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Kyoto University
Rademacher Series and Self-affine functions

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0. Introduction.

Let \( R_n(x) \) (\( n=1,2, \cdots \)) be the \( n \)th Rademacher function, that is,
\[ R_n(x) = 1 - 2 \varepsilon_n(x), \]
where \( \varepsilon_n(x) \) is the \( n \)th digit of the (finite) binary expansion of \( x \in [0,1) \). In this note we will deal with the Rademacher series
\[ f_r(x) = \sum_{n=1}^{\infty} r^n R_n(x) \quad (0 < r < 1). \]

1) The distribution of \( f_r \).

Let \( f \) be a real function defined on \([0,1]\). We will consider \( f \) as a random variable on the probability space \(([0,1], dx)\) and
define its probability distribution by

$$\mu_f(E) = |\{(x \in [0,1); f(x) \in E)\}| \text{ for } E \in \mathcal{B}(\mathbb{R}),$$

where $|\cdot|$ denotes the one-dimensional Lebesgue measure. If $\mu_f$ is absolutely continuous with respect to the Lebesgue measure, we denote the density by $\alpha_f$.

Theorem (Jessen and Wintner (1935)). The distribution $\mu_{fr}$ is either absolutely continuous or singular with respect to the Lebesgue measure.

It is well-known that $\mu_{fr}$ is singular for $0 < r < 1/2$ and that $\mu_{fr}$ is absolutely continuous for $r = 2^{-1/d}$ with a positive integer $d$.

Theorem (Erdös (1939)). Let $1/2 < r < 1$. If $r^{-1}$ is a Pisot number, that is, an algebraic number whose other conjugates lie inside the unit circle, then $\mu_{fr}$ is singular.

Theorem (Salem (1943)). Let $1/2 < r < 1$. Then $r^{-1}$ is a Pisot number if and only if the Fourier-Stielties transform $\hat{\mu}_{fr}(\xi)$ of $\mu_{fr}$ does not tend to zero as $|\xi| \to \infty$.

It is unknown for which values of $r$ the distribution $\mu_{fr}$ is
absolutely continuous. Garsia (1962) gave several sufficient conditions for $\mu_f$ to be absolutely continuous or to be singular.

2) Hausdorff dimension of level sets of $f_r$.

The following result is referred in Mandelbrot's (1982) book.

**Theorem (Beyer (1962)).** If $r = 2^{-1/d}$ with a positive integer $d$, then the Hausdorff dimension of the level set of $f_r$ is equal to $1 - (1/d)$.

3) Fat baker's transformation.

Alexander and Yorke (1984) introduced the notion of the fat baker's transformation and point out its connection to our Rademacher series.

1. Self-affine functions and Dimensions.

**Definition (Kôno (1986)).** Let $m$ be a positive integer $> 1$. A real function $g$ defined on $[0,1]$ is said to be a self-affine function with the scale parameter $0 < H \leq 1$ to base $m$ if the relation

$$g((j+x)m^{-N}) - g(jm^{-N}) = T_N, j^{-NH} (g(x) - g(0))$$

holds for any $jm^{-N}$ ($j=0,1,\ldots,m^{N-1}$, $N=1,2,\ldots$) and $x \in [0,1)$, where
\( f_{N,j} = 1 \text{ or } -1. \)

We denote the graph of a function \( f \) by \( G(f) \), the Hausdorff dimension of a set \( E \) by \( \dim_H(E) \) and the packing dimension (Taylor and Tricot (1985)) of a set \( E \) by \( \dim_P(E) \). We note that \( \dim_H(E) \leq \dim_P(E) \) for any set \( E \).

Theorem (Kôno (1986)). Let \( g \) be a self-affine function with the scale parameter \( 0 < H \leq 1 \). Assume that the distribution \( \mu_g \) is absolutely continuous with respect to the Lebesgue measure and that the density \( \alpha_g \) belongs to \( L^p(-\infty, \infty) \) for every \( 1 < p < \infty \). Then we have \( \dim_H(G(g)) \geq 2 - H \).

Theorem. Let \( g \) be a bounded self-affine function with the scale parameter \( 0 < H \leq 1 \). Then we have \( \dim_P(G(g)) \leq 2 - H \).

Proposition. The function \( f_r \) \( (0 < r < 1) \) is a bounded self-affine function with the scale parameter \( H = \log_2(1/r) \) to base 2. Here we set \( f_r(1) = f_r(1-) \).

Theorem. Let \( 1/2 < r < 1 \). Assume that the distribution \( \mu_{f_r} \) is absolutely continuous with respect to the Lebesgue measure and that the density \( \alpha_{f_r} \) belongs to \( L^p(-\infty, \infty) \) for any \( 1 < p < \infty \). Then we have \( \dim_H(G(f_r)) = \dim_P(G(f_r)) = 2 - \log_2(1/r) \).

