On the Finite Church-Rosser Property of Nonlinear Term Rewriting Systems *
(Preliminary report)

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Abstract

This paper proves that an infinitely nonoverlapping (possibly nonlinear) TRS is finitely Church-Rosser. The condition infinitely nonoverlapping is a nonoverlapping condition under unification with infinite terms. The property finitely Church-Rosser is equivalent to uniquely normalizing with respect to equality (i.e. $x = y \Rightarrow x \equiv y$ for any normal forms $x, y$), and is an intermediate property between Church-Rosser and uniquely normalizing with respect to reduction.

1 Introduction

Equational logic has been applied in the program-specification and the other logical frameworks. A Term Rewriting System (TRS), intuitively which is a set of directed equations (deduction rules), have been adopted for an execution model of equational logic. That is, a TRS converts expressions using equations only forward, whereas equational logic permits using them both forward and backward. For these purposes, one of the important properties of a TRS is confluence-related properties, namely, Church-Rosser property, unique normalizability, etc. Church-Rosser property, which is equivalent to confluence, guarantees that the congruent relation (equality) among expressions will be examined without back-track. Unique normalizability, which is deduced from confluence, guarantees that the result of an execution is uniquely determined if terminates.

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Several criteria have been proposed for the confluence of a TRS and its variations \[4,5,6,8,9,10\]. Generally speaking, neither confluence, unique-normalizability, nor termination is decidable. Most of known sufficient conditions for confluence of a TRS are restricted to nonoverlapping TRSs \[4,5,6\].

Intuitively speaking, the nonoverlapping property means that no reducible expressions (redexes) overlap on one term. This property seems to provide the implicit commutativity of reductions. Thus, a nonoverlapping TRS would be confluent.

In fact, a nonoverlapping TRS is known to be confluent if either left-linear or strongly-normalizing, where a TRS is said to be left-linear if every variables appear at most once on left-hand side of reduction rules, and said to be strongly-normalizing if every reduction paths are terminating.

Nevertheless, both possibly nonterminating and nonlinear TRSs may be neither confluent nor uniquely-normalizing even if nonoverlapping. Typical nonconfluent cases are shown in the following three examples.

**Example 1** \[9\]

\[
R_1 \overset{\text{def}}{=} \left\{ \begin{array}{ll}
    d(x,x) & \rightarrow 0 \\
    d(x,f(x)) & \rightarrow 1 \\
    2 & \rightarrow f(2)
\end{array} \right.
\]

(Critical on \(d(2,2) \rightarrow d(2,f(2))\).)

**Example 2**

\[
R_2 \overset{\text{def}}{=} \left\{ \begin{array}{ll}
    d(f(x),x) & \rightarrow 0 \\
    d(x,g(x)) & \rightarrow 1 \\
    2 & \rightarrow f(3) \\
    3 & \rightarrow g(2)
\end{array} \right.
\]

(Critical on \(d(2,3) \rightarrow d(f(3),3), d(2,g(2))\).)

![Fig.1 R overlapping example \(R_1\)](image)

![Fig.2 R overlapping example \(R_2\)](image)

Examples 1 and 2 show the existence of unmeetable branches on nonlinear reduc-
tion paths, although no redexes overlap. For instance, in Example 1 reductions on leaves of some redex will produce another redex corresponding to a different reduction rule. In fact, a redex $d(2, 2)$ of the first rule is converted to a different redex $d(2, f(2))$ of the second rule by the reduction $2 \rightarrow f(2)$ at a leave of $d(2, 2)$. Further in Example 2, reductions in subterms of some non-redex term will produces different redexes. In fact, a non-redex term $d(2, 3)$ is reduced to either a redex $d(f(3), 3)$ of the first rule or a redex $d(2, g(2))$ of the second rule. These are said to be $R$-overlapping. This concept will be further discussed in Section 4.2.

Example 3

$$R_3 \overset{\text{def}}{=} \left\{ \begin{array}{l} d(x, x) \rightarrow 0 \\ f(x) \rightarrow d(x, f(x)) \\ 1 \rightarrow f(1) \end{array} \right\}$$

(Critical on $f(f(1)).$)

$$1 \rightarrow f^n(1) = f(f^{n-1}(1))$$
$$\rightarrow d(f^{n-1}(1), f^n(1))$$
$$\rightarrow d(f^n(1), f^n(1))$$
$$\rightarrow 0$$

Example 3 shows the case where once some redex is modified by reductions at its leaves, then it will be never recovered as a redex. Note that such a reduction path will never terminate. This example is analogous to the nonconfluent example of $\lambda$-calculus with a nonlinear $\delta$-reduction rule, namely $\delta xx \rightarrow \varepsilon$ by Staples [1].

One of the previously investigated approaches is restricting reduction strategies. That is, a reduction rule is applied only when accompanied conditions are satisfied. This is called a conditional TRS. The main known result is that a restricted nonlinear membership conditional TRS is confluent [9]. Intuitively speaking, the restricted nonlinear membership conditional TRS imposes that a nonlinear reduction rule is applied only after subterms of all nonlinear occurrences reach normal forms, by analogy of $\lambda$-calculus with Church's $\delta$ [1].

