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On the Irregularity of $\mathcal{D}_X$-module.

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0. Introduction.

In the theory of singularities of linear differential equations, it is important to know whether a system of linear differential equations is regular singular or irregular singular. To a given system of linear differential equations and a point, we can assign a number, called the irregularity, so that the system is regular singular at the point, if and only if the number is equal to 0. In this article, we investigate the irregularity of linear differential equations.

1. Irregularity of a Linear Ordinary Differential Equation.

Consider a linear ordinary differential equation of the m-th order $Pu=(\sum_{i=0}^{m}a_i(x)(d/dx)^i)u=0$ at the origin in the complex plane, where $a_i(x)$ is holomorphic at a point x for any $i=0,1,...,m$. Without loss of generality, we may suppose that the point x is the origin. P can be regarded as a linear operator of various functional spaces. Let $\mathcal{O}$ be the ring of convergent power-series at the origin and $\hat{\mathcal{O}}$ be the ring of formal power-series at the origin. Let K and $\hat{K}$ be the field of fractions of $\mathcal{O}$ and $\hat{\mathcal{O}}$, respectively, and let E be the ring of convergent Laurent series at the origin. $\mathcal{O}$, K and E are the rings of germs of holomorphic, meromorphic and essentially singular fucntions at the origin, respectively. Malgrange [16] proved

Theorem I. *For* $M=\mathcal{O}$, $\hat{\mathcal{O}}$, K, $\hat{K}$ and E, P *is a linear operator from* M *to itself with the index* $\chi(P,M)$ = dim Ker$(P:M\to M)$ − dim Coker$(P:M\to M)$: $\chi(P,\mathcal{O}) = m-\mathrm{v}(a_m)$, $\chi(P,\hat{\mathcal{O}}) = \sup(i-\mathrm{v}(a_i))$, $\chi(P,K) = -\sup(i-\mathrm{v}(a_i))+(m-\mathrm{v}(a_m))$, $\chi(P,\hat{K}) = 0$ *and* $\chi(P,E) = 0$.

Then, he defined the irregularity $i(P)_0$ of P at the origin as the number $\sup(i-\mathrm{v}(a_i))-(m-\mathrm{v}(a_m))$, which is equal to $\chi(P,\hat{\mathcal{O}})-\chi(P,\mathcal{O})$, $\chi(P,\hat{\mathcal{O}}/\mathcal{O})$, $\chi(P,\hat{K})-\chi(P,K)$, $-\chi(P,K)$, $\chi(P,\hat{K}/K)$, $\chi(P,E)-\chi(P,K)$, and $\chi(P,E/K)$.

Notice that the irregularity is also equal to $\chi(P,E/\mathcal{O})-\chi(P,K/\mathcal{O})$ and that $E/\mathcal{O}$ and $K/\mathcal{O}$ are the rings of hyperfunctions and distributions with support at the origin, respectively.

Let $\mathcal{A}_0$ be the sheaf of germs of functions asymptotic to the formal series 0 over the real blow-up $(C^{\sim},pr)$ of C with the center at the origin. Then, we have $H^1(pr^{-1}(0),\mathcal{A}_0)=\hat{\mathcal{O}}/\mathcal{O}$ by a theorem of Malgrange [17] , P can be also considered as a linear operator from $\mathcal{A}_0$ into itself and

$$0\to\mathcal{K}er(P,\mathcal{A}_0)\to\mathcal{A}_0\to\mathcal{A}_0\to 0\ ,$$

is an exact sequence of sheaves by a theorem of Hukuhara [6] , Malmquist [18] and Trjitzinsky [27] . From these facts, the irreregularity of P at the origin is also equal to dim $H^1(pr^{-1}(0),\mathcal{K}er(P,\mathcal{A}_0))$. On the other hand, by a Theorem of Hukuhara [6] , Malmquist [18] and Trjitzinsky [27] , the function which assigns to $\theta\in pr^{-1}(0)$ the dimension of $\mathcal{K}er(P,\mathcal{A}_0)_\theta$, has only a finite number of discontinuous points. And so, the irregularity is also equal to (1/2)(the total variation of the function: $\theta\in pr^{-1}(0)\to$ dim $\mathcal{K}er(P,\mathcal{A}_0)_\theta$). These are due to Malgrange and Deligne(cf. Bertrand [1]).

