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NORMAL HILBERT POLYNOMIALS

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1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughout this note $(A, m)$ will be a Cohen-Macaulay local ring of dimension $d \geq 2$, and $I$ will be a parameter ideal for $A$, i.e., $I$ is an $m$-primary ideal generated by $d$ elements. Assume that $A$ is analytically unramified and $A/m$ is infinite. For an ideal $J$ in $A$, $\overline{J}$ denotes the integral closure of $J$, i.e., $\overline{J} = \{x \in A| x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in J^i\}$.

It is well known that there exist uniquely determined integers $\overline{e}_0(I), \ldots, \overline{e}_d(I)$ such that

$$length_A(A/\overline{I}^{n+1}) = \overline{e}_0(I) \binom{n+d}{d} - \overline{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I)$$

for all large $n$. We say that

$$P(I, n) = \overline{e}_0(I) \binom{n+d}{d} - \overline{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I)$$

is the normal Hilbert polynomial of $I$. $\overline{e}_0(I)$ is a well-known number called the multiplicity of $I$ i.e., $\overline{e}_0(I) = e(I) = length_A(A/I)$. Our purpose of this note is to report some properties of $\overline{e}_1(I), \overline{e}_2(I)$ and $\overline{e}_3(I)$. Our results are contained in the following two theorems.

**Theorem 1.**

1. $\overline{e}_1(I) - length_A(\overline{I}/I) \geq length_A(\overline{I^2}/I\overline{I})$, and the equality holds if and only if $\overline{I^{n+2}} = I^n \overline{I^2}$ for every $n \geq 0$.

2. $\overline{e}_2(I) \geq \overline{e}_1(I) - length_A(\overline{I}/I)$, and the equality holds if and only if $\overline{I^{n+2}} = I^n \overline{I^2}$ for every $n \geq 0$.

**Theorem 2.**

1. $\overline{e}_3(I) \geq 0$, and if $\overline{e}_3(I) = 0$, then $\overline{I^{n+2}}$ is contained in $I^n$ for every $n \geq 0$.

2. Assume that $A$ is Gorenstein and $\overline{I} = m$. Then $\overline{e}_3(I) = 0$ if and only if $\overline{I^{n+2}} = I^n \overline{I^2}$ for every $n \geq 0$. 


Ooishi called the number $\bar{e}_1(I) - \text{length}_A(\bar{I}/I)$ the normal sectional genus of $I$ and denoted it by $\overline{g}_s(I)$.

In two dimensional case, Huneke remarked in his paper that

\[
\text{length}_A(A/I^{n+1}) = \text{length}_A(A/I) \binom{n+2}{2} - \left( \sum_{r \geq 0} \text{length}_A(I^{r+1}/I) \right) \binom{n+1}{1} + \sum_{r \geq 1} \text{length}_A(I^{r+1}/I) r 
\]

for all large $n$ ([3]). Therefore

\[
\sum_{r \geq 1} \text{length}_A(I^{r+1}/I) r \geq \overline{g}_s(I) = \sum_{r \geq 1} \text{length}_A(I^{r+1}/I) r \geq \text{length}_A(\bar{I}^{2}/\bar{I})
\]

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

2. Key Theorem.

Detailed studies in $\bar{e}_i(I)$'s are based on the following theorem and lemma.

**Theorem 3.** There exists a system of generators $x_1, \ldots, x_d$ of $I$ such that, if we put $C = A(T)/\left( \sum_i x_i T_i \right)$ and $J = IC$, where $A(T) = A[T][T]$ with $T = (T_1, \ldots, T_d)$ indeterminates, then

1. $J^n \cap A = \bar{I}^n$ for every $n \geq 0$;
2. $J = \bar{I}C$;
3. $J^n = \bar{I}^n C \cong \bar{I}^n A(T)/(\sum_i x_i T_i) \bar{I}^{n-1} A(T)$ for all large $n$;
4. $C$ is normal if $A$ is analytically normal and $\dim A \geq 3$.

**Lemma 4.** Choose a system of minimal generators $x_1, \ldots, x_d$ of $I$, and put $B = A[x_1/x_2], R = R(A, I) = A[It, t^{-1}]$ (the Rees ring of $I$), $D = R_P$, where $P = (t^{-1}, m)R$, and $nD$ is the maximal ideal of $D$. Let $h : B \rightarrow D$ be a canonical homomorphism which maps $x_1/x_2$ to $x_1 t/x_2 t$. Then

1. $nD \cap B = m[x_1/x_2]$.