This paper investigates the finite Church-Rosser property of an infinitely nonoverlapping TRS. Finite Church-Rosser property is Church-Rosser property on normalizable terms. That is, congruence between two terms is examined by syntactical comparison between their normal forms (if exist). This property implies uniqueness of the normal form. The assumption, infinitely nonoverlapping, is a natural extension of the left-linear nonoverlapping, and is decidable. The only difference between them is that the unification with infinite terms [2,3,7] is applied instead of usual unifications.
The main theorem investigated here is

*An infinitely nonoverlapping TRS is finitely Church-Rosser.*

For the investigation, a key concept $R$-nonoverlapping is also proposed. A TRS $R$ is said to be $R$-nonoverlapping iff all reduction paths have no substantially separated branches.

First, an infinitely nonoverlapping TRS $R$ is shown to be $R$-nonoverlapping. Second, an $R$-nonoverlapping TRS is shown to be finitely Church-Rosser.

Note that Example 1 and 2 above are both infinitely overlapping, though nonoverlapping. In fact, $d(x, x)$ and $d(x, f(x))$ have an infinite unifier $x = f(f(\cdots))$. Also, $d(f(x), x)$ and $d(x, g(x))$ have an infinite unifier $x = f(g(f(g(\cdots))))$.

Example 3 is infinitely nonoverlapping and is not confluent. This causes from that once a reduction path enters an unmeetable path, the reduction sequence always falls into an infinite loop, and never terminates. Thus, by restricting discussion to normalizable terms, $R_3$ is shown to be finitely Church-Rosser.

### 2 Unification with infinite terms

#### 2.1 Variation of unifications

Unifications are classified into following three classes. They are,

- Unification without occur check.
- Unification with occur check.
- Unification with infinite terms (called *infinite unification*).

Unification without occur check does not care on name conflicts. Thus, even for finite terms, this is not correct for non-linear terms. For instance, $f(x, x)$ and $f(g(y), h(y))$ are unified as $\{x = g(y), \ x = h(y)\}$. In other words, consistency of binding environments is not preserved.

In contrast, unification with occur check treats name conflicts as unification failed. This is correct on finite terms, but not correct on infinite terms. For instance, unification between $f(x, x)$ and $f(z, g(z))$ is failed, though it can be unified with the infinite term $f(g(g(g(\cdots))), g(g(g(\cdots))))$.

There have been proposed several algorithms for unification with infinite terms [2,3,7]. The substantial difference on infinite unification is that expressions defining a binding environment can refer the environment itself recursively. Therefore, a looped infinite term such as $g(g(g(\cdots)))$ (the solution for $x = g(x)$) is permitted as a unifier. A looped infinite term can be represented by a cyclic finite graph as an internal form.
Thus, the algorithm of infinite unification terminates as same as usual unification algorithms do.

In the next section, the algorithm called \textit{UNIFY0} for unification with infinite terms by Martelli and Rossi is briefly introduced. For details, refer [7].

\section*{2.2 Algorithm of unification with infinite terms}

The algorithm \textit{UNIFY0} computes the \textit{common parts} and \textit{frontiers} iteratively. This terminates when frontiers reach \textit{solved forms} or \textit{fails} during the iterative processes.

The common part of a set of terms $M$ is a dual concept to the usual unifier. Intuitively, the common part is obtained by superposing all terms of $M$ and by taking the part which is common to all of them starting from the root.

For instance, the common part of $M = \{f(x, g(h(a), v), y), f(h(y), g(x, b), z)\}$ is $C : f(x, g(x, v), y)$, where variables are noted by \(x, y, z, u, v\) and constants are noted by \(a, b, c\). Notice that the common part does not exist iff two terms have different function symbols at the roots, such as $M = \{f(x, y), g(x, u)\}$.

The frontier is intuitively an environment for variables in the common part. More specifically, the frontier is a set of multiequations (which are pairs \(\{S_i = M_i\}\) of a set of variables \(S_i(\neq \phi)\) and a set of non-variable terms \(M_i\) ), where every multiequation is associated with a leaf of the common part and consists of all subterms corresponding to that leaf.

For instance, the frontier of $M$ above is $F : \{\{x\} = \{h(y), h(a)\}, \{v\} = b, \{y, z\} = \phi\}$. In $F$, $\{y, z\} = \phi$ means $y = z$, but no non-variable terms are substituted.

With definitions above, the unification algorithm \textit{UNIFY0} starts with a set of multiequations and repeatedly applies transformations until all multiequations become \textit{solved forms}, or \textit{fails} during the iterative processes. A frontier \(\{S_i = M_i\}\) is said to be a \textit{solved form} iff \(S_i \cap S_j = \phi\) for \(\forall i, j\) s.t. \(i \neq j\) and \(\text{card}(M_i) = 1\) for \(\forall i\).

Transformers produce equivalent multiequations, which means a set of all unifiers is preserved. In \textit{UNIFY0}, the following two transformations are used.