Put $\hat{E}=E+\hat{K}$. Then, by the isomorphisms $\hat{\mathcal{O}}/\mathcal{O} = \hat{K}/K = \hat{E}/E$ and $\hat{E}/\hat{K}$ =

E/K, we have the following

Proposition 1. *The irregularity of P is also equal to* $\chi(P,\hat{E})-\chi(P,E)$, $\chi(P,\hat{E})-\chi(P,\hat{K})$, $\chi(P,\hat{E}/E)$, $\chi(P,\hat{E}/\hat{K})$ and $\chi(P,\hat{E})$.

In particular, the dimension of the formal Laurent series solution space to the differential equation $Pu=0$ is not less than the irregularity of P at the origin. (cf. Dwork [4]).

Notice that $\chi(P,\hat{E})$ is equal to $-\chi(P,K)$ if $\chi(P,E)=\chi(P,\hat{K})=0$.

Let P^* be the adjoint operator of P, i.e.

$P^*=\sum_{i=0}^{m}(-1)^i(d/dx)^i(a_i(x)\cdot)$.

Then, by the form of P^* and Theorem I, we have

Proposition 2. $\chi(P,\hat{\mathcal{O}})=\chi(P^*,\hat{\mathcal{O}})$ *and* $\chi(P,\mathcal{O})=\chi(P^*,\mathcal{O})$. *In paticular,* $\chi(P,\hat{\mathcal{O}})-\chi(P,\mathcal{O})=\chi(P^*,\hat{\mathcal{O}})-\chi(P^*,\mathcal{O})$ *and* $i(P)_0=i(P^*)_0$.

By using the fact that $E/\mathcal{O}$ is the dual space of $\mathcal{O}$ and that $K/\mathcal{O}$ is the dual space of $\hat{\mathcal{O}}$, we have

Proposition 3. $\chi(P,\hat{\mathcal{O}})=-\chi(P^*,K/\mathcal{O})$ *and* $\chi(P,\mathcal{O})=-\chi(P^*,E/\mathcal{O})$. *In particular* $\chi(P,\hat{\mathcal{O}})-\chi(P,\mathcal{O})=\chi(P^*,E/\mathcal{O})-\chi(P^*,K/\mathcal{O})$.

By the above results, all the following numbers are equal to the irregularity of P at the origin: $\chi(P,\hat{\mathcal{O}})-\chi(P,\mathcal{O})$, $\chi(P^*,\hat{\mathcal{O}})-\chi(P^*,\mathcal{O})$, $\chi(P,E)-\chi(P,K)$, $\chi(P^*,E)-\chi(P^*,K)$, $\chi(P,E/\mathcal{O})-\chi(P,K/\mathcal{O})$, $\chi(P^*,E/\mathcal{O})-\chi(P^*,K/\mathcal{O})$, $\chi(P,\hat{E}/E)$, $\chi(P^*,\hat{E}/E)$, $\chi(P,\hat{K})-\chi(P,K)$, $\chi(P^*,\hat{K})-\chi(P^*,K)$, $-\chi(P,K)$, $-\chi(P^*,K)$, $\chi(P,\hat{K}/K)$, $\chi(P^*,\hat{K}/K)$, $\chi(P,E/K)$, $\chi(P^*,E/K)$, dim $H^1(pr^{-1}(0),\mathfrak{Ker}(P,\mathcal{A}_0))$, dim $H^1(pr^{-1}(0),\mathfrak{Ker}(P^*,\mathcal{A}_0))$, (1/2)(the total variation of the function: $\theta\in pr^{-1}(0)\to$ dim $\mathfrak{Ker}(P,\mathcal{A}_0)_\theta$) and (1/2)(the total variation of the function: $\theta\in pr^{-1}(0)\to$ dim $\mathfrak{Ker}(P^*,\mathcal{A}_0)_\theta$).