We here put $C = Bm[x_1/x_2]$ and $J = IC$. Then

1. $J^n \cap A = \bar{I}^n$ for every $n \geq 0$,
2. $J = \bar{I}C$ and
(3) $I^{n+r} \subseteq I^n$ for every $n \geq 0$ if and only if $J^{n+r} \subseteq J^n$ for every $n \geq 0$.

Applying the theorem (and the lemma), we have the following results.

**Proposition 5.** With the same notation as in Theorem 3, we have the following assertions.

1. $\overline{e}_i(I) = \overline{e}_i(J)$ for every $i \leq d-1$;
2. $\overline{g}_s(I) = \overline{g}_s(J)$;
3. $\overline{T^2A(T)}/\overline{IA(T)}$ is a submodule of $\overline{T^2/JI}$, in particular $\text{length}_A(\overline{T^2}/\overline{I}) \geq \text{length}_C(\overline{T^2}/\overline{J})$.
4. $I^{n+r}$ is contained in $I^n$ for every $n \geq 0$ if and only if $J^{n+r}$ is contained in $J^n$ for every $n \geq 0$.

**Proof.** We put $z = \sum_i x_i T_i$ for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that $\overline{T^2A(T)} \cap (\overline{I}, I)A(T) = \overline{IA(T)}$. $\overline{T^2A(T)} \cap (\overline{I}, z)A(T) = \overline{IA(T)} + \overline{T^2A(T)} \cap zA(T) = \overline{IA(T)} + \overline{IA(T)}z = \overline{IA(T)}$. (4) follows from Lemma 4(3).

It is known that $\overline{e}_2(I) \geq \overline{g}_s(I)$ if $\text{dim} A = 2$; therefore by the induction on $d = \text{dim} A$, we have

**Corollary 6.** $\overline{e}_2(I) \geq \overline{g}_s(I)$.

As proved in [4, Proposition 10],

\[ (*) \text{length}_A(A/\overline{I^{n+1}}) \geq \text{length}_A(A/\overline{I^{n+1}}/\overline{T^2}) \]
\[ = \text{length}_A(A/I) \left( \frac{n+d}{d} \right) - (\text{length}_A(\overline{I}/I) + \text{length}_A(\overline{T^2}/\overline{I})) \left( \frac{n+d-1}{d-1} \right) \]
\[ + \text{length}_A(\overline{T^2}/\overline{I}) \left( \frac{n+d-2}{d-2} \right) \]

for all $n$. Therefore

\[ \overline{e}_1(I) \geq \text{length}_A(\overline{I}/I) + \text{length}_A(\overline{T^2}/\overline{I}) \text{ i.e.,} \]
\[ \overline{g}_s(I) = \overline{e}_1(I) - \text{length}_A(\overline{I}/I) \geq \text{length}_A(\overline{T^2}/\overline{I}). \]

We give here the proof of (1) in Theorem 1.

**Proposition 7.** (1) $\overline{g}_s(I) \geq \text{length}_A(\overline{T^2}/\overline{I})$. 

(2) $\bar{g}_s(I) = \text{length}_A(\overline{I^{2}}/\overline{I}I)$ if and only if $\overline{I^{n+2}} = I^n\overline{I^{2}}$ for every $n \geq 0$.

**Proof.** (2) We may assume that $A/m$ is infinite. We use the induction on $d = \text{dim} A$. If $d = 2$, the assertion clearly holds. So assume that $d > 2$. If part follows from (*). only if part: Choose a system of generators $x_1, \ldots, x_d$ of $I$ satisfying the conditions of Theorem 3, and put $z = \sum_i x_iT_i, C = A(T)/zA(T)$ and $J = IC$. Since $\text{length}_A(\overline{I^{2}}/\overline{I}I) \leq \text{length}_C(J^2/JJ) \leq \bar{g}_s(J) = \bar{g}_s(I)$ by Proposition 5, we have $\text{length}_C(J^2/JJ) = \bar{g}_s(J)$ and $J^2 = \overline{I^{2}}C$. Thus $\overline{J^{n+2}} = J^n\overline{J^{2}}$ for every $n \geq 0$. Then $\overline{I^{n+2}}A(T) = \overline{I^{n+2}}A(T)$ is contained in $I^n\overline{I^{2}}A(T) + zA(T)$, and hence $\overline{I^{n+2}}A(T) = \overline{I^{n+2}}A(T) \cap (I^n\overline{I^{2}}A(T) + zA(T)) = I^n\overline{I^{2}}A(T) + z\overline{I^{n+1}}A(T)$. By the induction on $n$, $\overline{I^{n+2}}A(T) = I^n\overline{I^{2}}A(T)$, and therefore $\overline{I^{n+2}} = I^n\overline{I^{2}}$.