\textbf{COMPACTON} \quad Given a set $L$ containing two multiequations $S = M$ and $S' = M'$, with $S \cap S' \neq \phi$. The new set $L'$ of multiequations is obtained by replacing these two multiequations with a multi equation $S \cup S' = M \cup M'$.

\textbf{REDUCTION} \quad Given a set $L$ containing a multiequation $S = M$, such that $M \neq \phi$ and $M$ has a common part $C$ and a frontier $F$. The new set of multiequations $L'$ is obtained by replacing $S = M$ with the multiequation $S = \{C\}$ and with all multiequations of $F$. If there does not exist the common part, then stop with failed.
**Algorithm : Unify0**  Let $P$, $Q$ be terms. Set $L$ as all frontiers of a pair $(P, Q)$, perform on $L$ any of the following actions. If neither applies, then stop with *success*. When success, $P$ and $Q$ are said to be **infinitely unifiable**.

- If there are two multiequations $S = M$ and $S' = M'$ with $S \cap S' \neq \phi$, then apply **Compaction**.
- If there is a multiequation $S = M$ such that $M$ includes more than two terms, then compute the common part and the frontier of $M$. And then if $M$ has no common part then stop with *failure*. Else apply **Reduction**.

**Remark**  Note that every right hand side of frontiers are subterms of either given terms $P$ or $Q$.

**Example**  Unify two terms $P = g(x, f(z, h(x)), x), Q = g(f(h(y), z), y, y)$. Then, the common part $C$ of $M = \{P, Q\}$ is $\{g(x, y, x)\}$, and the frontier $F^{(0)}$ of them is

\[
F^{(0)} = \{ 
\{x\} = \{f(h(y), z)\}, \\
\{y\} = \{f(z, h(x))\}, \\
\{x, y\} = \phi \}
\]

Then,

**Step 1a Compaction**  $F^{(1)} = \{ 
\{x, y\} = \{f(h(y), z), f(z, h(x))\} \}$

**Step 1b Reduction**  $F^{(2)} = \{ 
\{x, y\} = \{f(z, z)\}, \\
\{z\} = \{h(y)\}, \\
\{z\} = \{h(x)\} \}$

**Step 2a Compaction**  $F^{(3)} = \{ 
\{x, y\} = \{f(z, z)\}, \\
\{z\} = \{h(x), h(y)\} \}$

**Step 2b Reduction**  $F^{(4)} = \{ 
\{x, y\} = \{f(z, z)\}, \\
\{x, y\} = \phi, \\
\{z\} = \{h(x)\} \}$

**Step 3a Compaction**  $F^{(5)} = \{ 
\{x, y\} = \{f(z, z)\}, \\
\{z\} = \{h(x)\} \text{ (solved form)} \}$

**Finish**  $M$ and $N$ are unified to $g(f(h(f\cdots), h(f\cdots)), f(h(f\cdots), h(f\cdots)))$. 


3 Basic definitions and results on confluence

A reduction system is a structure $R = \langle A, \rightarrow \rangle$ consisting of an object set $A$ and any binary relation $\rightarrow$ on $A$ (i.e., $\rightarrow \subseteq A \times A$), called a reduction relation. A reduction (starting with $x_0$) in $R$ is a finite or an infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$. The transitive closure of $\rightarrow$ is noted as $\Rightarrow$. A less-than $n$-step reduction is defined as $x \Rightarrow y$ iff $\exists m \leq n \exists z_1, z_2, \cdots, z_{m-1}$ s.t. $x \leftrightarrow z_1 \leftrightarrow z_2 \leftrightarrow \cdots \leftrightarrow z_{m-1} \leftrightarrow y$.

A congruent relation $\Rightarrow^*$ in $R$ is the transitive reflexive closure of the binary relation $\rightarrow$ where $x \leftrightarrow y$ is defined to be $x \rightarrow y \lor y \rightarrow x$. A less-than $n$-step congruent relation is defined as $x \Rightarrow^* y$ iff $\exists m \leq n \exists z_1, z_2, \cdots, z_{m-1}$ s.t. $x \leftrightarrow z_1 \leftrightarrow z_2 \leftrightarrow \cdots \leftrightarrow z_{m-1} \leftrightarrow y$.

A set of normal forms of $R$ is defined as $NF(R) = \{ x \in A | \neg \exists y \text{ s.t. } x \rightarrow y \}$.

The important properties of a reduction system $R = \langle A, \rightarrow \rangle$ are termination-related properties (e.g. weakly-normalizing, strongly-normalizing), and confluence-related properties (e.g. confluent, Church-Rosser, uniquely-normalizing).

Definition A reduction system $R = \langle A, \rightarrow \rangle$ is said to be weakly-normalizing (WN) iff $\forall x \in A \exists y \in NF(R)$ s.t. $x \Rightarrow y$. A TRS $R = \langle A, \rightarrow \rangle$ is said to be strongly normalizing (SN) iff all reduction paths are terminating, i.e. $\forall x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \exists n \text{ s.t. } x_n \in NF(R)$.