Malgrange [16] proved also that the irregularity is equal to dim Ker$(P,\hat{\mathcal{O}}/\mathcal{O})$ and to dim Ker$(P,E/K)$.

2. Holonomic Differential Modules.

We shall summarize some results on holonomic differential modules which will be used in the latter section. The results are due to Kashiwara [7-10], Kashiwara-Kawai [11], Brilinski-Kashiwara [2], Mebkhout [19-23] (cf. Lê-Mebkhout [12]), Ramis [25-26] and Noumi [24].

Let $\mathcal{D}_X$ be the sheaf of germs of holomorphic linear differential operators of finite order over a complex analytic manifold X of dimension n. Let Ω^i be the sheaf of germs of holomorphic i-forms over X for $i=0,1,\ldots,n$. We write $\mathcal{O}_X$ for Ω^0. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module, namely a coherent $\mathcal{D}_X$-module such that $\mathcal{M}$ has a projective resolution of the form

$$0 \to \mathcal{D}_X^{N_{2n}} \to \cdots \to \mathcal{D}_X^{N_0} \to \mathcal{M} \to 0,$$

and that $R^i\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) = 0$ $(i \neq n)$. Denote by $\mathcal{M}^*$ the adjoint $\mathcal{D}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\Omega^n, \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X))$ of $\mathcal{M}$. Then, we have the relations $\mathcal{M}^{**} = \mathcal{M}$ and $\mathcal{O}^* = \mathcal{O}$.

Theorem II. *For two holonomic $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$,*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{N}^*).$$

In particular,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O}_X).$$

Let Y be an analytic subset of X and $\mathcal{J}_Y$ be the defining ideal. Denote by $\mathcal{O}_{X\hat{|}Y}$ the Zariski formal completion of $\mathcal{O}_X$ along Y: $\mathcal{O}_{X\hat{|}Y} = \operatorname{proj}\lim_{k\to\infty} \mathcal{O}_X(\mathcal{J}_Y^k)$.

Put

$$\Gamma_{[X-Y]}\mathcal{M} = \operatorname{ind}\lim_{k\to\infty} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_Y^k;\mathcal{M}),$$

$$\Gamma_{[Y]}\mathcal{M} = \operatorname{ind}\lim_{k\to\infty} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_Y^k;\mathcal{M}),$$

and denote by $R\Gamma_{[X-Y]}$ and $R\Gamma_{[Y]}$ the right derived functor of $\Gamma_{[X-Y]}$ and $\Gamma_{[Y]}$, repectively, in the derived category.

$R^q\Gamma_{[X-Y]}\mathcal{M} = \mathcal{H}^q_{[X-Y]}(\mathcal{M}) = \text{ind}\lim\limits_{k\to\infty} \mathcal{E}xt\mathcal{g}_{x}(\mathcal{J}^k_Y;\mathcal{M})$,

$R^q\Gamma_{[Y]}\mathcal{M} = \mathcal{H}^q_{[Y]}(\mathcal{M}) = \text{ind}\lim\limits_{k\to\infty} \mathcal{E}xt\mathcal{g}_{x}(\mathcal{O}_X/\mathcal{J}^k_Y;\mathcal{M})$,

are called the q-th algebraic local cohomology sheaf of $\mathcal{M}$. Then, we have the following triangles

$$\begin{array}{ccc} & R\Gamma_{[Y]}\mathcal{M} & \\ {}^{+1}\nearrow & & \searrow \\ R\Gamma_{[X-Y]}\mathcal{M} & \longleftarrow & \mathcal{M}, \end{array}$$

$$\begin{array}{ccc} & R\Gamma_{[X-Y_1\cap Y_2]}\mathcal{M} & \\ {}^{+1}\nearrow & & \searrow \\ R\Gamma_{[X-Y_1]}\mathcal{M}\oplus R\Gamma_{[X-Y_2]}\mathcal{M} & \longleftarrow & R\Gamma_{[X-Y_1\cup Y_2]}\mathcal{M}, \end{array}$$

and we have the relations

$R\Gamma_{[X-Y_1]}R\Gamma_{[X-Y_2]}\mathcal{M} = R\Gamma_{[X-Y_1\cup Y_2]}\mathcal{M}$,

