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On product of weakly infinite-dimensional spaces

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1. Introduction

In this note we study with the following open question.

Question. *Is the product of two compact metric w.i.d. spaces w.i.d. ?*

In §2, we introduce the several partial answers of this question. In §3, we discuss the Pol's example which showed that this question is negative if neither of the two spaces are compact. All spaces are assumed to be normal T_1, and all mappings continuous. The letter $\mathbb{N}$ denotes the positive integers. Our terminology and notation follows **[2]** and **[5]**.

Let A and B be disjoint closed subsets of a space X. A closed subset T of X is said to be a *partition in* X *between* A *and* B if there exist open subsets U and V such that $A \subset U$, $B \subset V$, $U \cap V = \phi$, and $X - (U \cup V) = T$.

Definitions of infinite-dimensional spaces

1. A space is called ***countable dimensional (c.d.)*** if it can be written as the countable union of finite-dimensional subspaces.

2. A space X is called ***A-weakly infinite-dimensional (A-w.i.d.)*** if for every sequence $\{(A_i, B_i) : i \in \mathbb{N}\}$ of pairs of disjoint closed subsets of X there exist closed sets L_i, $i \in \mathbb{N}$, such that each L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \phi$. If $\bigcap_{i=1}^{n} L_i = \phi$ for some $n \in \mathbb{N}$, we call the space ***S-weakly infinite-dimensional (S-w.i.d.)***. In the realm of compact spaces both notions coincide, and we use the abbreviation ***weakly infinite-dimensional (w.i.d.)***.

3. A space X is said to be a ***C-space (to have property C)*** if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of open collections in X such that

(1) each $\mathcal{V}_n$ is pairwise disjoint,

(2) each V in $\mathcal{V}_n$ is contained in some U in $\mathcal{V}_n$, and

(3) $\cup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a cover of X.

Remark. For metric spaces we have the following implications:

countable dimensional ⟶ property C ⟶ A-weak infinite dimensionality

The converse of the first implication does not hold (See §3), but the converse of the second implication remains still open even for compact spaces.

2. Partial answers

It is easy to see that the product of two metric c.d. space is c.d., then A-w.i.d. B. T. Levšenko obtained the following mapping and product theorem.

Theorem [4]. *If* $f\colon X \longrightarrow Y$ *is a closed mapping of a strongly paracompact space* X *onto a space* Y, *which can be written as a union of countable number of closed subspaces having ind, such that* $f^{-1}(y)$ *is S-w.i.d. for each* $y \in Y$, *then* X *is A-w.i.d.*

Corollary [4]. *The product of a w.i.d. compact space with a compact space which has ind is w.i.d.*

For these results, the author obtained the similar results for spaces having Ind, and for c.d.

Theorem [10]. *Let* $f\colon X \longrightarrow Y$ *be a closed mapping of a countably paracompact of hereditarily normal space* X *onto a paracompact space* Y, *which can be written as a union of countable number of closed subspaces having Ind such that* $f^{-1}(y)$ *is A-w.i.d. for each* $y \in Y$, *then* X *is A-w.i.d.*

Corollary [10]. *The product of a w.i.d. compact space with a metric space which has Ind or is c.d. is A-w.i.d.*

On the other hand, for the product of two C-spaces, Addis and Gresham proved the following.

Theorem [1]. *Let* X *be a C-space such that every open cover of* X *has a precise open* F_σ *refinement. Let* Y *be a compact C-space such that*

(a) $X \times Y$ *is hereditarily collectionwise normal.*

(b) *There is a basis* $\mathcal{B}$ *for* Y *such that for all* $B \in \mathcal{B}$ $X \times \mathrm{Bd}B$ *is a C-space.*

Then $X \times Y$ *is a C-space.*

Recently, Y. Hattori, and the author obtained the dimension-lowering closed mapping theorem for C-spaces. Thus, the product theorem is obtained from the theorem, which is a generalization of the above corollaries.

Theorem [3]. *Let* f *be a closed mapping from a paracompact space* X *onto a C-space* Y. *If* $f^{-1}(y)$ *has property C for each* $y \in Y$, *then* X *is a C-space.*

Corollary [3,9]. *The product of a paracompact C-space with a compact C-space is also a C-space.*

Simultaneously, the following results were obtained.

Theorem [3]. *Let* f *be a closed mapping from a countably paracompact space* X *onto a C-space* Y. *If* $f^{-1}(y)$ *is A-w.i.d. for each* $y \in Y$, *then* X *is A-w.i.d.*

Corollary [3,9]. *The product of a compact w.i.d. space with a countably paracompact C-space is A-w.i.d.*

3. Examples

R. Pol constructed the following interesting example.

Example [7]. *There is a compact metric w.i.d. space (in fact C-space)* P *such that* P *contains a totally disconnected, not A-w.i.d. dense* G_δ *subspaces* Y, *and the remainder* M = P - Y *is c.d.*

Since countable dimensionality is hereditarily, the space P is not c.d. Thus, the space P shows that the converse of the first implication in §1 does not hold. Furthermore, using the space P, R. Pol and E. Pol showed the following.

Example [8]. *There are separable metric A-w.i.d. spaces (in fact C-spaces)* X_1 *and* X_2 *such that the product* $X_1 \times X_2$ *is not A-w.i.d.*

Example [6]. *There exists a separable metric A-w.i.d. space* X_3 *such that the product* $X_3 \times B$ *of* X_3 *with a certain subspace* B *of the space of irrationals is not A-w.i.d. Moreover,* X_3 *is a C-space.*

Example [6]. *Under the assumption of the Continuum Hypothesis there exists a separable metric A-w.i.d. space* X_4 *such that the product* $X_4 \times \mathbb{P}$ *of* X *with the space of irrationals* $\mathbb{P}$ *is not A-w.i.d. Moreover,* X_4 *is a C-space.*

These examples show that the product theorem for two C-spaces does not hold if neither of the spaces are compact. In particular, E. Pol's examples show the product theorem for a C-space with a

0-dimensional space does not hold if neither of the spaces are compact. On the other hand, it is easy to see that the product of a metric S-w.i.d. space with a metric c.d. space is A-w.i.d. (Remark: E. Pol's examples also show that the product of a separable metric S-w.i.d. space with a separable metric C-space is not necessary A-w.i.d.) Thus, neither of the space X_3 nor X_4 is not S-w.i.d. For this result, we have the question whether the product of two metric S-w.i.d. spaces is A-w.i.d. or not. For this question, the following fact shows that it is equivalent to the question in §1.

Proposition. *The following are equivalent.*

(1) *The product of two compact metric w.i.d. spaces is w.i.d.*

(2) *The product of a compact metric w.i.d. space with a metric S-w.i.d. is A-w.i.d.*

(3) *The product of two metric S-w.i.d. spaces is A-w.i.d.*

Thus, if both the space X_1 and X_2 are S-w.i.d., the question in §1 is answered negatively. But, we can prove the following unfortunately.

Theorem. *Neither the space* X_1 *and* X_2 *is S-w.i.d.*

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