As we remarked in [4, Proposition 3],

(**) if $\overline{I^{n+2}} = I^n\overline{I^{2}}$ for every $n \geq 0$, then $G = R'/t^{-1}R'$ is Cohen-Macaulay.

Since

$$\text{length}_A(A/I^{n+1}) = \text{length}_A(A/I^{n+1})$$

$$- \sum_{0 \leq r \leq n} \text{length}_A(I^{n-r}\overline{I^{2}+1/I^{n-r+1}I^{2}})$$

and

$$\text{length}_A(I^{n-r}\overline{I^{2}+1/I^{n-r+1}I^{2}})$$

$$\leq \text{length}_A((\overline{I^{2}+1/I^{2}}) \otimes (I^{n-r}/I^{n-r+1}I^{2}))$$

$$= \text{length}_A(\overline{I^{2}+1/I^{2}}) \binom{n-r+d-1}{d-1}$$

$$= \text{length}_A(\overline{I^{2}+1/I^{2}}) \binom{n+d-1}{d-1} - r \binom{n-r+d-2}{d-2} + \text{lower degree terms},$$

we have

$$\overline{e}_1(I) \leq \sum_{r \geq 0} \text{length}_A(\overline{I^{r+1}/I^{2}}) \ i.e.,$$

$$\bar{g}_s(I) = \overline{e}_1(I) - \text{length}_A(\overline{I}/I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}/I^{2}}).$$

**Proposition 8.** (1) $\bar{g}_s(I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}/I^{2}}).$
(2) If \( \text{depth}_M R' \geq d \), then \( \bar{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}) \) and \( \bar{e}_2(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r})r^f \), where \( M = (t^{-1}, It)R \).

3. \( \bar{e}_2(I) \).

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

**Proposition 9.** \( \bar{e}_2(I) = \bar{g}_s(I) \) if and only if \( \overline{I^{n+2}} = I^n\overline{I^2} \) for every \( n \geq 0 \).

If part of the above proposition clearly holds by \((*)\). Only if part follows from the following proposition.

**Proposition 10.** Assume that \( d \geq 3 \), and choose a system of generators \( x_1, \ldots, x_d \) satisfying the conditions of Theorem 3, and put \( z = \sum_i x_i T_i, C = A(T)/zA(T) \) and \( J = IC \). If \( J^{n+2} = J^nJ^2 \) for every \( n \geq 0 \), then \( I^{n+2} = I^nI^2 \) for every \( n \geq 0 \).

**Proposition 11.** Let \( r \) be either 1 or 2. Then the following assertions are equivalent.

1. \( \overline{I^{n+r}} = I^n\overline{I^r} \) for every \( n \geq 0 \).
2. \( [H_{N'}^{i}(R')]_j = 0 \) for \( i + j \geq r + 1 \).
3. \( [H_{N'}^{i}(R')]_j = 0 \) for \( i + j = r + 1 \).

**Proof of Proposition 10:** Assume that \( d \geq 3 \), and choose a system of generators \( x_1, \ldots, x_d \) satisfying the conditions of Theorem 3, and put \( z = \sum_i x_i T_i, C = A(T)/zA(T) \), as in Theorem 3. \( S = A(T) \otimes R(= R(A(T), I_A(T))) \), \( S' = A(T) \otimes R'(= R'(A(T), I_A(T))) \), \( N' = ItS, M' = (t^{-1}, It)S' \), \( F = S/ztS(= R(C, J)) \) and \( F' = \sum \overline{I^{n}C}t^n(= R'(C, J)) \). Suppose that \( [H_{N'}^{i}(F')]_j = 0 \) for \( i + j = 3 \). We shall prove \( [H_{N'}^{i}(R')]_j = 0 \) for \( i + j = 3 \). Since \( A \rightarrow A(T) \) is faithfully flat, \( H_{N'}^{i}(R') = H_{N'}^{i}(S') \); thus it is sufficient to prove that \( [H_{N'}^{i}(S')]_j = 0 \) for \( i + j = 3 \). We first prove that \( [H_{N'}^{i-1}(S'/ztS')]_{j+1} = 0 \) for \( 3 \leq i \leq d \) and \( j \geq 3 - i \). If this is proved, then \( ([H_{N'}^{i}(S')]_j) = [H_{N'}^{i}(S'(-1))]_j+1 \) is injective; since every element of \( H_{N'}^{i}(S') \) is annihilated by some power of \( zt \), \( [H_{N'}^{i}(S')]_j \) must be 0. Since \( \dim F'/(S'/ztS') = 0 \), we have \( H_{N'}^{i}(S'/ztS') = F'/(S'/ztS') \) and \( H_{N'}^{i}(S'/ztS') = H_{N'}^{i}(F') \) for \( i \geq 2 \). Therefore, for \( 3 \leq i \leq d \), \( H_{N'}^{i-1}(S'/ztS')_{j+1} = H_{N'}^{i-1}(F')_{j+1} = 0 \), since \( F' \) is a Cohen-Macaulay ring. We next prove that \( H_{N'}^{3}(S')_1 = 0 \). It is known that \( H_{N'}^{2}(S')_0 = 0 \); therefore
0 = H^2_N(R')_0 \xrightarrow{zt} H^2_N(R'/ztR')_1 = 0; hence H^2_N(R')_1 = 0. It is also known that H^0_M(S') = H^1_M(S') = 0 and H^i_M(S') = H^i_N(S') for i = 0, 1. Therefore H^i_N(S') = 0 for i = 0, 1.

