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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1990, 712: 73-92</td>
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<tr>
<td>Issue Date</td>
<td>1990-02</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/101696">http://hdl.handle.net/2433/101696</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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Invariant Hyperfunctions on a Rank One Semisimple Symmetric Space

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1. Introduction

Let $G$ be a connected semisimple Lie group, $\sigma$ an involutive automorphism of $G$ and $H$ an open subgroup of the group of fixed points of $\sigma$. Then $G/H$ is called a semisimple symmetric space. Let $g$ be the Lie algebra of $G$ and $g = \mathfrak{h} \oplus \mathfrak{q}$ the decomposition into $+1$ and $-1$ eigenspaces with respect to $\sigma$. Choose a maximal abelian subspace $a \subset \mathfrak{q}$ consisting of semisimple elements. Then all such $a$ have the same dimension, which we call the rank of $G/H$. In what follows, we assume that the rank of $G/H$ is one. It is clear that $\mathfrak{q}$ can be identified with the tangent space at the origin of $G/H$. Moreover, $H$ acts on $\mathfrak{q}$ by means of the restriction of the adjoint representation to $\mathfrak{q}$.

If $E$ is a real vector space, we put $E_C$ the complexification of $E$ and $S(E_C)$ the symmetric algebra of $E_C$. For any $p \in S(q_C)$, we define, as usual, the differential operator $\partial(p)$ on $q$. Let $B(\quad, \quad)$ be the Killing form on $g$, and write $\omega(X) = B(X, X)$ for all $X \in \mathfrak{q}$, the Casimir polynomial of $\mathfrak{q}$, then the polynomial $\omega$ is an $H$-invariant nondegenerate
quadratic form on $q$. Identifying $q$ with its dual by means of the Killing form, $\square = \partial(\omega)$ can be viewed as an $H$-invariant constant coefficients second order differential operator on $q$. We call $\square$ the pseudo-Laplacian of $q$. We set

$$(\tilde{H})_1 = \{ T \in GL(q); \omega(Tx) = \omega(x) \text{ for all } x \in q \}.$$ 

Let $(\tilde{H})_0$ be the identity component of $(\tilde{H})_1$, and $\mathfrak{h}$ be the Lie algebra of $(\tilde{H})_0$, then $\text{ad}_q(\mathfrak{h}) \subset \mathfrak{h}$.

The dimensions of $H$-invariant eigendistributions of $\square$ on $q$ is given by careful considerations to the radial part of pseudo-Laplacian $\square$ by van Dijk [2]. On the other hand by a result of Cerezo [1] we can also give the dimensions of $\tilde{H}$-invariant eigenhyperfunction of $\square$ on $q$. By comparison of these results, Kowata introduce the conjecture that every $H$-invariant eigenhyperfunction of $\square$ on $q$ is $(\tilde{H})_0$-invariant, and he has shown that this conjecture holds if the eigenvalue of $\square$ is not equal to zero.

In this paper we give the answer to this conjecture except for the case of one of the exceptional type $F_{4(4)}/\text{Spin}(4,5)$. In §2 we rewrite the problem in terms of the systems of differential equations ($\mathcal{D}$-modules) and in §3 we compare these two $\mathcal{D}$-modules related to $H$ and $(\tilde{H})_0$ in the case of the classical type. In §4 we make more precise investigation by using the $H$-orbits decomposition of $q$. We must remark that the conjecture of Kowata does not hold in one case (Proposition 18).

I express my sincere gratitude to Prof. T.Oshima, whose constant encouragements and guidances made me write this paper. I would also like to thank Prof. T.Kobayashi and Prof. N.Tose for many advices.
2. Preliminaries

Let $\tilde{H}$ be the subgroup of $GL(q)$ generated by $(\tilde{H})_0$ and $\text{Ad}_q(H)$. Then $H$ is contained in $\tilde{H}$ and $(\tilde{H})_0$ is the identity component of $\tilde{H}$. Let $\mathcal{B}$ be the sheaf of hyperfunctions on $q$.

We define $B^H(q), B^H_\lambda(q), B^{\tilde{H}}(q)$ and $B^{\overline{H}}_\lambda(q)$ for $\lambda \in \mathbb{C}$ by

$B^H(q) = \{ f \in \mathcal{B}(q); f \text{ is } H\text{-invariant} \}$

$B^H_\lambda(q) = \{ f \in B^H(q); (\Box - \lambda)f = 0 \}$

$B^{\tilde{H}}(q) = \{ f \in \mathcal{B}(q); f \text{ is } \tilde{H}\text{-invariant} \}$

$B^{\overline{H}}_\lambda(q) = B^{\tilde{H}}(q) \cap B^H_\lambda(q)$.

A hyperfunction $u \in B^H(q)$ is said to be $H$-invariant and a hyperfunction $u \in B^H_\lambda(q)$ is called an invariant spherical hyperfunction. In general, we have $B^{\tilde{H}}_\lambda(q) \subset B^H_\lambda(q)$ and $B^{\overline{H}}(q) \subset B^H(q)$.