Definition $R = \langle A, \rightarrow \rangle$ is said to be confluent iff $\forall x, y, z \in A$ s.t. $x \Rightarrow y \land x \Rightarrow z \Rightarrow y \downarrow z$ (i.e. $\exists w \in A$ s.t. $y \Rightarrow w$ and $z \Rightarrow w$).

$R = \langle A, \rightarrow \rangle$ is said to be Church-Rosser (CR) iff $\forall x, y, z \in A$ s.t. $x \Rightarrow y \Rightarrow x \downarrow y$.

Definition $R = \langle A, \rightarrow \rangle$ is said to be uniquely-normalizing (UN) iff $\forall x \in A \forall y, z \in NF(R)$ s.t. $x \rightarrow y \land x \rightarrow z \Rightarrow y \equiv z$. ($x \equiv y$ iff $x$ and $y$ are syntactically same.)

Definition $R = \langle A, \rightarrow \rangle$ is said to be locally-confluent iff $\forall x, y, z \in A$ s.t. $x \rightarrow y \land x \rightarrow z \Rightarrow y \downarrow z$.

Fact 1 UN $\land$ WN $\quad \Rightarrow \quad$ CR $\quad \Rightarrow \quad$ UN

local-confluent $\land$ SN $\quad \Rightarrow \quad$ confluent $\quad \Rightarrow \quad$ locally-confluent

However, the inverse of implication arrows above are not satisfied [4].

Definition An occurrence $occur(M, N)$ of a subterm $N$ in a term $M$ is defined inductively as
occur\( (M,N) \) \( \overset{\text{def}}{=} \begin{cases} 
eq & \text{if } N = M \\ i \cdot u & \text{if } u = \text{occur}(N_i, N) \text{ and } M = f(N_1, \cdots, N_n) \end{cases} \)

The subterm \( N \) of \( M \) at occurrence \( u \) is noted as \( M/u \). (That is, \( u = \text{occur}(M,N) \).

**Definition**  The order on occurrences \( u,v \) is defined as \( u \preceq v \iff \exists w \text{ s.t. } v = u \cdot w \).

If \( u \preceq v \land u \neq v \) then it is noted as \( u \prec v \). The occurrences \( u, v \) is said to be disjoint and noted \( u \upharpoonright v \) if \( u \not\preceq v \) and \( v \not\prec u \).

**Notation**  \( V(M) \) \( \overset{\text{def}}{=} \{ x \mid \text{variable } x \text{ which is contained in } M \} \)

\( V_{\text{NL}}(M) \) \( \overset{\text{def}}{=} \{ x \mid \text{variable } x \text{ which occur more than once in } M \} \)

\( O(M) \) \( \overset{\text{def}}{=} \{ \text{occur}(M,N) \mid \forall N : \text{subterm of } M \} \)

\( O_\alpha(M) \) \( \overset{\text{def}}{=} \{ \text{occur}(M,N) \mid N \not\in V(M) \} \)

\( O_{\text{NL}}(M,x) \) \( \overset{\text{def}}{=} \{ \text{occur}(M,x) \mid x \in V_{\text{NL}}(M) \} \)

\( u \cdot V \) \( \overset{\text{def}}{=} \{ u \cdot v \mid v \in V \} \)

\( U \cdot V \) \( \overset{\text{def}}{=} \{ u \cdot v \mid u \in U \} \)

\( U \cdot V \) \( \overset{\text{def}}{=} \{ u \cdot v \mid u \in U, v \in V \} \)

\( \text{Min}(U) \) \( \overset{\text{def}}{=} \{ w \in U \mid w \not\prec w \text{ for } \forall w' \in U \} \)

where \( u,v \in O(M) \) and \( U,V \subseteq O(M) \) for a term \( M \).

**Definition**  A finite set \( R = \{ (\alpha_i, \beta_i) \} \) of ordered pairs of two terms is said to be a Term Rewriting System (TRS) iff each \( \alpha_i \) is not a variable and \( V(\alpha_i) \supseteq V(\beta_i) \) is satisfied for \( \forall i \).

A reduction is defined on a term \( M \) as \( M \rightarrow N \) at \( u \) iff there exists a substitution \( \sigma \) and an occurrence \( u \in O(M) \) s.t. \( \sigma(\alpha_i) \equiv M/u \) and \( N \equiv M[u \leftarrow \sigma(\beta_i)] \).

A congruent relation is defined on terms \( M \) and \( N \) as \( M \leftrightarrow N \) at \( u \) iff \( M \rightarrow N \) at \( u \) or \( N \rightarrow M \) at \( u \).

In the situation above, \( M/u \) is said to be a redex. A set of all occurrences of redexes for \( \alpha_i \rightarrow \beta_i \) in \( M \) is noted as \( \text{Redex}(M,\alpha_i) \), and \( \text{Redex}(M) \overset{\text{def}}{=} \bigcup_i \text{Redex}(M,\alpha_i) \).

