

A note on a theorem of Fukasawa-Gel'fond

by

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§ 1. Introduction

In 1915, G. Pólya [5] showed that an entire function f satisfying $f(\mathbb{N}_0) \subset \mathbb{Z}$ and $\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r} < \log 2$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$|f|_r := \max_{|z| \leq r} |f(z)|$, is a polynomial. Because of the existence of the en-

tire function 2^z , the value $\log 2$ in the above result is best possible. Let $\ell \in \mathbb{N}_0$ and let $f^{(k)}(z)$ for $k \in \mathbb{N}_0$ denote k -th derivative of $f(z)$. Then A. Gel'fond [2], in 1929, proved that an entire function f which satisfies

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r} < (\ell+1) \log \left\{ 1 + e^{-\ell/(\ell+1)} \right\} \text{ and } f^{(k)}(\mathbb{N}_0) \subset \mathbb{Z} \text{ for all}$$

$k=0, 1, \dots, \ell$ is a polynomial. A. Selberg [6] showed that the above upper bound can be replaced by $(\ell+1) \log \omega_\ell$ with some $\omega_\ell > 1 + e^{-\ell/(\ell+1)}$ when $\ell \geq 1$.

In another direction, S. Fukasawa [1], in 1926, studied entire functions satisfying $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$, and in 1929, A. Gel'fond [3] refined the result of Fukasawa and obtained: There exists a real number $\alpha > 0$ such that

if f is an entire function satisfying $\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} < \alpha$ and

$f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$, then f is a polynomial.

Several authors have tried to determine the exact value of α , and finally in 1981, F. Gramain [4], proved a more general theorem to show that the best possible value of α is equal to $\pi/2e$:

Theorem. (F. Gramain) Let \mathbb{K} be any imaginary quadratic number field whose discriminant is $-\Delta$ and let $a := \sqrt{\Delta}/2$ be the area of the fundamental parallelogram of the lattice of integers $\mathcal{O}_{\mathbb{K}}$ in \mathbb{K} .

(i) If f is an entire function satisfying

$$f(\Theta_{\mathbb{K}}) \subset \Theta_{\mathbb{K}} \quad (1)$$

and

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} < \frac{\pi}{2ea}, \quad (2)$$

then f is a polynomial.

(ii) There exists an entire function f such that $f(\Theta_{\mathbb{K}}) \subset \Theta_{\mathbb{K}}$ and

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} = \frac{\pi}{2ea}.$$

In particular, f is not a polynomial.

In this note, we shall prove the following generalization of part

(ii) of Gramain's theorem:

Theorem. Let \mathbb{K} and $\Theta_{\mathbb{K}}$ be as above, then there exists an entire function f such that

$$\frac{1}{k!} f^{(k)}(\Theta_{\mathbb{K}}) \subset \Theta_{\mathbb{K}} \quad \text{for all } k = 0, 1, \dots, \ell, \quad (3)$$

and

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} = \frac{(\ell+1)\pi}{2ea}.$$

It follows from our theorem that when the condition (1) in Gramain's theorem is replaced by (3), the upper bound which corresponds to the right-hand side of (2) does not exceed $(\ell+1)\pi/2ea$.

§ 2. Lemmas

In this section we prepare some notions and lemmas.

Let $\Lambda = \{\zeta_m\}_{m \in \mathbb{N}_0}$ be any homogeneous lattice in $\mathbb{R}^2 = \mathbb{C}$, whose elements are arranged in the following way: $m < n$ ($m, n \in \mathbb{N}_0$) if and only if we have either $|\zeta_m| < |\zeta_n|$ or $|\zeta_m| = |\zeta_n|$ with $\arg \zeta_m < \arg \zeta_n$.

Lemma 1. ([4] lemma 2) Let a be the area of the fundamental parallelogram of Λ , then we have for any $n \in \mathbb{N}_0$,

$$\left| |\zeta_n| - \sqrt{\frac{an}{\pi}} \right| \leq c_1.$$

Here and in the sequel c_1, c_2, \dots denote effectively computable positive constants depending only on Λ .

Lemma 2. ([4] lemma 3) Let $n \in \mathbb{N}$ with $n \geq 2$ and let $z \in \mathbb{C}$, if we define $\theta \geq 0$ by $|z| = \theta |\zeta_n|$, then

$$\left| \log \prod_{j=0}^n |z - \zeta_j| - \frac{1}{2} n \log n - n w(\theta) \right| \leq c_2 \max(1, \theta) \sqrt{n \log n},$$

$$|z - \zeta_j| \geq 1$$

where

$$w(\theta) := \begin{cases} \log \theta - \frac{1}{2} \log \frac{\pi}{a} & \text{if } \theta \geq 1, \\ \frac{\theta^2}{2} - \frac{1}{2} - \frac{1}{2} \log \frac{\pi}{a} & \text{if } \theta \leq 1. \end{cases}$$

In what follows we assume that k is always an integer with $0 \leq k \leq \ell$

The following lemma 3 is a generalization of lemma 7 in [4].