4. $\overline{\epsilon}_3(I)$.

In general, it is known that

$$(-1)^d \overline{\epsilon}_d(I) = F(0) = \sum (-1)^i \text{length}_A([H^i_N(R')]_0).$$

If $d = 3$, then $\overline{\epsilon}_3(I) = \text{length}_A([H^3_N(R')]_0) = \text{length}_A(H^2(X, O_X))$, because $[H^3_N(R')]_0 = [H^2_M(R')]_0 = 0$. Therefore by Proposition 5,

$$\text{length}_A([H^i_N(R')]_0) = 0.$$

It is then natural to ask when $\overline{\epsilon}_3 = 0$. It follows from [4, Appendix 2] that there exists a canonical graded homomorphism $\alpha : H^d_N(R') \rightarrow H^d_m(A)[t, t^{-1}]$. We denote by $\alpha_j$ the graded part of degree $j$ of $\alpha$. Then we have

\text{LEMMA 12.} $\alpha_j = 0$ (i.e., \[H^d_M(R')]_j = [H^d_N(R')]_j\] if and only if $\overline{I^{n+d-1+j}} \subseteq I^n$ for every $n \geq 0$.

\text{Proof of Theorem 2.} (1): By Proposition 5, we may assume that $d = \text{dim} A = 3$. Then the assertion follows from (*** and Lemma 12. (2): This follows from the following proposition.

\text{PROPOSITION 13.} Assume that $A$ is a Gorenstein local ring and $\overline{I^2} = I \overline{I}$. If $\overline{I^{n+2}}$ and $m \overline{I^{n+1}}$ are contained in $I^n$ for every $n \geq 0$, then $\text{length}_A(\overline{I^2}/\overline{I}) = 1$ and $\overline{I^{n+2}} = I^n \overline{I^2}$ for every $n \geq 0$.

\text{Proof.} Since $m \overline{I^{n+1}} \subseteq I^n$, we have $(I^n : m)/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n \overline{I}$, and hence $\text{length}_A(\overline{I^{n+1}}/I^n \overline{I}) \leq \text{length}_A((I^n : m)/I^n) = (n^{-1+d-1})$, because $A$ is Gorenstein. Therefore

$$\text{length}_A(\overline{I^n}/I^n) = \text{length}_A(A/I^n) - \text{length}_A((I^n \overline{I}/I^{n+1}) - \text{length}_A(\overline{I^n}/I^{n+1})$$

$$\geq \text{length}_A(A/I) \binom{n+d}{d} - \text{length}_A(\overline{I}/I) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1}$$

$$= \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\overline{I}/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}.$$. 

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We have already proved in [4, Proposition 10] that \( \text{length}_A((I^n\overline{I}/I^{n+1}) = \text{length}_A(\overline{I}/I)(^{n+d-1})d-1). \) Thus by (*), \( \text{length}_A(\overline{I}/I) = 1 \) and \( \text{length}_A(A/I^{n+1}) = \text{length}_A(A/I)(^{n+d-1}) -(\text{length}_A(\overline{I}/I) + 1)(^{n+d-1}) + (^{n+d-2}), \) and in particular, \( \overline{I^{n+1}} = I^{n-1}\overline{I^{2}}. \)

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

**Conjecture:** Assume that \( A \) is Gorenstein and \( d = \text{dim} A \geq 3. \) Then \( \overline{e}_3(I) = 0 \) if and only if \( \overline{I^{n+2}} = I^n\overline{I^2} \) for every \( n \geq 0. \)

Assume that \( d = 3 \) and \( \overline{e}_3(I) = 0: \) By Proposition 11, if \( [H_M^2(R')]_1 = [H_N^2(R')]_1 = 0, \) then \( \overline{I^{n+2}} = I^n\overline{I^2} \) for every \( n \geq 0. \)

**References**