**Definition**  A pair of reduction rules \( \alpha_i \rightarrow \beta_i \) and \( \alpha_j \rightarrow \beta_j \) is said to be nonoverlapping iff \( \exists u \in \overline{O}(\alpha_i) \text{ s.t. } \alpha_i/u \text{ and } \alpha_j \text{ are unifiable } \iff i = j \text{ and } u = e \)" is satisfied. A TRS \( R \) is said to be nonoverlapping iff all pairs of reduction rules are nonoverlapping.

**Remark**  If a TRS is nonoverlapping, then \( \text{Redex}(M,\alpha_i) \cap \text{Redex}(M,\alpha_j) = \phi \) for \( \forall M \forall i,j \text{ s.t. } i \neq j \).

**Definition**  A reduction rule \( \alpha_i \rightarrow \beta_i \) is said to be left-linear iff \( \forall x \in V(\alpha_i) \) appears only once in \( \alpha_i \). A TRS \( R \) is said to be left-linear iff all reduction rules in \( R \) are left-linear.
Fact 2  A nonoverlapping TRS is locally-confluent.
A left-linear nonoverlapping TRS is confluent[4].

4 Finite Church-Rosser Property of a TRS

4.1 Finite Church-Rosser property of an infinitely nonoverlapping TRS

Definition  A TRS $R$ is said to be \textit{finitely confluent} iff $\forall x,y,z$ s.t. $x \rightarrow y \wedge x \rightarrow z$ satisfy the condition

\[(\exists y', z' \in NF(R) \text{ s.t. } y \rightarrow y' \wedge z \rightarrow z') \text{ implies } y \downarrow z.\]

Remark  Finitely confluent is equivalent to uniquely normalizing. Then, for a strongly-normalizing TRS $R$, finitely confluent, locally confluent, and confluent are equivalent.

As an analogy to the relation between \textit{confluence} and \textit{finite confluence}, finite Church-Rosser property is defined as follows.

Definition  A TRS $R$ is said to \textit{finitely Church-Rosser} iff $\forall x,y,x' y'$ s.t. $x \rightarrow^* x' \wedge y \rightarrow^* y'$ satisfy the condition

\[(x \rightarrow^* y \wedge x', y' \in NF(R)) \text{ implies } x' \equiv y'.\]

Note that, \textit{Finite Church-Rosser property} is equivalent to "UN in [8]", and \textit{finite confluence} is equivalent to "UN" in [8].

Remark  Church-Rosser $\Rightarrow$ finitely Church-Rosser $\Rightarrow$ finitely confluent,
However, Church-Rosser $\not\Rightarrow$ finitely Church-Rosser $\not\Rightarrow$ finitely confluent.

For instance, the example $R_3$ is finitely Church-Rosser, but not Church-Rosser (See section 4.2). And, the example $R_4$ is finitely confluent, but not finitely Church-Rosser (See Fig.4). Note that $R_4$ is overlapping.

Example 4

$$R_4 \overset{\text{def}}{=} \begin{cases} f(2) \rightarrow 0 \\ f(3) \rightarrow 1 \\ f(x) \rightarrow f(f(4)) \end{cases}$$

Fig.4 A finitely confluent, but not finitely Church-Rosser example $R_4$. 

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If a TRS $R$ is weakly-normalizing, then
\[\text{Church-Rosser} \iff \text{finitely Church-Rosser} \iff \text{finitely confluent}.\]
Note that neither $R_3$ nor $R_4$ is weakly-normalizing.

**Definition** A pair of reduction rules $\alpha_i \rightarrow \beta_i$ and $\alpha_j \rightarrow \beta_j$ is said to be infinitely nonoverlapping iff \(\exists u \in \overline{O}(\alpha_i)\) s.t. $\alpha_i/u$ and $\alpha_j$ are infinitely unifiable $\iff i = j$ and $u = \varepsilon$" is satisfied. A TRS $R$ is said to be infinitely nonoverlapping iff all pairs of reduction rules are infinitely nonoverlapping.

An infinitely nonoverlapping TRS is nonoverlapping. And in case of a left-linear TRS, infinitely nonoverlapping is equivalent to nonoverlapping. Thus, a class of infinitely nonoverlapping TRSs is a natural extension of left-linear nonoverlapping TRSs to nonlinear TRSs. Our main conjecture is the next claim.

**Conjecture** An infinitely nonoverlapping TRS is finitely Church-Rosser.

In the following sections, we will prove this conjecture. Among these investigation, R-nonoverlapping is a key concept.

Intuitively, a TRS $R$ is said to be R-nonoverlapping if there do not exist branches of reduction paths in which applications of reduction rules are implicitly overlapping. They are the cases of reduction paths starting with $d(2,2)$ in Example 1 and $d(2,3)$ in Example 2. (See Fig.1 and 2)

First, an infinitely nonoverlapping TRS $R$ is proved to be R-nonoverlapping. Second, an R-nonoverlapping TRS $R$ is proved to be finitely Church-Rosser.