Lemma 3. Let $\Lambda = \{\zeta_m\}_{m \in \mathbb{N}_0}$ and a be as in lemma 1 and let f be an entire function. Define for $n \in \mathbb{N}_0$,

$$P_{n,k}(z) := \prod_{m=0}^{n-1} (z - \zeta_m)^{\ell+1} (z - \zeta_n)^k$$

with the convention that $P_{0,k}(z) := z^k$, and let

$$a_{n,k} := \frac{1}{2\pi i} \int_{C_n} \frac{f(\zeta)}{P_{n,k+1}(\zeta)} d\zeta, \quad (4)$$

where C_n is a closed curve containing the points $\zeta_0, \zeta_1, \dots, \zeta_n$ in its interior. Then the following formula holds for all $z \in \mathbb{C}$ contained in the interior of C_n :

$$f(z) = \sum_{n=0}^N \sum_{k=0}^{\ell} a_{n,k} P_{n,k}(z) + \frac{P_{N+1,0}(z)}{2\pi i} \int_{C_N} \frac{f(\zeta)}{P_{N+1,0}(\zeta)(\zeta-z)} d\zeta. \quad (5)$$

(i) If f satisfies

$$\tau := \overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} < \frac{(\ell+1)\pi}{2a}, \quad (6)$$

then the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\ell} a_{n,k} P_{n,k}(z) \quad (7)$$

converges uniformly to f on any compact set in \mathbb{C} , and the coefficients $a_{n,k}$ satisfy

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log |a_{n,k}| + \frac{\ell+1}{2} n \log n}{n} \leq \frac{\ell+1}{2} \left\{ 1 + \log \left(\frac{\tau}{\ell+1} \right) \right\}. \quad (8)$$

(ii) If $\{b_{n,k}; n \in \mathbb{N}_0, 0 \leq k \leq \ell\}$ is a sequence of complex numbers satisfying

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log |b_{n,k}| + \frac{\ell+1}{2} n \log n}{n} =: \lambda < \frac{\ell+1}{2} \left(1 + \log \frac{\pi}{a}\right), \quad (9)$$

then the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\ell} b_{n,k} P_{n,k}(z) \quad (10)$$

converges uniformly on any compact set in \mathbb{C} and defines an entire function g satisfying

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |g|_r}{r^2} \leq \frac{\ell+1}{2} \exp\left(\frac{2\lambda}{\ell+1} - 1\right). \quad (11)$$

Remark. From the conclusions of both parts of lemma 3, the inequalities (8) and (11) can be replaced by equalities.

Proof. We first prove part (ii) of lemma 3. Let $\rho_n := |\zeta_n|$ and fix λ' such that $\lambda < \lambda' < \frac{\ell+1}{2}(1 + \log \frac{\pi}{a})$, and choose a sufficiently small $\theta \in]0, 1[$ satisfying

$$\lambda' + \frac{\ell+1}{2} (\theta^{2-1} - \log \frac{\pi}{a}) < 0. \quad (12)$$

By the assumption (9), there exists an integer n_1 such that

$\log |b_{n,k}| + \frac{\ell+1}{2} n \log n \leq \lambda' n$ for all $n \geq n_1$. For any $R \geq 0$, there exists some integer $n_0 \geq n_1$ such that $\theta \rho_{n_0} \geq R$. Hence it follows from lemma 2 that

$$\log |b_{n,k} P_{n,k}(z)| \leq \left\{ \lambda' + \frac{\ell+1}{2} (\theta^{2-1} - \log \frac{\pi}{a}) \right\} n + o(n) \leq -C_4 n$$

for all $n > n_0$ and all $z \in \mathbb{C}$ with $|z| \leq R$. Therefore the series (10) converges uniformly on any compact set in \mathbb{C} and defines an entire function g of which we consider the rate of growth.