Let $\mathcal{D}$ be the sheaf of the ring of holomorphic coefficient differential operators on $q_{\mathbb{C}}$. For any $X \in gl(q)$, define a vector field $D_X$ on $q$ by

$$(D_X f)(y) = \frac{d}{dt} f(e^{-tX}y) \bigg|_{t=0} \quad (y \in q, f \in C^\infty(q)).$$

Then $D_X$ is a first order differential operator with polynomial coefficients on $q_{\mathbb{C}}$. We define the systems of differential equations defining $H$-invariants and $\tilde{H}$-invariants

$$\mathcal{M} = \mathcal{D}/\mathcal{I} = \mathcal{D}/ \sum_{X \in \mathfrak{h}} \mathcal{D}D_X$$

$$\mathcal{N} = \mathcal{D}/\mathcal{J} = \mathcal{D}/ \sum_{\sim, X \in \mathfrak{h}} \mathcal{D}D_X,$$

and also define
\[ \mathcal{M}_\lambda = \mathcal{D}/\mathcal{I}_\lambda = \mathcal{D}/(\mathcal{I}_\lambda + \mathcal{D}(\Box - \lambda)) \]
\[ \mathcal{N}_\lambda = \mathcal{D}/\mathcal{J}_\lambda = \mathcal{D}/(\mathcal{J}_\lambda + \mathcal{D}(\Box - \lambda)). \]

In general, we have surjective homomorphisms \( \mathcal{M} \to \mathcal{N} \) and \( \mathcal{M}_\lambda \to \mathcal{N}_\lambda \).

In these notations, our problems are written as follows.

**Problem I.** When is \( B^H_\lambda(q) = B^{\tilde{H}}_\lambda(q) \)?

**Problem II.** When is \( B^H(q) = B^{\tilde{H}}(q) \)?

**Problem III.** When is \( \mathcal{M}_\lambda \stackrel{\sim}{\to} \mathcal{N}_\lambda \)?

**Problem IV.** When is \( \mathcal{M} \stackrel{\sim}{\to} \mathcal{N} \)?

**Remark 1.**

1. If (IV) holds, then (II) and (III) hold. If (II) or (III) hold, then (I) hold.
2. Problem III and IV depend only on the complexification \((\mathfrak{g}_C, \mathfrak{h}_C)\) of \((\mathfrak{g}, \mathfrak{h})\).

Our approach to these problems is case by case calculation. The following table shows the non-Riemannian semisimple symmetric spaces \( G/H \) of rank one ([4, Chapter X], [2, § 6.3]).
TABLE 2.

<table>
<thead>
<tr>
<th>No.</th>
<th>( G/H )</th>
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<tbody>
<tr>
<td>1°</td>
<td>( SO_0(p,q+1)/SO_0(p,q) )</td>
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<tr>
<td>2°</td>
<td>( SO(p,q+1)/S(O(p,q) \times O(1)) )</td>
</tr>
<tr>
<td>3°</td>
<td>( SU(p,q+1)/SU(p,q) \times U(1) )</td>
</tr>
<tr>
<td>4°</td>
<td>( Sp(p,q+1)/Sp(p,q) \times Sp(1) )</td>
</tr>
<tr>
<td>5°</td>
<td>( F_4(-20)/Spin(1,8) )</td>
</tr>
<tr>
<td>6°</td>
<td>( SL(m+1,R)/S(GL_+(m,R) \times GL_+(1,R)) )</td>
</tr>
<tr>
<td>7°</td>
<td>( SL(m+1,R)/S(GL(m,R) \times GL(1,R)) )</td>
</tr>
<tr>
<td>8°</td>
<td>( Sp(m+1,R)/Sp(m,R) \times Sp(1,R) )</td>
</tr>
<tr>
<td>9°</td>
<td>( F_4(4)/Spin(4,5) )</td>
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</table>

Here \( p \geq 1, q \geq 1, m \geq 2 \).

We sometimes write \( p + q = m \) in cases 1°, 2°, 3°, 4°.

In cases 1° and 2°, then we have \( \text{ad}_q(\mathfrak{h}) = \tilde{\mathfrak{h}} \), hence Problem IV and therefore all problems are true. On the other hands, we have \( \text{ad}_q(\mathfrak{h}) \neq \tilde{\mathfrak{h}} \) in the rest cases, hence the whole problems have meanings.

3. Problem III and IV

3.1. The Spaces 6°: \( G/H = SL(m+1,R)/S(GL_+(m,R) \times GL_+(1,R)) \).

Let \((x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_m)\) be a coordinate system in \( q \cong R^{2m} \). We denote the action of \( T \in GL(q) \) on \( \begin{pmatrix} x \\ y \end{pmatrix} \in q \) by left multiplication. The Casimir polynomial is \( \omega = x_1y_1 + \cdots + x_my_m \). We realize \( \mathfrak{h} \) and \( \tilde{\mathfrak{h}} \) as subalgebras of \( \mathfrak{gl}(q) \).

\[
\begin{align*}
\mathfrak{h} &= \mathfrak{gl}(m,R) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & -^tA \end{pmatrix} : A \in M(m,R) \right\}. \\
\tilde{\mathfrak{h}} &= \mathfrak{so}(m,m) \\
&= \left\{ \begin{pmatrix} A & B \\ C & -^tA \end{pmatrix} : A, B, C \in M(m,R), \begin{array}{c} 'B = -B, 'C = -C \end{array} \right\}.
\end{align*}
\]

First we deal with Problem IV.
PROPOSITION 3. If \( m \geq 3 \), then \( \mathcal{M} \sim \mathcal{N} \).