$R\Gamma_{[X-Y]}R\Gamma_{[Y]}\mathcal{M} = R\Gamma_{[Y]}R\Gamma_{[X-Y]}\mathcal{M} = 0$,

$R\Gamma_{[X-Y_1]}R\Gamma_{[Y_2]}\mathcal{M} = R\Gamma_{[Y_2]}R\Gamma_{[X-Y_1]}\mathcal{M}$,

$R\Gamma_{[Y_1]}R\Gamma_{[Y_2]}\mathcal{M}O = R\Gamma_{[Y_1\cap Y_2]}\mathcal{M}$.

If Y is a hypersurface defined by $f=0$ with a holomorphic function f, then $R\Gamma_{[X-Y]}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{O}(*Y)$, $R\Gamma_{[Y]}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{O}(*Y)/\mathcal{M}[-1]$, i.e.,

$R^0\Gamma_{[X-Y]}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{O}(*Y)$, $R^q\Gamma_{[X-Y]}\mathcal{M} = 0$ $(q\neq 0)$,

$R^1\Gamma_{[Y]}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{O}(*Y)/\mathcal{M}$, $R^q\Gamma_{[Y]}\mathcal{M} = 0$ $(q\neq 1)$,

where $\mathcal{O}(*Y)$ denotes the sheaf of germs of meromorphic functions which are holomorphic in $X-Y$ and have poles on Y.

Theorem III. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$,

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_{X\uparrow Y}) = R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathcal{M},\mathcal{O}_X)$.

Theorem IV. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$,

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[X-Y]}\mathcal{N}) = R\mathcal{H}om_{\mathcal{D}_X}((R\Gamma_{[X-Y]}\mathcal{M}^*)^*,\mathcal{N})$,

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[Y]}\mathcal{N}) = R\mathcal{H}om_{\mathcal{D}_X}((R\Gamma_{[Y]}\mathcal{M}^*)^*,\mathcal{N})$.

Denote by Γ_{X-Y} and Γ_Y the sheaves of sections of $\mathcal{M}$ with support in $X-Y$ and Y, respectively. Denote by $R\Gamma_{X-Y}$ and $R\Gamma_Y$ the right derived functors of Γ_{X-Y} and Γ_Y, respectively. $R^q\Gamma_{X-Y}\mathcal{M} = \mathcal{H}^q_{X-Y}(\mathcal{M})$ $R^q\Gamma_Y\mathcal{M} = \mathcal{H}^q_Y(\mathcal{M})$ are called the transcendental local cohomology sheaves. Then, we have the following triangle

$$\begin{array}{ccc} & R\Gamma_Y\mathcal{M} & \\ {}^{+1}\nearrow & & \searrow \\ R\Gamma_{X-Y}\mathcal{M} & \longleftarrow & \mathcal{M}. \end{array}$$

If Y is a hypersurface defined by $f=0$ with a holomorphic function f, then $R\Gamma_{X-Y}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X} j_*j^{-1}\mathcal{O}_X$, $R\Gamma_Y\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X} j_*j^{-1}\mathcal{O}_X/\mathcal{M}[-1]$, i.e.,

$$R^0\Gamma_{X-Y}\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X} j_*j^{-1}\mathcal{O}_X\ ,\ R^q\Gamma_{X-Y}\mathcal{M} = 0\ \ (q\neq 0),$$

$$R^1\Gamma_Y\mathcal{M} = \mathcal{M}\otimes_{\mathcal{O}_X} j_*j^{-1}\mathcal{O}_X/\mathcal{M},\ R^q\Gamma_Y\mathcal{M} = 0\ \ (q\neq 1)\ ,$$

where j is the inclusion of $X-Y$ to X and $j_*j^{-1}\mathcal{O}_X$ is the sheaf of germs holomorphic functions in $X-Y$ eventually with essential singularities on Y.