Corollary. If \( r = 2^{-1/d} \) with a positive integer \( d \), then we
have \( \dim_H(G(f_r)) = \dim_p(G(f_r)) = 2 - (1/d) \).

Remark. 1) Przytycki and Urbański proved that if \( \mu_{f_r} \) is absolutely continuous, then we have \( \dim_H(G(f_r)) = 2 - \log_2(1/r) \). We note that their result requires no assumption on the density \( \alpha_{f_r} \). Indeed, it is remarked in their paper that the absolute continuity of \( \mu_{f_r} \) implies the boundedness of \( \alpha_{f_r} \). Furthermore, by the use of Erdős's result, they obtained the following: If \( r^{-1} \) is a Pisot number, then \( 1 < \dim_H(G(f_r)) < 2 - \log_2(1/r) \).

2) For a continuous self-affine function, Kôno (1988) gave a necessary and sufficient condition for the distribution to be absolutely continuous and then, by the use of Kôno's result, Urbański obtained the exact formula of the Hausdorff dimension of the graph of a self-affine function. On the other hand, Bertoin computed the Hausdorff dimension of the level set of a continuous self-affine function.


Theorem (Hutchinson (1981)). Let \((X,d)\) be a complete metric space and \(\{f_j\}_{1 \leq j \leq m}, 2 \leq m < \infty\) be a set of the contractions on \(X\).

1) There is a unique compact subset \(K = K(f_1, f_2, \ldots, f_m)\) of \(X\) such that the equality \(K = \bigcup_{j=1}^{m} f_j(K)\) holds.

2) For any compact subset \(E\) of \(X\), we have \(\lim_{n \to \infty} F_n^*(E) = K\) in the

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Hausdorff metric, where $F$ is defined by $F(E) = \bigcup_{j=1}^{m} f_j(E)$.

In the following we suppose that a function $G$ may take two values $G(jm^{-N})$ and $G(jm^{-N}-)$ at $x = jm^{-N}$ (N=1,2,\ldots).

Theorem (de Rham (1957), Hata (1984)). Suppose that $(f_j)_{1 \leq j \leq m}$ is the same as in the above theorem.

1) The functional equation

$$G(x) = \begin{cases} f_1(G(mx)) & \text{for } 0 \leq x \leq \frac{1}{m}, \\ \vdots \\ f_m(G(mx-(m-1))) & \text{for } \frac{m-1}{m} \leq x \leq 1. \end{cases}$$

has a unique continuous solution $G : [0,1] \to X$ if and only if

$$f_{j+1}(\text{Fix}(f_1)) = f_j(\text{Fix}(f_m))$$

holds for every $1 \leq j \leq m-1$. In this case the graph $G([0,1])$ is compact.

2) Suppose that the above functional equation has a solution $G : [0,1] \to X$, whose range $G([0,1])$ is compact in $X$. Then we have $G([0,1]) = K(f_1,f_2,\ldots,f_m)$. Furthermore, if $X$ is a vector space in addition, we can approximate $G([0,1])$ by broken-lines: Define

$$G^0(x) = (\text{Fix}(f_m)-\text{Fix}(f_1))x + \text{Fix}(f_1)$$
and, for $n = 1, 2, \ldots$,

$$G^{(n)}(x) = \begin{cases} f_1(G^{(n-1)}(mx)) & \text{for } 0 \leq x \leq \frac{1}{m}, \\ \vdots & \\ f_m(G^{(n-1)}(mx-(m-1))) & \text{for } \frac{m-1}{m} \leq x \leq 1. \end{cases}$$

Then we have $K(f_1, f_2, \ldots, f_m) = \lim_{n \to \infty} G^{(n)}([0,1])$ in the Hausdorff metric.

We now consider a characterization of self-affine function by a functional equation. We suppose that a self-affine function $g$ satisfies the equality

$$g((j+1)m^{-N}) - g(jm^{-N}) = T_{N,j/m}^{-NH}(g(1) - g(0))$$

for any $jm^{-N}$ ($j=0, 1, \ldots, m^{N-1}$, $N=1, 2, \ldots$).

Theorem. A real function $g$ defined on $[0,1]$ with $g(0) = 0$ is a self-affine function with the scale parameter $0 < H \leq 1$ to base $m$ if and only if it satisfies the following functional equations:

$$g(x) = \begin{cases} a_1 m^{-H}g(mx) & \text{for } 0 \leq x \leq \frac{1}{m}, \\ a_2 m^{-H}g(mx-1) + b_2 & \text{for } \frac{1}{m} \leq x \leq \frac{2}{m}, \\ \vdots & \\ a_m m^{-H}g(mx-(m-1)) + b_m & \text{for } \frac{m-1}{m} \leq x \leq 1, \end{cases}$$
where \( a_j = 1 \) or \(-1\) and \( b_j \) is a constant for each \( j \).

It is also possible to treat the self-affine function as a curve in the plane.