PROOF: We write \( \mathcal{I} \) and \( \mathcal{J} \) explicitly on \( q_{\mathbb{C}} \) with the above coordinates

\[
\mathcal{I} = \sum_{i,j=1}^{m} \mathcal{D}(x_i \partial_{x_j} - y_j \partial_{y_i}).
\]

\[
\mathcal{J} = \mathcal{I} + \sum_{i,j=1}^{m} \mathcal{D}(x_i \partial_{y_j} - x_j \partial_{y_i}) + \sum_{i,j=1}^{m} \mathcal{D}(y_i \partial_{x_j} - y_j \partial_{x_i}).
\]

We will show \( x_i \partial_{y_j} - x_j \partial_{y_i} \in \mathcal{I} \) for \( 1 \leq i < j \leq m \). Since \( m \geq 3 \), there is an integer \( k \) (\( 1 \leq k \leq m \)) such that \( i, j, k \) are different to one another. Since the right hand side of the formula

\[
x_i \partial_{y_j} - x_j \partial_{y_i} = (x_j \partial_{y_k} - x_k \partial_{y_j})(x_i \partial_{x_k} - y_k \partial_{y_i})
+ (x_k \partial_{y_i} - x_i \partial_{y_k})(x_j \partial_{x_k} - y_k \partial_{y_j})
+ (x_i \partial_{y_j} - x_j \partial_{y_i})(x_k \partial_{x_k} - y_k \partial_{y_k})
\]

is contained in \( \mathcal{I} \), then we have \( x_i \partial_{y_j} - x_j \partial_{y_i} \in \mathcal{I} \). Hence Proposition 3 is proved.

Next we deal with the case \( m = 2 \). We define two left \( \mathcal{D} \)-Modules

\[
B_{\{x=0\}|q_{\mathbb{C}}} = \mathcal{D}/\mathcal{D}(x_1, x_2, \partial_{y_1}, \partial_{y_2})
\]

\[
B_{\{y=0\}|q_{\mathbb{C}}} = \mathcal{D}/\mathcal{D}(y_1, y_2, \partial_{x_1}, \partial_{x_2}),
\]

and two morphisms as left \( \mathcal{D} \)-Modules

\[
(3.1) \quad B_{\{x=0\}|q_{\mathbb{C}}} \to \mathcal{M}, \quad P \mapsto P(y_1 \partial_{x_2} - y_2 \partial_{x_1})
\]

\[
(3.2) \quad B_{\{y=0\}|q_{\mathbb{C}}} \to \mathcal{M}, \quad P \mapsto P(x_1 \partial_{y_2} - x_2 \partial_{y_1}).
\]
By (3.1) and (3.2), we get the complex of left \( \mathcal{D} \)-modules.

\[
(3.3) \quad 0 \longrightarrow B_{\{x=0\}|q_\mathbb{C}} \oplus B_{\{y=0\}|q_\mathbb{C}} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0.
\]

**PROPOSITION 4.** The complex (3.3) is exact.

**PROOF:** It is sufficient to prove the injectivity of the first morphism, which follows from Proposition 6 below.

Next we consider Problem III.

**PROPOSITION 5.** If \( m = 2 \) and \( \lambda \neq 0 \), then \( \mathcal{M}_\lambda \longrightarrow \mathcal{N}_\lambda \).

**PROOF:** Since the pseudo-Laplacian \( \square \) is written as \( \square = \sum_{i=1}^{m} \partial_{x_i} \partial_{y_i} \) by the explicit coordinates, then \( \square - \lambda = \partial_{x_1} \partial_{y_1} + \partial_{x_2} \partial_{y_2} - \lambda \). We find that \( (y_1 \partial_{x_2} - y_2 \partial_{x_1}) \in \mathcal{I}_\lambda \) by the formula

\[
\lambda(y_1 \partial_{x_2} - y_2 \partial_{x_1}) = -(y_1 \partial_{x_2} - y_2 \partial_{x_1})(\square - \lambda)
+ \partial_{x_1}^2(x_1 \partial_{x_2} - y_2 \partial_{y_1}) - \partial_{x_2}^2(x_2 \partial_{x_1} - y_1 \partial_{y_2})
- \partial_{x_1} \partial_{x_2}(x_1 \partial_{x_1} - y_1 \partial_{y_1}) + \partial_{x_1} \partial_{x_2}(x_2 \partial_{x_2} - y_2 \partial_{y_2}).
\]

Hence Proposition 5 is proved.

The rest is in case \( \lambda = 0 \).

**PROPOSITION 6.** The following complex (3.4) induced by the complex (3.3) is exact.

\[
(3.4) \quad 0 \longrightarrow B_{\{x=0\}|q_\mathbb{C}} \oplus B_{\{y=0\}|q_\mathbb{C}} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{N}_0 \longrightarrow 0.
\]

**PROOF:** It is sufficient to prove the injectivity of the first morphism. First we consider the complex (3.4) outside the origin. We may assume
\( y_2 \neq 0 \) without loss of generality. Then remark that \( B_{\{y=0\}|q_C} = 0 \) on the open set \( \{y_2 \neq 0\} \). Moreover the following coordinate transformation makes sense on the open set \( \{y_2 \neq 0\} : \)

\[
\begin{align*}
\tilde{x}_1 &= x_1 \\
\tilde{x}_2 &= x_1 y_1 + x_2 y_2 \\
\tilde{y}_1 &= y_1 \\
\tilde{y}_2 &= y_2
\end{align*}
\]