Theorem V. *For two holonomic $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, $R^q\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M};\mathcal{N}) = \mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})$ are constructible, i.e., $\dim_{\mathbb{C}}\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})_x$ is finite for any point x in X and there exists a stratification on X on each of whose stratum $\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})$ is locally constant.*

By this theorem, the character of the complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})$ at x in X,

$$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{N}))_x = \sum_q(-1)^q\dim_{\mathbb{C}}\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\mathcal{N})_x\ ,$$

is defined as a finite number.

Theorem VI. *For a holonomic $\mathcal{D}_X$-modules $\mathcal{M}$ and a point y in X, $\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\mathcal{O}_{X,y})$ and $\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{M};\hat{\mathcal{O}}_{X,y})$ are the dual vector spaces of $\mathcal{E}xt^{n-q}_{\mathcal{D}_X}(\mathcal{M}^*;\mathcal{B}^\infty_{\{x\}|X})$ and $\mathcal{E}xt^{n-q}_{\mathcal{D}_X}(\mathcal{M}^*;\mathcal{B}_{\{y\}|X})$, respectively, where $\hat{\mathcal{O}}_{X,y}$ is the ring of formal power-series at y, $\mathcal{B}^\infty_{\{y\}|X} = \mathcal{H}^n_y(\mathcal{O}_X)$ and $\mathcal{B}_{\{y\}|X} = \mathcal{H}^n_{[y]}(\mathcal{O}_X)$. Therefore,*

$$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M};\mathcal{O}_X))_y = -\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*;R\Gamma_y\mathcal{O}_X))_y\ ,$$

$$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M};\mathcal{O}_{X\hat{|}y}))_y = -\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*;R\Gamma_{[y]}\mathcal{O}_X))_y\ .$$

Theorem VII. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$,*

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O})$ *and* $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O})$ are mutually Verdier dual, i.e.

$R\mathcal{H}om_{C_X}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O})),C_X) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O})$,

$R\mathcal{H}om_{C_X}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O})),C_X) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O})$.

Moreover, we have

$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O}_X))_x$.

Theorem VIII. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$,*

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_Y(\mathcal{O}_X))$ *and* $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O}_X)_{X|Y}$, are mutually Verdier dual, i.e.

$R\mathcal{H}om_{C_X}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,R\Gamma_Y(\mathcal{O}_X)),C_X) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O})_{X|Y}$,

$R\mathcal{H}om_{C_X}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O}_X)_{X|Y}),C_X) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_Y(\mathcal{O}_X))$.

Moreover, we have

$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_Y(\mathcal{O}_X)))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,\mathcal{O}_X)_{X|Y})_x$.

3. Irregularities of Holonomic $\mathcal{D}_X$-modules.

For a holonomic $\mathcal{D}_X$-module with singular points on Y, by using the derived functors $R\mathcal{H}om$, $R\Gamma_Y$, $R\Gamma_{X-Y}$, $R\Gamma_{[Y]}$ and $R\Gamma_{[X-Y]}$, we define the irregularities of $\mathcal{M}$ at x in X as follow:

$i^Y_{f/c}(\mathcal{M})_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_{X\hat{|}Y}))_x - \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)_{X|Y})_x$,

$i^Y_{e/m}(\mathcal{M})_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{X-Y}\mathcal{O}_X))_x - \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[X-Y]}\mathcal{O}_X))_x$,

$i^Y_{h/d}(\mathcal{M})_x = -\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_Y\mathcal{O}_X))_x + \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[Y]}\mathcal{O}_X))_x$.