Let $z \in \mathbb{C}$ satisfy $|z| =: r > \rho_{n_0}$, then, using lemmas 1 and 2, we get for all n with $\rho_n \geq r$

$$\log |b_{n,k} P_{n,k}(z)| \leq \left\{ \lambda' + \frac{\ell+1}{2} \left(\frac{r^2}{\rho_n^2} - 1 - \log \frac{\pi}{a} \right) \right\} n + O(\sqrt{n} \log n), \quad (13)$$

and also for all $n \geq n_1$ with $\rho_n \leq r$

$$\log |b_{n,k} P_{n,k}(z)| \leq \left\{ \lambda' + (\ell+1) \left(\log \frac{r}{\rho_n} - \frac{1}{2} \log \frac{\pi}{a} \right) \right\} n + O(r \log r) \quad (14)$$

If we define s_0, s_1 and s_2 by

$$\begin{aligned}
 s_0 &:= \left| \sum_{0 \leq n \leq n_1} \sum_{0 \leq k \leq \ell} b_{n,k} P_{n,k}(z) \right|_r, \\
 s_1 &:= \left| \sum_{\substack{n \geq n_1 \\ \rho_n \leq r}} \sum_{0 \leq k \leq \ell} b_{n,k} P_{n,k}(z) \right|_r, \\
 s_2 &:= \left| \sum_{\rho_n > r} \sum_{0 \leq k \leq \ell} b_{n,k} P_{n,k}(z) \right|_r,
 \end{aligned}$$

then we have

$$|g|_r \leq s_0 + s_1 + s_2. \tag{15}$$

Making use of lemmas 1, 2 and (13), we get

$$s_2 \leq e^{(\ell+1)\pi r^2/2a} \sum_{\rho_n > r} \exp \left[\left\{ \lambda' - \frac{\ell+1}{2} \left(1 + \log \frac{\pi}{a} \right) \right\} n + O(\sqrt{n} \log n) \right],$$

and thus, by (12) and the fact that $\frac{\pi}{a} \left(\lambda' - \frac{\ell+1}{2} \log \frac{\pi}{a} \right) \leq \frac{\ell+1}{2} \exp\left(\frac{2\lambda'}{\ell+1} - 1\right)$,

$$\log s_2 \leq \frac{\ell+1}{2} r^2 \exp\left(\frac{2\lambda'}{\ell+1} - 1\right) + O(r \log r). \tag{16}$$

Since we have, by (14),

$$s_1 \leq \sum_{\rho_n \leq r} \exp \left[n \left\{ \lambda' + (\ell+1) \left(\log r - \frac{1}{2} \log n \right) \right\} + O(r \log r) \right],$$

we obtain from lemma 1 and the fact that $\max_{x>0} x \left\{ \lambda' + (\ell+1) \left(\log r - \frac{1}{2} \log x \right) \right\}$

$$= \frac{\ell+1}{2} r^2 \exp\left(\frac{2\lambda'}{\ell+1} - 1\right),$$

$$\log s_1 \leq \frac{\ell+1}{2} r^2 \exp\left(\frac{\lambda'}{\ell+1} - 1\right) + o(r^2). \tag{17}$$

Therefore, from (15), combining the estimates (16), (17) and $\log s_0 \leq c_3 \log r$ and then letting $\lambda' \rightarrow \lambda$, we get the conclusion (11) of part (ii) of lemma 3.

We next prove part (i) of lemma 3. Let τ' with $\tau < \tau' < (\ell+1)\pi/2a$ be fixed and let C_n in (4) be the circle with center 0 and radius $\theta \rho_n$ where $\theta > 1$ is a parameter which will be chosen later. Then, by (4), (6) and lemma 2, we get

$$\begin{aligned}
 \log |a_{n,k}| &\leq \tau' (\theta \rho_n)^2 + \log \theta \rho_n - \\
 &\quad - (\ell+1) \left\{ \frac{1}{2} n \log n + n \left(\log \theta - \frac{1}{2} \log \frac{\pi}{a} \right) \right\}
 \end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$, and hence, using lemma 1, we obtain

$$\begin{aligned}
 \frac{1}{n} \left\{ \log |a_{n,k}| + \frac{\ell+1}{2} n \log n \right\} &\leq \frac{\tau' \theta^2 a}{\pi} - \\
 &\quad - (\ell+1) \left(\log \theta - \frac{1}{2} \log \frac{\pi}{a} \right) + o(1).
 \end{aligned}$$

Since the right-hand side of the above inequality attains its minimal value when $\theta^2 = (\ell + 1)\pi / 2a\tau'$, we obtain

$$\overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \left\{ \log |a_{n,k}| + \frac{\ell + 1}{2} n \log n \right\} \leq \frac{\ell + 1}{2} \left\{ 1 + \log \left(\frac{2\tau'}{\ell + 1} \right) \right\},$$

which yields (8).