\( (3.5) \)

\[
\begin{align*}
x_1 &= \tilde{x}_1 \\
x_2 &= \frac{\tilde{x}_2 - \tilde{x}_1 \tilde{y}_1}{\tilde{y}_2} \\
y_1 &= \tilde{y}_1 \\
y_2 &= \tilde{y}_2
\end{align*}
\]

We rewrite everything in the new coordinates. Then

\[
B_{\{z=0\}|q_C} = D/\mathcal{D}(\tilde{x}_1, \tilde{x}_2, \partial_{\tilde{y}_1}, \tilde{y}_2 \partial_{\tilde{y}_2} - 1),
\]

\[
\mathcal{M}_0 = D/\mathcal{D}(\partial_{\tilde{y}_1}, \partial_{\tilde{y}_2}, \tilde{x}_1, \tilde{x}_2, \partial_{\tilde{x}_1}, \partial_{\tilde{x}_2}^2),
\]

and the morphism \((3.1)\) is the right multiplication by \( \tilde{y}_2 \partial_{\tilde{x}_1} \), which is the composition of the following two injective morphisms

\[
B_{\{z=0\}|q_C} \xrightarrow{\tilde{y}_2} D/\mathcal{D}(\tilde{x}_1, \tilde{x}_2, \partial_{\tilde{y}_1}, \partial_{\tilde{y}_2}) \xrightarrow{\partial_{\tilde{x}_1}} \mathcal{M}_0.
\]

This implies the complex \((3.4)\) is exact outside the origin, i.e. the kernel of \( B_{\{z=0\}|q_C} \oplus B_{\{y=0\}|q_C} \longrightarrow \mathcal{M}_0 \) is supported in the origin. Moreover it follows from the involutivity of the characteristic varieties that the kernel must be zero. Thus we finish the proof of Proposition 6 and Proposition 4. \( \blacksquare \)
3.2. The Spaces $3^\circ : SU(p, q + 1)/S(U(p, q) \times U(1))$

By Remark 1, we can get the following Proposition 7 for spaces $3^\circ$ as a corollary of the previous results for spaces $6^\circ$. Let $m = p + q$, $q = C^m$.

We take coordinates $z = (z_1, \ldots, z_m)$ of $q$ and those of $q_C$ as $(z, w) \in C^{2m}$ with $z, w \in C^m$, $q = \{(z, w) \in q_C; z = \bar{w}\}$.

PROPOSITION 7.

(1) If $m \geq 3$, then $\mathcal{M} \xrightarrow{\sim} \mathcal{N}$.

If $m = 2$, then we have the following exact sequence,

$$0 \longrightarrow B_{\{z=0\}|q_{\mathbb{C}}} \oplus B_{\{w=0\}|q_{\mathbb{C}}} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0.$$

(2) If $m \geq 3$ or $\lambda \neq 0$, then $\mathcal{M}_{\lambda} \xrightarrow{\sim} \mathcal{N}_{\lambda}$.

If $m = 2$ and $\lambda = 0$, then we have the exact sequence,

$$0 \longrightarrow B_{\{z=0\}|q_{\mathbb{C}}} \oplus B_{\{w=0\}|q_{\mathbb{C}}} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{N}_0 \longrightarrow 0.$$

(3) We have an isomorphism $\underline{Hom}_{\mathcal{D}}(\mathcal{N}, B) \xrightarrow{\sim} \underline{Hom}_{\mathcal{D}}(\mathcal{M}, B)$.

In particular, $B^H(q) = B^H(q)$ and $B^\overline{H}_{\lambda}(q) = B^H_{\lambda}(q)$.

PROOF: (1) and (2) follow from Proposition 3, 4, 5 and 6.

(3) is led by $\underline{Hom}_{\mathcal{D}}(B_{\{z=0\}|q_{\mathbb{C}}}, B) = \underline{Hom}_{\mathcal{D}}(B_{\{w=0\}|q_{\mathbb{C}}}, B) = 0$. \[\square\]
3.3. The Spaces $8^0 : Sp(m+1, R)/Sp(m, R) \times Sp(1, R)$.

We set $Sp(m, R) = \{ g \in SL(2m, R); g \left( \begin{array}{ll} 0 & 1_m \\ -1_m & 0 \end{array} \right)^t g = \left( \begin{array}{ll} 0 & 1_m \\ -1_m & 0 \end{array} \right) \}$. We take coordinates

$$\left( \begin{array}{l} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array} \right) \in M(2m, 2; R) \cong q, \quad x_1 = \left( \begin{array}{l} x_{11} \\ \vdots \\ x_{m1} \end{array} \right), \quad x_2, y_1, y_2 \in R^m.$$

In this coordinates, the Casimir polynomial $\omega$ is written as

$$\omega = t \left( \begin{array}{l} x_1 \\ y_1 \end{array} \right) \left( \begin{array}{ll} 0 & 1_m \\ -1_m & 0 \end{array} \right) \left( \begin{array}{l} x_2 \\ y_2 \end{array} \right) = t x_1 \cdot y_2 - t y_2 \cdot x_2 = \sum_{k=1}^{m} (x_{k1}y_{k2} - y_{k1}x_{k2}).$$

The group $H = Sp(m, R) \times Sp(1, R)$ acts on $q$ by

$$(3.6) \quad (A, B) \cdot C = ACB^{-1} \quad (A \in Sp(m, R), B \in Sp(1, R), C \in q).$$