By the triangles, we have

$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[X-Y]}\mathcal{O}_X))_x + \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{[Y]}\mathcal{O}_X))_x$,

$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_{X-Y}\mathcal{O}_X))_x + \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},R\Gamma_Y\mathcal{O}_X))_x$,

and so

$i^Y_{h/d}(\mathcal{M})_x = i^Y_{e/m}(\mathcal{M})_x$.

Lemma 1.

$\chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*,R\Gamma_{[Y]}\mathcal{O}_X))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_{X\hat{|}Y}))_x$.

Proof of Lemma 1. By Theorem III,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X|Y}) = R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathcal{M}, \mathcal{O}_X),$$

and by Theorem IV,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^*, R\Gamma_{[Y]}\mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_X}((R\Gamma_{[Y]}\mathcal{M})^*, \mathcal{O}_X).$$

On the other hand, by Theorem VII,

$$\chi(R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathcal{M}, \mathcal{O}_X))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}((R\Gamma_{[Y]}\mathcal{M})^*, \mathcal{O}_X))_x .$$

Hence, we obtain the equality mentioned above. Q.E.D.

Lemma 2. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singular points at most on Y, $i^Y_{f/c}(\mathcal{M})_x = i^Y_{h/d}(\mathcal{M}^*)_x$ and $i^Y_{f/c}(\mathcal{M}^*)_x = i^Y_{h/d}(\mathcal{M})_x$.*

Proof of Lemma 2. By Theorem VIII and Lemma 1, we obtain the equalities. Q.E.D.

For a holonomic $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ defined by a linear ordinary differential operator P, the irregularities coincides with the irregularity defined in the first section. Moreover, we have

Theorem 1. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singular points on Y in the complex analytic manifold X of dimension 1, $i^Y_{f/c}(\mathcal{M})_x$, $i^Y_{h/d}(\mathcal{M})_x$, $i^Y_{e/m}(\mathcal{M})_x$, $i^Y_{f/c}(\mathcal{M}^*)_x$, $i^Y_{e/m}(\mathcal{M}^*)_x$* and *$i^Y_{h/d}(\mathcal{M}^*)_x$ are equal for any point x in X.*

In order to prove this theorem, we use the following facts:

Lemma 3. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ defined by a linear ordinary differential operator P, $\mathcal{M}^* = \mathcal{D}_X/\mathcal{D}_X P^*$. Therefore, $i^Y_{f/c}(\mathcal{M})_x$, $i^Y_{h/d}(\mathcal{M})_x$, $i^Y_{e/m}(\mathcal{M})_x$, $i^Y_{f/c}(\mathcal{M}^*)_x$, $i^Y_{e/m}(\mathcal{M}^*)_x$* and *$i^Y_{h/d}(\mathcal{M}^*)_x$ are equal for any point x in X, where singular points of P are included in Y.*

Proof of Lemma 3. As $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}) = \mathcal{D}_X/P\mathcal{D}_X$, i.e.

$$0 \to \mathcal{D}_X \to \mathcal{D}_X \to \mathcal{D}_X/P\mathcal{D}_X \to 0,$$

is an exact sequence of right $\mathcal{D}_X$-modules. From this sequence and $\mathcal{E}xt^1_{\mathcal{D}_X}(\Omega^1, \mathcal{D}_X) = 0$, we obtain $\mathcal{M}^* = \mathcal{D}_X/\mathcal{D}_X P^*$. Q.E.D.

Lemma 4. *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ in the one-dimensional complex analytic manifold X, is locally generated by one element, and for any* *point x in X and a neighborhood U of x, there exist a linear ordinary differential operator P and a positive number l such that*

$$0 \to \mathcal{O}_X^l \to \mathcal{D}_X/\mathcal{D}_X P \to \mathcal{M} \to 0,$$

is an exact sequence of holonomic $\mathcal{D}_X$-modules in U.

Proof of Lemma 4. See Kashiwara-Kawai [11], Lemma 6.4.2.

