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NORMAL HILBERT POLYNOMIALS

by Shiroh ITOH

1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughout this note $(A, m)$ will be a Cohen-Macaulay local ring of dimension $d \geq 2$, and $I$ will be a parameter ideal for $A$, i.e. $I$ is an $m$-primary ideal generated by $d$ elements. Assume that $A$ is analytically unramified and $A/m$ is infinite. For an ideal $J$ in $A$, $\overline{J}$ denotes the integral closure of $J$, i.e., $\overline{J} = \{x \in A | x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in J^i\}$.

It is well known that there exist uniquely determined integers $\overline{e}_0(I), \cdots, \overline{e}_d(I)$ such that

$$\text{length}_A(A/\overline{I}^{n+1}) = \overline{e}_0(I) \left( \begin{array}{l} n + d \\ d \end{array} \right) - \overline{e}_1(I) \left( \begin{array}{l} n + d - 1 \\ d - 1 \end{array} \right) + \cdots + (-1)^d \overline{e}_d(I)$$

for all large $n$. We say that

$$P(I, n) = \overline{e}_0(I) \left( \begin{array}{l} n + d \\ d \end{array} \right) - \overline{e}_1(I) \left( \begin{array}{l} n + d - 1 \\ d - 1 \end{array} \right) + \cdots + (-1)^d \overline{e}_d(I)$$

is the normal Hilbert polynomial of $I$. $\overline{e}_0(I)$ is a well-known number called the multiplicity of $I$, i.e., $\overline{e}_0(I) = e(I) = \text{length}_A(A/I)$. Our purpose of this note is to report some properties of $\overline{e}_1(I), \overline{e}_2(I)$ and $\overline{e}_3(I)$. Our results are contained in the following two theorems.

**Theorem 1.** (1) $\overline{e}_1(I) - \text{length}_A(I/I) \geq \text{length}_A(\overline{I}/I)$, and the equality holds if and only if $\overline{I}^{n+2} = I^n \overline{I}^2$ for every $n \geq 0$.

(2) $\overline{e}_2(I) \geq \overline{e}_1(I) - \text{length}_A(I/I)$, and the equality holds if and only if $\overline{I}^{n+2} = I^n \overline{I}^2$ for every $n \geq 0$.

**Theorem 2.** (1) $\overline{e}_3(I) \geq 0$, and if $\overline{e}_3(I) = 0$, then $\overline{I}^{n+2}$ is contained in $I^n$ for every $n \geq 0$.

(2) Assume that $A$ is Gorenstein and $\overline{I} = m$. Then $\overline{e}_3(I) = 0$ if and only if $\overline{I}^{n+2} = I^n \overline{I}^2$ for every $n \geq 0$. 

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Ooishi called the number $\overline{e}_1(I) - \text{length}_A(\overline{I}/I)$ the normal sectional genus of $I$ and denoted it by $\overline{g}_s(I)$.

In two dimensional case, Huneke remarked in his paper that

$$\text{length}_A(A/\overline{I^{n+1}}) = \text{length}_A(A/I)\binom{n+2}{2} - \left( \sum_{r \geq 0} \text{length}_A(I^{r+1}/I^{r})\binom{n+1}{1} \right) + \sum_{r \geq 1} \text{length}_A(I^{r+1}/I^{r})r$$

for all large $n$ ([3]). Therefore

$$\sum_{r \geq 1} \text{length}_A(I^{r+1}/I^{r})r \geq \overline{g}_s(I) = \sum_{r \geq 1} \text{length}_A(I^{r+1}/I^{r}) \geq \text{length}_A(I^2/I)$$

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

2. Key Theorem.

Detailed studies in $\overline{e}_i(I)$'s are based on the following theorem and lemma.

**Theorem 3.** There exists a system of generators $x_1, \cdots, x_d$ of $I$ such that, if we put $C = A(T)/(\sum_i x_i T_i)$ and $J = IC$, where $A(T) = A[T]_{m[T]}$ with $T = (T_1, \cdots, T_d)$ $d$ indeterminates, then

1. $\overline{J^n} \cap A = \overline{I^n}$ for every $n \geq 0$;
2. $\overline{J} = \overline{I}C$;
3. $\overline{J^n} = \overline{I^n}C \cong \overline{I^n}A(T)/(\sum_i x_i T_i)\overline{I^{n-1}}A(T)$ for all large $n$;
4. $C$ is normal if $A$ is analytically normal and $\text{dim} A \geq 3$.

**Lemma 4.** Choose a system of minimal generators $x_1, \cdots, x_d$ of $I$, and put $B = A[x_1/x_2], R = R(A, I) = A[It, t^{-1}]$ (the Rees ring of $I$), $D = R_P$, where $P = (t^{-1}, m)R$, and $nD$ is the maximal ideal of $D$. Let $h : B \rightarrow D$ be a canonical homomorphism which maps $x_1/x_2$ to $x_1t/x_2t$. Then

0. $nD \cap B = m[x_1/x_2]$.