**Theorem 8.** For $m \geq 1$, we have $M \sim N$.

**Proof:** First we consider the problem outside the origin. This step contains rather tedious calculation. We may assume $x_{11} \neq 0$ and the following coordinate transformation makes sense on the locus $\{ x_{11} \neq 0 \}$:

$$\begin{align*}
\tilde{x}_{j1} &= x_{j1}, \\
\tilde{y}_{j1} &= y_{j1} \\
\tilde{x}_{12} &= x_{12}, \\
\tilde{y}_{12} &= \omega(x, y) \\
\tilde{x}_{k2} &= x_{k2} - \frac{x_{12}x_{k1}}{x_{11}}, \\
\tilde{y}_{k2} &= y_{k2} - \frac{x_{12}y_{k1}}{x_{11}}
\end{align*}$$

(1 $\leq j \leq m$, 2 $\leq k \leq m$).
We recall the generators of $\mathcal{I}$ by old coordinates (cf. (3.6)).

\[
\begin{align*}
(a) & : \ x_j (i, j) - y_j (i, j) + x_j (i, j) - y_j (i, j) \\
(b) & : \ y_j (i, j) + x_j (i, j) + y_j (i, j) + y_j (i, j) \\
(c) & : \ x_j (i, j) + x_j (i, j) + x_j (i, j) + x_j (i, j) \\
(d) & : \ \sum_{i=1}^{m} (x_i (i, j) + y_i (i, j) - x_i (i, j) - y_i (i, j)) \\
(e) & : \ \sum_{i=1}^{m} (x_i (i, j) + y_i (i, j)) \\
(f) & : \ \sum_{i=1}^{m} (x_i (i, j) + y_i (i, j)).
\end{align*}
\]

In new coordinates, we exchange generators of $\mathcal{I}$:

\[
\begin{align*}
(e) & = x_1 \partial_{\overline{x}_{12}}, \\
((c), i = j = 1) & = 2x_1 \partial_{\overline{y}_{11}}, \\
((a), i = j = 1) & = x_1 \partial_{\overline{z}_{11}} + x_1 \partial_{\overline{y}_{11}} - y_1 \partial_{\overline{y}_{11}}, \\
((c), i \geq 2, j = 1) & = x_1 \partial_{\overline{y}_{11}}, \\
((a), i \geq 2, j = 1) & = x_1 \partial_{\overline{z}_{11}} - y_1 \partial_{\overline{y}_{11}},
\end{align*}
\]

(3.7)

(3.8) \quad (d) = - \sum_{k=2}^{m} (\overline{x}_{k2} \partial_{\overline{y}_{k2}} + \overline{y}_{k2} \partial_{\overline{y}_{k2}})

+ \sum_{k=2}^{m} (x_k \partial_{\overline{z}_{k1}} + y_k \partial_{\overline{y}_{k1}}) + x_{11} \partial_{\overline{z}_{11}} + y_{11} \partial_{\overline{y}_{11}},

\sum_{k=2}^{m} (\overline{x}_{k2} \partial_{\overline{y}_{k2}} + \overline{y}_{k2} \partial_{\overline{y}_{k2}}) \in \mathcal{I}.
Since $\omega$ is $\mathcal{H}$-invariant, then

$$\mathcal{J} \subset D(\partial_{\bar{x}_{ij}}, \partial_{\bar{y}_{ij}}; 1 \leq i \leq m, 1 \leq j \leq 2, \text{except for } \partial_{\bar{y}_{12}}).$$

If we can prove

$$\mathcal{I} \supset \mathcal{D}(\partial_{\bar{x}_{ij}}, \partial_{\bar{y}_{ij}}; 1 \leq i \leq m, 1 \leq j \leq 2, \text{except for } \partial_{\bar{y}_{12}}),$$

then $\mathcal{I} = \mathcal{J}$. We prepare two Lemmas.

**Lemma 9.** Two groups $Sp(n, \mathbb{R}) \subset SL(2n, \mathbb{R})$ act on $\mathbb{R}^{2n}$ by left multiplication ($n \geq 1$). Then the system $\mathcal{M}_1$ of differential equations defining $Sp(n, \mathbb{R})$-invariants equals the system $\mathcal{M}_2$ of differential equations defining $SL(2n, \mathbb{R})$-invariants.

**Proof:** We set $\mathcal{M}_1 = D/I_1$, $\mathcal{M}_2 = D/I_2$. We can express $I_1$ and $I_2$
in coordinates.

\[
\begin{align*}
\mathcal{I}_1 &= \mathcal{D} \left( x_i \partial_{x_j} - y_j \partial_{y_i} ; \ 1 \leq i, j \leq m \right) \\
\mathcal{I}_2 &= \mathcal{D} \left( x_i \partial_{y_j} \ y_j \partial_{x_i} ; \ 1 \leq i, j \leq m \right) \\
\end{align*}
\]

One after another,

\[
\begin{align*}
x_i^2 \partial_{x_i} &= x_i(x_i \partial_{x_i} - y_i \partial_{y_i}) + y_i x_i \partial_{y_i} \in \mathcal{I}_1. \\
x_i^2 \partial_{y_j} &= x_i(x_i \partial_{y_j} + x_j \partial_{y_i}) - x_j x_i \partial_{y_i} \in \mathcal{I}_1. \\
x_i \partial_{y_j} &= \frac{1}{2}(\partial_{x_i} x_i^2 \partial_{y_j} - \partial_{y_j} x_i^2 \partial_{x_i}) \in \mathcal{I}_1, \ y_i \partial_{x_j} \in \mathcal{I}_1.
\end{align*}
\]

Lie brackets of these elements in \( \mathcal{I}_1 \) span all generators of \( \mathcal{I}_2 \).  

