad \alpha(q_1,x') \not\equiv 0 \text{ for any } q_1 \text{ near } p_1,$$

(see Lemma 2.3). If there is a point q such that $\alpha(q) = 0$, then an inverse matrix $\mathfrak{A}^{-1}(q_1,x',D')$ of $\mathfrak{A}(q_1,x',D')$ $(q = (q_1,q'))$ has singular coefficients at q'. Therefore the mapping $(\#)_q$ is not surjective, which contradicts the well-posedness of $(A,\mu)_q$ for any q near p. Thus we have proved

$$(3.15) \qquad \det_\sigma \mathfrak{A}(x,D') \equiv \alpha(x) \neq 0 \quad \text{in a neighbourhood of } p.$$

Now it is obvious that $\mathfrak{A}(x,D') \in GL_{|s|}(\mathcal{D}_p)$ implies $\det A_0(p) \neq 0$, which asserts that the hyperplane $x_1 = p_1$ is not characteristic at p. This completes the proof.

4. <u>An example</u>. Let $n=2$ and consider the following matrix.

$$(4.1) \qquad A(x,D) = \begin{pmatrix} D_1 & a(x,D_2) \\ b(x,D_2) & D_1 \end{pmatrix} \qquad (\text{order } a + \text{order } b = 2),$$

where a and b are p.d.op. with holomorphic coefficients in a neighbourhood of the origin. Obviously, A is non-degenerate of $\text{order}_D A = 2$ and hyperplanes $x_1 = p_1$ are all non-characteristic. We consider the Cauchy problem $(A,\mu)_p$ with $\mu = (2,0)$. In this case, we take $\{s_i, t_j\}$ as follows.

$$s_1 = 2, \; s_2 = 1, \; t_1 = 3, \; t_2 = 2.$$

Then we have

$$A_0 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in GL_2(\mathcal{D}(\Omega)), \quad \mathfrak{A}(x,D_2) = \begin{pmatrix} 0 & a & 0 \\ 1 & D_1(a) & a \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, $\det_\sigma \mathfrak{A} = -\sigma(a)(x,\xi_2)$. Therefore, the Cauchy problem (A,μ) is well-posed in a neighbourhood of the origin if and only if $a \equiv a(x) \neq 0$ in a neighbourhood of the origin.

Next, we consider the Cauchy problem $(A,\mu)_p$ at a fixed ponit $p = (p_1,p_2)$. As mentioned in the proof of Theorem 1, the Cauchy problem $(A,\mu)_p$ is well-posed if and only if the mapping $(\#)_p$ is bijective, and it is equivalent to the bijectiveness of the mapping,

$$(\#)'_p \qquad a(p_1,x_2,D_2) : \mathcal{O}_{p_2} \longrightarrow \mathcal{O}_{p_2}.$$

We consider the following three cases of a, in each case the Cauchy problem $(A,\mu)_o$ is well-posed.

1) $a = x_1 D_2 + 1$. In this case, the Cauchy problem $(A,\mu)_p$ is well-posed if and only if $p_1 = 0$, and at $p_1 \neq 0$ the Cauchy

problem has infinitely many solutions.

2) $a = x_2 D_2 + 1$. In this case, $(A,\mu)_p$ is well-posed if and only if $p_2 = 0$, and at $p_2 \neq 0$ the Cauchy problem has infinitely many solutions. We note that the bijectivity of the mapping $(\#)'_o$ is obvious, since $x_2 = 0$ is a regular singular point as an ordinary differential operator.

3) $a = x_2^2 D_2^2 - D_2 + 1$. In this case also the Cauchy problem is well-posed if and only if $p_2 = 0$. We note that a has an irregular singular point $x_2 = 0$ as an ordinary differential operator. By Komatsu [5] or Malgrange [6], we know for the mapping $(\#)'_o$

$$\dim_{\mathbb{C}} \operatorname{Ker} a - \operatorname{codim}_{\mathbb{C}} \operatorname{Im} a = 0.$$

On the other hand, by the construction of a, we easily see that $\dim_{\mathbb{C}} \operatorname{Ker} a = 0$. Therefore we obtain the bijectiveness of $(\#)'_o$. We remark that we know by Malgrange [6] the mapping

$$a : \hat{\mathcal{O}} \longrightarrow \hat{\mathcal{O}} \quad (\hat{\mathcal{O}} \text{ is the set of formal power series})$$

has an index

$$\dim_{\mathbb{C}} \operatorname{Ker} a - \operatorname{codim}_{\mathbb{C}} \operatorname{Im} a = 1 \quad (\operatorname{Ker} a \cong \mathbb{C}).$$

Therefore we see that the Cauchy problem $(A,\mu)_o$ has infinitely many formal power series solutions, but has a unique holomorphic solution.

Referenes

[1] Andronikov, E., Le déterminant selon M. Sato et M. Kashiwara, Séminaire Vaillant, Université de Paris VI, 1979-80.

[2] ______________ , Sur une question de S. Mizohata, C.R. Acad. Sci. Paris, 299 (1984), 923-926.

[3] Dieudonné, J., Les déterminants sur une corps non commutatif, Bull. Soc. Math. France, 71 (1943), 27-45.

[4] Kitagawa, K. and T. Sadamatsu, Sur le problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. Kyoto Univ., 21 (1981), 85-103.

[5] Komatsu, H., On the index of ordinary differential operators, J. Fac. Sci. Univ. Tokyo, 1 (1971), 379-398.

[6] Malgrange, B., Remarques sur le point singuliers des équations différentielles, Enseignement Math., 20 (1974), 147-176.

[7] Miyake, M., Characterisations of the well-posed Cauchy problem for a system in a complex domain, Preprint series, No. 4 (1986), Nagoya Univ. (College of General Education)

[8] Sadamatsu, T., On the Cauchy-Kowalevski theorem for general system of differential equations, J. Math. Kyoto Univ., 24 (1984), 593-609.

[9] Sato, M. and M. Kashiwara, The determinant of pseudo-differential operators, Proc. Japan Acad., 51 (1975), 17-19.

[10] Schapira, P., Microdifferential systems in the complex domain, Springer-Verlag, Berlin-Heiderberg-New York-Tokyo, 1985.

[11] Wagschal, C., Diverses formulations du problème de Cauchy pour un système d'équations aux dérivées partielles, J. Math. pures et appl., 53 (1974), 51-70.

[12] Hufford, G., On the characteristic matrix of differential operators, J. of. Diff. Eq., 1 (1965), 27-38.