Thus, by the assumption $\tau < (\ell + 1)\pi/2a$ as well as the inequality (8), it follows from part (ii) of lemma 3 that the series (7) converges on any compact set and defines an entire function $g(z)$ satisfying $\overline{\lim}_{r \rightarrow +\infty} \frac{\log |g|_r}{r^2} \leq \tau$ and further, from the formula (5), $f^{(k)}(\zeta) = g^{(k)}(\zeta)$ holds for all $k = 0, 1, \dots, \ell$ and all $\zeta \in \Lambda$. Hence, to complete the proof of part (i) of lemma 3, it is sufficient to show that an entire function ϕ satisfying

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\log |\phi|_r}{r^2} =: \tau < \frac{(\ell + 1)\pi}{2a} \quad (18)$$

and $\phi^{(k)}(\Lambda) = \{0\}$ for all $k = 0, 1, \dots, \ell$, is identically zero. Assume that ϕ is not zero, then the Taylor expansion of ϕ at $z=0$ has the form $\alpha_h z^h + \alpha_{h+1} z^{h+1} + \dots$ with some $\alpha_h \neq 0$ ($h \geq 0$) and, by Jensen's formula, we have

$$\log |\phi|_r \geq \log |\alpha_h| + h \log r + (\ell + 1) \cdot \sum_{0 < |\zeta_j| \leq r} \log \frac{r}{|\zeta_j|}. \quad (19)$$

Applying lemmas 2 and 3 to the right-hand side of (19), we obtain $\tau \geq (\ell + 1)\pi/2a$, which contradicts the assumption (18).

§ 3. Proof of Theorem

We can deduce the conclusion of our theorem by taking $\Lambda = D = \mathcal{O}_{\mathbb{K}}$ in the following lemma 4:

Lemma 4. Let the lattice $\Lambda \subset \mathbb{R}^2 = \mathbb{C}$ be as in lemma 3 and let D be a subset of \mathbb{C} which has the following property: there exists a constant $\delta > 0$ such that for any $z \in \mathbb{C}$, we can find some $d \in D$ satisfying $|z - d| \leq \delta$. Then there exists an entire function satisfying $\frac{1}{k!} f^{(k)}(\Lambda) \subset D$ for all

$$k = 0, 1, \dots, \ell \quad \text{and} \quad \overline{\lim}_{r \rightarrow +\infty} \frac{\log |f|_r}{r^2} = \frac{(\ell + 1)\pi}{2ea}.$$

Proof. We use the same notation as in lemma 3. Since the coefficients of a generalized interpolation series of f at the points of Λ are given by (4), we obtain from the residue theorem

$$a_{n,k} = \sum_{m=0}^{n-1} \sum_{h=0}^{\ell} \frac{f^{(h)}(\zeta_n)}{h!(\ell-h)!} \left[\left(\frac{d}{d\zeta}\right)^{\ell-h} \frac{(\zeta - \zeta_m)^{\ell+1}}{P_{n,k+1}(\zeta)} \right]_{\zeta = \zeta_m} +$$

$$+ \sum_{h=0}^k \frac{f^{(h)}(\zeta_n)}{h!(k-h)!} \left[\left(\frac{d}{d\zeta}\right)^{k-h} \frac{1}{P_{n,0}(\zeta)} \right]_{\zeta = \zeta_n}.$$

Thus we have

$$P_{n,0}(\zeta_n) a_{n,k} = \sum_{m=0}^{n-1} \sum_{h=0}^{\ell} p_{h,k,m,n} \frac{f^{(h)}(\zeta_m)}{h!} + \sum_{h=0}^{k-1} q_{h,k,n} \frac{f^{(h)}(\zeta_n)}{h!} +$$

$$+ \frac{f^{(k)}(\zeta_n)}{k!}.$$

with some $p_{h,k,m,n}$ and $q_{h,k,n} \in \mathbb{C}$. Therefore, if we define $a_{n,k}$ by choosing $\frac{1}{k!} f^{(k)}(\zeta_n) \in D$ such that

$$|P_{n,0}(\zeta_n) a_{n,k} - 2\delta| \leq \delta, \quad (20)$$

then we get from lemmas 1 and 2

$$\log |a_{n,k}| = -\frac{\ell+1}{2} (n \log n - n \log \frac{\pi}{a}) + o(n),$$

and thus, from part (ii) of lemma 3, the series (10) converges and defines an entire function f which satisfies the required conditions; and the proof of lemma 4 is completed.

Remark. As in the remark of lemma 8 in [4], we can construct infinitely (even uncountably) many functions f , by choosing $\frac{1}{k!} f^{(k)}(\zeta_n) \in D$ such that, for example $|a_{n,k} P_{n,k}(\zeta_n) - 4\delta| \leq 3\delta$ instead of (20).

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