**Lemma 10.** Let us consider that the group \( GL_+(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \) by left multiplication. Then the system of differential equations \( \mathcal{M}_1 \) defining \( GL_+(n, \mathbb{R}) \)-invariants is de Rham system.

**Proof:** We set \( \mathcal{M}_1 = \mathcal{D}/\mathcal{I}_1, \mathcal{I}_1 = \sum_{i,j=1}^{n} \mathcal{D}(x_i \partial_j). \)

\[
\partial_j = \partial_i x_i \partial_j - \partial_j x_i \partial_i \in \mathcal{I}_1 \quad (\text{for } i \neq j).
\]

Then \( \mathcal{I}_1 = \sum_{j=1}^{n} \mathcal{D} \partial_j. \)

Now we return to the proof of (3.10). Applying Lemma 9 to (3.9) and then applying Lemma 10 to (3.8), we conclude

(3.11) \( \partial_{x_{i2}}, \ \partial_{y_{i2}} \in \mathcal{I} \quad (2 \leq i \leq m). \)
(3.7) and (3.11) imply (3.10). Thus $\mathcal{M}|_{q-\{0\}} \sim \mathcal{N}|_{q-\{0\}}$. This isomorphism and following Proposition 11 imply $\mathcal{M} \sim \mathcal{N}$. Hence Theorem 8 is proved. 

PROPOSITION 11. Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ and $H_1 \subset H_2 \subset SL(V)$ be two closed subgroups. Suppose $H_1$ and $H_2$ are reductive in $SL(V)$, by definition $^tH_1 = H_1$, $^tH_2 = H_2$, here $^tH_i = \{^t h^{-1} \in SL(V); h \in H_i\}$. Let $\mathcal{M}_i$ be the system of differential equations defining $H_i$-invariants ($i = 1, 2$). We assume

$$
(3.12) \quad \mathcal{M}_1|_{V_{\mathbb{C}}-\{0\}} \sim \mathcal{M}_2|_{V_{\mathbb{C}}-\{0\}},
$$

then $\mathcal{M}_1 \sim \mathcal{M}_2$.

PROOF: $\mathcal{M}_i = \mathcal{D}/\mathcal{I}_i$, $\mathcal{I}_i = \sum_{X \in \mathfrak{h}_i} \mathcal{D}D_X$, here $\mathfrak{h}_i$ is the Lie algebra of $H_i$ ($i = 1, 2$).

Let $A$ be a Weyl algebra over $V_{\mathbb{C}}$ and $F : A \to A$ be the algebra automorphism defined by $x_i \mapsto -\partial_i$, $\partial_i \mapsto x_i$ (cf. [5]). By the assumption (3.12) and Hilbert zero point theorem, there exists a positive integer $N$ such that $(x_1, \ldots, x_n)^N D_X \subset \mathcal{I}_1$ for all $X \in \mathfrak{h}_2$, i.e. for any multi-index $\alpha$ satisfying ($|\alpha| \geq N$), there are $P_j(x, \partial) \in A$,

$$
x^\alpha D_X = \sum_{X_j \in \mathfrak{h}_1} P_j(x, \partial) D_{X_j}.
$$

Let $P^o_j(x, \partial) \in A$ be the homogeneous part of $P_j(x, \partial)$ of degree $-|\alpha|$, here we assign the degree 1 and $-1$ to the element $\partial_i$ and $x_i$ respectively. Then

$$
(3.13) \quad x^\alpha D_X = \sum_{X_j \in \mathfrak{h}_1} P^o_j(x, \partial) D_{X_j}.
$$
The transform of (3.13) by $F$ is

$$(-\partial)^\alpha D_{-\iota X} = \sum_{X_j \in b_1} P_j^\circ (-\partial, x) D_{-\iota_{X_j}}.$$ 

Then the left ideal \{ $P \in A; PD_X \in \mathcal{I}_1$ for any $X \in \mathfrak{h}_2$ \} of $A$ contains both $(x_1, \ldots, x_n)^N$ and $(\partial_1, \ldots, \partial_n)^N$, but such an ideal must be $A$ itself. This means that $D_X \in \mathcal{I}_1$ for any $X \in \mathfrak{h}_2$, hence Proposition 11 is proved.

3.4. The Spaces $4^\circ$ : $Sp(p, q+1)/Sp(p, q) \times Sp(1)$

As a corollary of Theorem 8 and Remark 1,

**Theorem 12.** For $p, q \geq 1$, we have $\mathcal{M} \sim \mathcal{N}$. 

4. Problem I and II

In this section, we treat Problem I and II with the $H$-orbit decomposition of $q$. Define $q_{reg} = \{ x \in q : \text{codim}_q H \cdot x = 1 \}$ and $S = q - q_{reg}$, then $q_{reg}$ is $H$-invariant open subset of $q$. Define $\omega' = \omega|_{q_{reg}}$ and $\omega_1 = \omega|_{q - \{0\}}$. 

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4.1.

We quote the following Proposition from [2, § 6.3].

**PROPOSITION 13.** In cases $3^\circ \sim 9^\circ$ except for the case $6^\circ$ and $m = 2$.

1. $\omega': q_{reg} \to \mathbb{R}$ gives $H$-orbit parametrization, by definition, $\text{grad } \omega' \neq 0$ everywhere and $\omega^{-1}(t)$ is $H$-orbit for any $t \in \mathbb{R}$.