Proof of Theorem 1. By Lemma 4 and $i^Y_{f/c}(\mathcal{O}^l)_x$ 0 ,

$$i^Y_{f/c}(\mathcal{M})_x = i^Y_{f/c}(\mathcal{D}_X/\mathcal{D}_X P)_x .$$

Again, by Lemma 4,

$$0 \to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X) \to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{D}/\mathcal{D}P,\mathcal{D}_X) \to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}^l,\mathcal{D}_X) \to 0,$$

is an exact sequence of right $\mathcal{D}_X$-modules. Therefore,

$$0 \to \mathcal{M}^* \to (\mathcal{D}/\mathcal{D}P)^* \to (\mathcal{O}^l)^*,$$

is exact. Hence,

$$i^Y_{f/c}(\mathcal{M}^*)_x = i^Y_{f/c}((\mathcal{D}_X/\mathcal{D}_X P)^*)_x .$$

By Lemma 3,

$$i^Y_{f/c}(\mathcal{D}_X/\mathcal{D}_X P)_x = i^Y_{f/c}((\mathcal{D}_X/\mathcal{D}_X P)^*)_x .$$

And so, we have

$$i^Y_{f/c}(\mathcal{M}^*)_x = i^Y_{f/c}(\mathcal{M})_x .$$

Combining the equalities in Lemma 2, we complete the proof. Q.E.D.

Then, we propose the following

Conjecture. *For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singularities along Y in the complex analytic manifold X of any dimension,* $i^Y_{f/c}(\mathcal{M})_x$, $i^Y_{h/d}(\mathcal{M})_x$, $i^Y_{e/m}(\mathcal{M})_x$, $i^Y_{f/c}(\mathcal{M}^*)_x$, $i^Y_{e/m}(\mathcal{M}^*)_x$ and $i^Y_{h/d}(\mathcal{M}^*)_x$ *are equal for any point x in X.*

By the equalities in Lemmas 1-2, and Theorem VII, the conjecture is equivalent to the validity of the following equality:

$$\chi(R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathcal{M},\mathcal{O}_X))_x = \chi(R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathcal{M}^*,\mathcal{O}_X))_x .$$

Proposition 4. *If $\mathcal{M} = R\Gamma_{[X-Y]}\mathcal{M}$ and $\mathcal{M}^* = R\Gamma_{[X-Y]}\mathcal{M}^*$, the conjecture* *is valid. In particular, If Y is a hypersurface and if $\mathfrak{M}$ and $\mathfrak{M}^*$ are meromorphic integrable connections with singularities along Y, the equality holds.*

Proof of Proposition 4. By the relations, $R\Gamma_{[Y]}R\Gamma_{[X-Y]}\mathfrak{M} = 0$ and $R\Gamma_{[Y]}R\Gamma_{[X-Y]}\mathfrak{M}^* = 0$, the equality is obviously valid.

If $\mathfrak{M}$ and $\mathfrak{M}^*$ are meromorphic integrable connections and if Y is a hypersurface, by the results in [11], the hypothesis is satisfied. Q.E.D.

Remark 1. In general, the following is *not* valid:

$R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathfrak{M}, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}\mathfrak{M}^*, \mathcal{O}_X)$,

$R\Gamma_{[Y]}\mathfrak{M}^* = (R\Gamma_{[Y]}\mathfrak{M})^*$.

For exemple, take $\mathcal{O}(*Y)$ and $\mathcal{O}$, respectively, for $\mathfrak{M}$. However, the characters coincide with each other.

Remark 2. At a *generic point* x in X the equality is valid. Because, in the case where dim X=1, the equality is valid, and for a generic point there exists a non-characteristic submanifold with respect to $\mathfrak{M}$ and we have a isomorphism theorem analogous to the theorem of Cauchy-Kowalewsky (cf. [11], Lemma 6.4.4.).

Moreover, if $\mathfrak{M}$ is a meromorphic integrable connection, then we can calculate the irregularity by useing asymptotic method. (cf. [13-15]).

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