### 4.2 Proof of conjecture

**Definition** Let $M \parallel N$ be $M \equiv M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_n \equiv N$ where $\forall i \ s.t. \ M_{i-1} \leftrightarrow M_i \ at \ u_i$. Then, reduced occurrence sequence $\overline{REDEX}(M \parallel N)$, reduced term sequence $\overline{TERM}(M \parallel N)$, invariant occurrences $O_{inv}(M \parallel N)$, and boundary occurrences $\partial O(M \parallel N)$, are defined as follows.

\begin{align*}
\overline{REDEX}(M \parallel N) & \overset{\text{def}}{=} (u_1, u_2, \cdots, u_n) \\
\overline{TERM}(M \parallel N) & \overset{\text{def}}{=} (M_0, M_1, M_2, \cdots, M_n) \\
O_{inv}(M \parallel N) & \overset{\text{def}}{=} \{ u \in \overline{O}(M) \mid u_i \not\leq u \ for \ \forall u_i \in \overline{REDEX}(M \parallel N) \} \\
\partial O(M \parallel N) & \overset{\text{def}}{=} \text{Min}(\overline{REDEX}(M \parallel N))
\end{align*}

**Definition** Assume $U = \{u_1, u_2, \cdots, u_k\} \subseteq \overline{O}(M)$ s.t. $i \neq j \implies u_i \not\leq u_j$ for $\forall u_i, u_j \in U$.

A parallel reduction is defined to be $M \rightarrow N$ at $U$ iff $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_k \equiv N$ where $\forall i \ s.t. \ M_{i-1} \rightarrow M_i \ at \ u_i$. A less-than n-step parallel reduction
is noted as $M \overset{n}{\rightarrow} N$.

A parallel congruent relation is defined to be $M \leftrightarrow N$ at $U$ iff $M \equiv M_0 \leftrightarrow M_1 \leftrightarrow M_2 \leftrightarrow \cdots \leftrightarrow M_k \equiv N$ where $\forall i$ s.t. $M_{i-1} \leftrightarrow M_i$ at $u_i$. A less-than $n$-step parallel congruent relation is noted as $M \overset{n}{\rightarrow} N$.

**Definition** Let $R$ be a TRS, and $M, N$ be a term s.t. $M \overset{n}{\rightarrow} N$ for $n \geq 0$. $R$ is said to be $(R, n)$-nonoverlapping at $M$, iff $\forall u, v \in O_{inv}(M \overset{n}{\rightarrow} N)$ s.t. $u \in Redex(M, \alpha_i)$, $v \in Redex(N, \alpha_j)$, $0 \leq m \leq n$ satisfies the following condition

$$(v \in u \cdot \bar{O}(\alpha_i) \vee u \in v \cdot \bar{O}(\alpha_j)) \Rightarrow (u = v \wedge i = j).$$

If $R$ is $(R, n)$-nonoverlapping for $\forall n \geq 0$, $R$ is said to be $R$-nonoverlapping.

**Proposition 1** An infinitely nonoverlapping TRS $R$ is $R$-nonoverlapping.

Before entering the proof, several technical lemmas should be prepared.

**Lemma 1** Let a TRS $R$ be $(R, n-1)$-nonoverlapping, and $M \overset{n}{\rightarrow} N$.

Assume $\exists u, v \in O_{inv}(M \overset{n}{\rightarrow} N)$, $\exists \alpha_i \rightarrow \beta_i, \alpha_j \rightarrow \beta_j \in R$ s.t. $u \in v \cdot \bar{O}(\alpha_j)$.

Then, $u \cdot \bar{O}(\alpha_i) \cap v \cdot \bar{O}(\alpha_j) \subseteq O_{inv}(M \overset{n}{\rightarrow} N)$.

**Lemma 2** Let a TRS $R$ be $(R, n)$-nonoverlapping.

Assume $\exists \sigma, \sigma' \exists \alpha_i \rightarrow \beta_i \in R$, $0 \leq m \leq n$ s.t. $\sigma(\alpha_i) \overset{m}{\rightarrow} \sigma'(\alpha_i)$.

Then, $\epsilon \in O_{inv}(\sigma(\alpha_i) \overset{m}{\rightarrow} \sigma'(\alpha_i)) \Rightarrow \sigma(\beta_i) \overset{m'}{\rightarrow} \sigma'(\beta_i)$ for some $m' < m$.

**Lemma 3** Let a TRS $R$ be $(R, n-1)$-nonoverlapping.

Assume $M \overset{m}{\rightarrow} N$ for some $m \leq n$, and $\exists u \in \bar{O}(M) \cap \bar{O}(N)$ s.t. $M/u$ and $N/u$ have different function symbols at the roots. Then, either (a) or (b) is satisfied for some $m', n' < m$ and some $v \leq u$ s.t. $m' + n' < m$.

(a) $\exists M', N' \in \overline{TERM}(M\overset{m'}{\rightarrow} N/v)$

\[ s.t. M\overset{m'}{\rightarrow} N' \rightarrow N/v \text{ and } M' \rightarrow N' \text{ at } \epsilon \in O_{inv}(M\overset{m'}{\rightarrow} M'). \]

(b) $\exists M', N' \in \overline{TERM}(M\overset{m'}{\rightarrow} N/v)$

\[ s.t. M\overset{m'}{\rightarrow} M' \leftarrow N' \rightarrow N/v \text{ and } N' \rightarrow M' \text{ at } \epsilon \in O_{inv}(N\overset{n'}{\rightarrow} N/v). \]

Furthermore, $\overline{TERM}(M\overset{m'}{\rightarrow} M' \leftarrow N') \subseteq \overline{TERM}(M\overset{m'}{\rightarrow} N/v)$.