2. $S \subset \omega^{-1}(0)$ and $S$ decomposes finite $H$-orbits.

3. $\omega_1: q - \{0\} \to \mathbb{R}$ gives $\tilde{H}$-orbit parametrization.

4. As a consequence,

$$
\begin{align*}
\{ & B^H(q_{reg}) = \omega^*(B_{\mathbb{R}}(\mathbb{R})) \\
B^{\tilde{H}}(q - \{0\}) = \omega_1^*(B_{\mathbb{R}}(\mathbb{R})).
\end{align*}
$$ (4.1)

We quote the following theorem due to Cerezo [1].

**THEOREM 14.** A connected Lie group $SO_0(p, q)$ acts on $\mathbb{R}^{p+q}$ naturally. If $(p, q) \neq (1, 1)$, then both $B^{SO_0(p, q)}(\mathbb{R}^{p+q}) \to B^{SO_0(p, q)}(\mathbb{R}^{p+q} - \{0\})$ and $B^{SO(p, q)}(\mathbb{R}^{p+q}) \to B^{SO(p, q)}(\mathbb{R}^{p+q} - \{0\})$ are surjective.

Now we prepare Lemma 15 for the proof of Proposition 16.

**LEMMA 15.** Set $\mathcal{F}[0] = \{u \in \mathcal{F}(q); \text{supp } u \subset \{0\}\}$. Then in case $1^\circ \sim 9^\circ$,

$$B^{\tilde{H}}[0] = B^H[0].$$

**PROOF:** After Fourier transformation, Lemma 15 is easily shown by the fact that the ring of $H$-invariant polynomial on $q$ is generated by Casimir polynomial $\omega$. □
PROPOSITION 16. In the same case in Proposition 13, if the assumption
(A) : $B^H(q - \{0\})[S - \{0\}] = 0$ holds, then we have $B^\tilde{H}(q) = B^H(q)$.

PROOF: Take $f \in B^H(q)$. For the restriction $f|_{q_{reg}} \in B^H(q_{reg})$, there
is a $g \in B(\mathbb{R})$ such that $f|_{q_{reg}} = g \omega'$ by (4.1). For $g \omega_1 \in B^\tilde{H}(q - \{0\})$,
there is an $\tilde{f} \in B^\tilde{H}(q)$ such that $\tilde{f}|_{q - \{0\}} = g \omega'$ by Theorem 14. Then $f - \tilde{f} \in B^H(q)[S]$. For $(f - \tilde{f})|_{q - \{0\}} \in B^H(q - \{0\})[S - \{0\}]$, it must
be zero by the assumption (A). Then $f - \tilde{f} \in B^H[0] = B^\tilde{H}[0]$, therefore
$f \in B^\tilde{H}(q)$. 

We can check the assumption (A) in some cases.

PROPOSITION 17.

(1) The singular set $S = \{0\}$ in cases $1^o, 2^o, 3^o, 4^o$ and $5^o$. In
particular, the assumption (A) is satisfied.

(2) The assumption (A) is satisfied in cases $6^o, 7^o$.

PROOF:

(1) This is the definition of isotropic symmetric pairs. [2]

(2) If $m \geq 3$, then Theorem 3 implies this. Thus we may assume
$m = 2$. Since the problem is outside the origin, we adopt the coor-
dinates (3.5). In this coordinates, $S = \{\bar{x}_1 = \bar{x}_2 = 0, \bar{y}_2 \neq 0\}$ and
$\mathcal{M} = \mathcal{D}/I = \mathcal{D}/(\partial_{y_1}, \partial_{y_2}, \bar{x}_1 \partial_{\bar{x}_1}, \bar{x}_2 \partial_{\bar{x}_1})$, then $\text{Hom}_D(\mathcal{M}, \Gamma_{S}\mathcal{B}) \subset 
\text{Hom}_D(\mathcal{D}/\mathcal{D}_{x_1, x_1, x_1=0}\partial_{\bar{x}_1}, \Gamma_{\{x_1=0\}}B) = 0$. 

Summing up the results, we have $B^\tilde{H}(q) = B^H(q)$ in cases $3^o\tilde{8}^o$ except
for the case $6^o$ and $m = 2$. 

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4.2. The Space: \( SL(3, \mathbb{R})/S(GL_{+}(2, \mathbb{R}) \times GL_{+}(1, \mathbb{R})) \)

We consider case 6°, \( m = 2 \). We adopt the notations in § 3.1. We introduce an element \( f \in \Gamma(\mathbb{R}^4 - \{0\}, B) \)

\[
\begin{align*}
\delta(x_1 y_1 + x_2 y_2) \{ Y(x_1)Y(y_2) + Y(x_2)Y(y_1) \} \\
\in \Gamma(\{x_1 \neq 0 \text{ or } y_2 \neq 0\}, B) \\
\delta(x_1 y_1 + x_2 y_2) \{ Y(x_2)Y(-y_1) + Y(-x_2)Y(y_1) \} \\
\in \Gamma(\{x_2 \neq 0 \text{ or } y_1 \neq 0\}, B)
\end{align*}
\]