**Corollary.** Define

\[
f_j: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1/m & 0 \\ 0 & a_j m^{-H} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} j-1/m \\ b_j \end{pmatrix}
\]

for \( j = 1, 2, \ldots, m \), where \( a_j = 1 \) or \(-1\) and \( b_j \) is a constant for each \( j \) except for \( b_0 = 0 \). Then \( g(x) \) is a self-affine function with the scale parameter \( 0 < H \leq 1 \) to base \( m \) if and only if \( G(x) = (x, g(x)) \) satisfies the functional equation

\[
G(x) = \begin{cases} 
  f_1(G(mx)) & \text{for } 0 \leq x \leq \frac{1}{m}, \\
  \vdots & \\
  \vdots \\
  f_m(G(mx-(m-1))) & \text{for } \frac{m-1}{m} \leq x \leq 1.
\end{cases}
\]

If \( G([0,1]) \), which is the graph of \( g \), is compact, then it coincides with the unique compact subset \( K = K(f_1, f_2, \ldots, f_m) \) satisfying \( K = \bigcup_{j=1}^{m} f_j(K) \).

We normalize \( f_x \) by setting

\[
g_r(x) = (1/2) - ((1-r)/2r)f_r(x) = ((1-r)/r)\sum_{n=1}^{\infty} r^n \varepsilon_n(x).
\]
Then \( g_r \) is also a self-affine function with the scale parameter \( H = \log_2(1/r) \) to base 2 and satisfies the following functional equation:

\[
g_r(x) = \begin{cases} 
  rg_r(2x) & \text{for } 0 \leq x \leq \frac{1}{2}, \\
  rg_r(2x-1) + (1-r) & \text{for } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

If we define

\[
f_1: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

\[
f_2: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1-r \end{pmatrix},
\]

then \( G_r(x) = (x, g_r(x)) \) satisfies the functional equation

\[
G_r(x) = \begin{cases} 
  f_1(G_r(2x)) & \text{for } 0 \leq x \leq \frac{1}{2}, \\
  f_2(G_r(2x-1)) & \text{for } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Since \( f_1 \) and \( f_2 \) are contractions, there is a unique compact subset \( K = K(f_1, f_2) \) of \( R^2 \) satisfying \( K = f_1(K) \cup f_2(K) \). In this case, \( f_1(\text{Fix}(f_2)) \neq f_2(\text{Fix}(f_1)) \) and hence the functional equation has no continuous solution \( G_r: [0,1] \rightarrow R^2 \). However, it is easily seen that \( G_r([0,1]) \) is compact. Hence we have \( K = G_r([0,1]) \) and \( \lim_{n \rightarrow \infty} G_r^{(n)}([0,1]) = K \) in the Hausdorff metric.

We next consider the relation between \( h_r \) and Lebesgue's singular
function. Define

$$\tilde{f}_1: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\tilde{f}_2: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & 1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ r \end{pmatrix}.$$ 

Since $\tilde{f}_1$ and $\tilde{f}_2$ are contractions, there is a unique compact subset $\bar{K} = \bar{K}(f_1,f_2)$ of $\mathbb{R}^2$ satisfying $\bar{K} = \tilde{f}_1(\bar{K}) \cup \tilde{f}_2(\bar{K})$. In this case, since $\bar{f}_2(\text{Fix}(\tilde{f}_1)) = \tilde{f}_1(\text{Fix}(\tilde{f}_2))$, the functional equation

$$\bar{G}_r(x) = \begin{cases} \tilde{f}_1(\bar{G}_r(2x)) & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \tilde{f}_2(\bar{G}_r(2x-1)) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

has a unique continuous solution $\bar{G}_r: [0,1] \rightarrow \mathbb{R}^2$ (de Rham (1957), Hata and Yamaguti (1984)). If we set $\bar{G}_r(x) = (x,\bar{G}_r(x))$, the above functional equation is equivalent to

$$\bar{G}_r(x) = \begin{cases} \tilde{G}_r(2x) & \text{for } 0 \leq x \leq \frac{1}{2}, \\ (1-r)\bar{G}_r(2x-1) + r & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The function $\bar{G}_r$ is so-called Lebesgue's singular function. Since $\bar{G}_r([0,1])$ is compact, $\lim_{n \rightarrow \infty} \bar{G}_r^{(n)}([0,1]) = K$ in the Hausdorff metric. This is nothing but a Salem's (1943) geometric construction of Lebesgue's singular function.
We finally remark that the distribution function $F_r(x) = \mu_{\mathcal{E}_r}((-\infty, x])$ satisfies the functional equation

$$F_r(x) = \frac{1}{2} (F_r\left(\frac{x}{r}\right) + F_r\left(-\frac{x}{r} - \frac{1-r}{r}\right));$$

$$F_r(x) = 0 \text{ for } x \leq 0; \quad F_r(x) = 1 \text{ for } x \geq 1.$$

References