**Proof of Lemma 3** The proof is due to the induction on $m$. For $m = 1$, the statement is obvious. Let the statement be satisfied when less than $m$.

From the assumption, $\exists v \in \bar{O}(M \overset{m}{\rightarrow} N)$ s.t. $v \leq u$. Then, $M/v \overset{m'}{\rightarrow} N/v$.

Assume any subsequence of $M/v \overset{m'}{\rightarrow} N/v$ satisfies neither (a) nor (b).

Then, $\exists M', M'', N', N'' \in \overline{TERM}(M/v \overset{m''}{\rightarrow} N/v)$
s.t. $M/v \overset{m''}{\rightarrow} N/v \equiv M/v \overset{m_1}{\rightarrow} M' \leftarrow M'' \overset{m''}{\rightarrow} N'' \rightarrow N' \overset{m_3}{\rightarrow} N/v$

for $M' \rightarrow M''$ and $N' \rightarrow N''$ both at $\epsilon \in O_{inv}(M/v \overset{m_1}{\rightarrow} M') \cap O_{inv}(N'/\overset{m_3}{\rightarrow} N/v)$.

Thus, there exist a subsequence $P' \leftarrow P \overset{m}{\rightarrow} Q \rightarrow Q'$ in $M' \leftarrow M'' \overset{m''}{\rightarrow} N'' \rightarrow N'$ for $P \rightarrow P'$ and $Q \rightarrow Q'$ both at $\epsilon \in O_{inv}(P \overset{m}{\rightarrow} Q)$.

From $\langle R, n-1 \rangle$-nonoverlapping property and the relation $m' < m \leq n$, there exist $\exists \sigma', \sigma \exists x \in R$ s.t. $P/v \equiv \sigma(\alpha_i)$ and $Q/v \equiv \sigma'(\alpha_i)$. Then, $\exists m'' < m$ s.t $M/v \overset{m''}{\rightarrow} N/v$ from lemma 2. Form the induction hypothesis, lemma 3 is proved.

(q.e.d.)

**Proof of proposition 1** we will prove that R is $\langle R, n \rangle$-nonoverlapping by induction on n. Since $\langle R, 0 \rangle$-nonoverlapping is equivalent to nonoverlapping, the statement is obvious for $n = 0$.

Assume $R$ be $\langle R, n-1 \rangle$-nonoverlapping as an induction hypothesis, and let $R$ be not $\langle R, n \rangle$-nonoverlapping. Then, $\exists M, N$ s.t. $M \overset{n}{\rightarrow} N$ and $\exists u, v \in O_{inv}(M \overset{n}{\rightarrow} N)$ s.t. $u \in \text{Redex}(M, \alpha_i) \land v \in \text{Redex}(N, \alpha_j) \land \neg(u = v \land i = j) \land u \in v \cdot \overline{O}(\alpha_j)$ (or $v \in u \cdot \overline{O}(\alpha_i)$).

From assumption, $\alpha_i$ and $\alpha_j$ are infinitely nonoverlapping (except $\alpha_i$ overlaps with itself at the root). Thus, along the execution of the infinite unification algorithm on $\alpha_i$ and $\alpha_j/w$ s.t. $u = v \cdot w$ and $\neg(w = \epsilon \land i = j)$, there exist non-variable subterms $P, P'$ of $\alpha_i$ or $\alpha_j/w$ s.t. some frontier $\{x\} = (P, P')$ fails. (That is, $P$ and $P'$ have different function symbols at their roots.)

There are three cases the frontier $\{x\} = (P, P')$ fails. Let $M/u \equiv \sigma(\alpha_i)$ and $N/v \equiv \sigma'(\alpha_j)$.

[case 1] $P \in \alpha_i, P' \in \alpha_j$.

i.e. $\exists s \in u \cdot \overline{O}(\alpha_i) \cap v \cdot \overline{O}(\alpha_j)$ s.t. $Q = \sigma(P) = M/s, Q' = \sigma'(P') = N/s$.

[case 2] $P, P' \in \alpha_i$.

i.e. $\exists s, s' \in u \cdot \overline{O}(\alpha_i), \exists r \in \overline{O}(\alpha_i), \exists t, t' \in v \cdot O_{NL}(\alpha_j, x)$

s.t. $\begin{cases} s = t \cdot r, s' = t' \cdot r, t \neq t', \\
Q = \sigma(P) = M/s, Q' = \sigma(P') = M/s', N/t \equiv N/t'.
\end{cases}$

[case 3] $P, P' \in \alpha_j$.

i.e. $\exists t, t' \in v \cdot \overline{O}(\alpha_j), \exists r \in \overline{O}(\alpha_j), \exists s, s' \in u \cdot O_{NL}(\alpha_i, x)$

s.t. $\begin{cases} t = s \cdot r, t' = s' \cdot r, s \neq s', \\
Q = \sigma'(P) = N/t, Q' = \sigma'(P') = N/t', M/s \equiv M/s'.
\end{cases}$

Then, contradiction will be deduced case-by-case from the fact that $Q$ and $Q'$ have different function symbols at their roots.