Then \( f \) is \( H \)-invariant, \( \square f = 0 \), and

\[
\begin{align*}
(x_1 \partial_{y_2} - x_2 \partial_{y_1}) f &= \delta(y) = \delta(y_1, y_2) \\
(y_1 \partial_{x_2} - y_2 \partial_{x_1}) f &= \delta(x)
\end{align*}
\]

thus \( f \) is not \( \tilde{H} \)-invariant. Let \( \theta = x_1 \partial_{x_1} + x_2 \partial_{x_2} + y_1 \partial_{y_1} + y_2 \partial_{y_2} \), then \( (\theta + 2)f = 0 \). We take \( \tilde{f} \in B(\mathbb{R}^4) \) such that \( \tilde{f}|_{\mathbb{R}^4 - \{0\}} = f \). Since the mapping \( (\theta + 2): B[0] \to B[0] \) is bijective, there is a unique \( g \in B[0] \) such that \( (\theta + 2)g = (\theta + 2)\tilde{f} \). We define \( f_0 = \tilde{f} - g \in B(q) \), then \( (\theta + 2)f_0 = 0 \). For any \( X \in \mathfrak{h} \), \( [D_X, \theta] = 0 \) implies that \( (\theta + 2)D_X f_0 = D_X (\theta + 2)f_0 = 0 \), then \( D_X f_0 = 0 \), i.e. \( f_0 \in B^H(q) \). Moreover \( \square f_0 \in B[0] \) is homogeneous of degree \(-4\), then \( \square f_0 = c\delta(x, y) \) by some constant \( c \in \mathbb{C} \). The relation \( (\square f_0)(-x_1, x_2, y_1, -y_2) = -\square f_0(x, y) \) and \( c\delta(-x_1, x_2, y_1, -y_2) = c\delta(x, y) \) imply \( \square f_0 = 0 \). Therefore \( f_0 \in B_0^H(q) \), \( f_0 \notin B^H(q) \) and supp(\( f_0 \)) = \( N_+^- = \{(x, y) \in \mathbb{R}^4; \omega(x, y) = 0, x_1 y_2 - x_2 y_1 \geq 0\} \). Conversely an element \( B^H(q - \{0\}) \) whose support is contained in \( N_+^- - \{0\} \) is a constant multiple of \( f_0|_{q - \{0\}} \).

Unfortunately \( \omega': \mathfrak{a}_{\text{reg}} \to \mathbb{R} \) does not give \( H \)-orbit parametrization because \( \omega'^{-1}(0) \) has two connected components \( N_{\pm} = \{(x, y) \in \mathbb{R}^4; \omega(x, y) = 0\} \).
$\omega(x, y) = 0, \pm(x_1y_2 - x_2y_1) > 0$, Therefore Proposition 13 must be modified in this case. Let $R'$ be the non-Hausdorff manifold obtained by taking copies of $R$, say $R_+, R_-$, and sticking together the positive parts of $R_+$ and $R_-$ and the negative parts of those [1]. Let $R' \xrightarrow{\rho} R$ be the canonical map and define $B_{R'} = \rho^{-1}B_R$, the sheaf of hyperfunctions on $R'$. We define a real analytic mapping $\omega'' : q_{reg} \to R'$ by $\omega''^{-1}(t) = \omega'^{-1}(t)$ for any $t \in R^\times$ and $\omega''^{-1}(0\pm) = (N_{\pm})$.

\[
\begin{array}{cc}
q_{reg} & \subset & q - \{0\} & \subset & q \\
\omega'' & \downarrow & \omega' & \searrow & \omega_1 & \nearrow & \omega \\
R' & \xrightarrow{\rho} & R
\end{array}
\]

Then $\omega'' : q_{reg} \to R'$ gives $H$-orbit parametrization, and therefore

(4.2) \[ B^H(q_{reg}) = \omega''^*(B_{R'}(R')). \]

**Proposition 18.** Retain the above notation.

(1) $B^H(q) = B^\tilde{H}(q) \oplus Cf_0$.

(2) $B_0^H(q) = B_0^\tilde{H}(q) \oplus Cf_0$.

**Proof:** (1) Take $f \in B^H(q)$. For $f|_{q_{reg}} \in B^H(q_{reg})$, there is a $g \in B(R')$ such that $f|_{q_{reg}} = g\omega''$ by (4.2). For $g|_{R_-} \in B_{R'}(R_-) = B_R(R)$, $(g|_{R_-})\omega_1 \in B^\tilde{H}(q - \{0\})$, then there is an $\tilde{f} \in B^\tilde{H}(q)$ such that $\tilde{f}|_{q - \{0\}} = (g|_{R_-})\omega_1$ by Theorem 14. Then $f - \tilde{f} \in B^H(q)[N_+]$. For $(f - \tilde{f})|_{q - \{0\}} \in B^H(q - \{0\})[\overline{N_+} - \{0\}]$, there is a constant $c \in C$ such that $(f - \tilde{f})|_{q - \{0\}} = cf_0$. Then $(f - \tilde{f} - cf_0) \in B^H[0] = B^\tilde{H}[0]$. Therefore $f \in B^\tilde{H}(q) \oplus Cf_0$.

(2) It is obvious from (1) because $f_0 \in B_0^H$. 

\[ \square \]
Because $f_0$ is a distribution and tempered, the table in [2, § 6.3] must be corrected: case 7, $m = 3$ and $\lambda = 0$, in his notation, then
\[
\dim D'_{\lambda, H}(q) = 3, \text{ and } \dim S'_{\lambda, H}(q) = 3.
\]

以前 [8] で記したとおりのうち、この § 4.2 に相当する部分は誤っています。正しくは、このようになります。おわびして訂正します。

REFERENCES