[case 1] $Q$ and $Q'$ have different function symbols at their roots. Then, $u \prec \emptyset \preceq s$
s.t. \( t \notin O_{\text{inv}}(M \xrightarrow{n} N) \). However, this contradicts to lemma 1 from the induction hypothesis.

[case 2] \( Q \) and \( Q' \) have different function symbols at their roots and \( S = S' \). Then, from lemma 1, there exists \( W \equiv M/p \) and \( W' \equiv M/p' \) s.t. \( t \preceq p \preceq s \), \( t' \preceq p' \preceq s' \) and \( W \xrightarrow{m} W' \) for \( 0 < m \leq n \).

From Lemma 3, there exist \( W'' \in \overline{\text{TERM}}(W \xrightarrow{m} W') \) s.t. \( W'' \xrightarrow{m'} W \) (or \( W' \)) and \( \exists r' \in \text{Redex}(W'', \alpha_k) \cap O_{\text{inv}}(W'' \xrightarrow{m'} W) \) (or \( W' \)) where \( r' \preceq r \), \( 0 \leq m' \leq m \). Then, \( \alpha_i \) and \( \alpha_k \) are \((R, m')\)-overlapping at \( s \cdot r' \) (or \( s' \cdot r' \)). This leads a contradiction.

[case 3] Same as in [case 2]. (q.e.d)

**Proposition 2** An \( R \)-nonoverlapping TRS \( R \) is finitely Church-Rosser.

**Proof** Let \( M, N \in NF(R) \) s.t. \( M \xrightarrow{\alpha} N \). We will prove \( M \equiv N \) by induction on \( n \).

Then, \( R \) is proved to be finitely Church-Rosser.

As an initial induction step, \( M \equiv N \) is obvious for \( n = 0 \).

As an induction hypothesis, let \( M \equiv N' \) hold for \( \forall m < n \) \( \forall N' \) s.t. \( M \xrightarrow{\alpha} N' \) and \( M, N' \in NF(R) \).

Assume \( M \xrightarrow{\alpha} N \) and \( M \neq N \) where \( M, N \in NF(R) \). From lemma 3, \( \exists m < n \) \( \exists u \in \partial O(M \xrightarrow{\alpha} N) \) s.t. \( M/u \xrightarrow{\alpha} M' \rightarrow N' \) and \( M' \rightarrow N' \) at \( \epsilon \in O_{\text{inv}}(M/u \xrightarrow{\alpha} M') \).

Let \( M' \rightarrow N' \) at \( \epsilon \) be by the rule \( \alpha_i \rightarrow \beta_i \). If \( \alpha_i \rightarrow \beta_i \) is a left-linear reduction rule, then \( R \)-nonoverlapping property and \( \epsilon \in O_{\text{inv}}(M/u \xrightarrow{\alpha} M') \) implies \( \epsilon \in \text{Redex}(M/u, \alpha_i) \). This contradicts to the assumption \( M/u \in NF(R) \).

Then, \( \alpha_i \rightarrow \beta_i \) must be a nonlinear rule. And from \( R \)-nonoverlapping property and \( M/u \in NF(R) \),

\[ \exists x \in V(\alpha_i) \quad \exists v, v' \in O_{NL}(\alpha_i, x) \quad \exists w, w' \in \partial O(M/u \xrightarrow{\alpha} M') \]

s.t. \( v \preceq w, v' \preceq w', v \neq v' \), and \( M/u \cdot w \neq M/u \cdot w' \).

Note that \( M'/v \equiv M'/v' \). Then, \( \exists p, q \) s.t. \( M/u \cdot v \xrightarrow{\alpha} M'/v \equiv M'/v' \xrightarrow{\alpha} M/u \cdot v' \) and \( p + q \leq m < n \). This contradicts to the induction hypothesis. (q.e.d.)

**Theorem** An infinitely nonoverlapping TRS is finitely Church-Rosser.

**Corollary 1** An infinitely nonoverlapping TRS \( R \) is uniquely-normalizing.

**Corollary 2** If an infinitely nonoverlapping TRS \( R \) is weakly-normalizing, then \( R \) is confluent.
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5 Conclusion

In this paper, the finite Church-Rosser property of a nonlinear TRS was investigated. Main result was

An infinitely nonoverlapping TRS is finitely Church-Rosser.

Finite Church-Rosser property guarantees that congruence between two terms is examined by syntactical comparison between their normal forms (if exists). The condition infinitely nonoverlapping is a natural extension of left-linear nonoverlapping. The difference between infinitely nonoverlapping and nonoverlapping is that the unification with infinite terms [2,3,7] is applied instead of a usual unification with occur check.

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References