We here put $C = Bm[x_1/x_2]$ and $J = IC$. Then

1. $\overline{J^n} \cap A = \overline{I^n}$ for every $n \geq 0$,
2. $\overline{J} = \overline{I}C$ and
(3) $\overline{I^{n+r}} \subseteq \overline{I^n}$ for every $n \geq 0$ if and only if $\overline{J^{n+r}} \subseteq \overline{J^n}$ for every $n \geq 0$.

Applying the theorem (and the lemma), we have the following results.

**Proposition 5.** With the same notation as in Theorem 3, we have the following assertions.

1. $\overline{e}_i(I) = \overline{e}_i(J)$ for every $i \leq d-1$;
2. $\overline{g}_s(I) = \overline{g}_s(J)$;
3. $\overline{T^2}A(T)/\overline{IA}(T)$ is a submodule of $\overline{T^2}/\overline{I}$, in particular $\text{length}_A(\overline{T^2}/\overline{I}) \geq \text{length}_C(\overline{T^2}/\overline{I})$.
4. $\overline{I^{n+r}}$ is contained in $\overline{I^n}$ for every $n \geq 0$ if and only if $\overline{J^{n+r}}$ is contained in $\overline{J^n}$ for every $n \geq 0$.

**Proof.** We put $z = \sum_i x_i T_i$ for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that $\overline{T^2}A(T) \cap (\overline{II}, z)A(T) = \overline{IA}(T)$. $\overline{T^2}A(T) \cap (\overline{II}, z)A(T) = \overline{IA}(T) + \overline{T^2}A(T) \cap zA(T) = \overline{IA}(T) + \overline{IA}(T)z = \overline{IA}(T)$. (4) follows from Lemma 4(3).

It is known that $\overline{e}_2(I) \geq \overline{g}_s(I)$ if $\text{dim} A = 2$; therefore by the induction on $d = \text{dim} A$, we have

**Corollary 6.** $\overline{e}_2(I) \geq \overline{g}_s(I)$.

As proved in [4, Proposition 10],

(\*) $\text{length}_A(A/\overline{I^{n+1}})$

\[
\geq \text{length}_A(A/\overline{I^{n+1}}) - (\text{length}_A(\overline{I}/I) + \text{length}_A(\overline{T^2}/\overline{I}))
\]

\[
= \text{length}_A(A/I) \left( \frac{n+d}{d} \right) - \text{length}_A(\overline{T^2}/\overline{I}) \left( \frac{n+d-1}{d-1} \right)
\]

\[+ \text{length}_A(\overline{T^2}/\overline{II}) \left( \frac{n+d-2}{d-2} \right)\]

for all $n$. Therefore

$\overline{e}_1(I) \geq \text{length}_A(\overline{I}/I) + \text{length}_A(\overline{T^2}/\overline{II})$ i.e.,

$\overline{g}_s(I) = \overline{e}_1(I) - \text{length}_A(\overline{I}/I) \geq \text{length}_A(\overline{T^2}/\overline{II})$.

We give here the proof of (1) in Theorem 1.

**Proposition 7.** (1) $\overline{g}_s(I) \geq \text{length}_A(\overline{T^2}/\overline{II})$. 
(2) \( \overline{g}_s(I) = \text{length}_A(\overline{T^2}/I\overline{T}) \) if and only if \( \overline{T^{n+2}} = I^n\overline{T^2} \) for every \( n \geq 0 \).

**Proof.** (2) We may assume that \( A/m \) is infinite. We use the induction on \( d = \text{dim} A \). If \( d = 2 \), the assertion clearly holds. So assume that \( d > 2 \). If part follows from (*), only if part: Choose a system of generators \( x_1, \ldots, x_d \) of \( I \) satisfying the conditions of Theorem 3, and put \( z = \sum_i x_iT_i, C = A(T)/zA(T) \) and \( J = IC \). Since \( \text{length}_A(\overline{T^2}/I\overline{T}) \leq \text{length}_C(\overline{J^2}/J\overline{J}) \leq \overline{g}_s(J) = \overline{g}_s(I) \) by Proposition 5, we have \( \text{length}_C(\overline{J^2}/J\overline{J}) = \overline{g}_s(J) \) and \( J^2 = I^2C \). Thus \( \overline{T^{n+2}} = J^nJ^2 \) for every \( n \geq 0 \). Then \( \overline{T^{n+2}}A(T) = \overline{T^{n+2}}A(T) \) is contained in \( I^n\overline{T^2}A(T) + zA(T) \), and hence \( \overline{T^{n+2}}A(T) = \overline{T^{n+2}}A(T) \cap (I^n\overline{T^2}A(T) + zA(T)) = I^n\overline{T^2}A(T) + \overline{T^{n+1}}A(T) \). By the induction on \( n \), \( \overline{T^{n+2}}A(T) = I^n\overline{T^2}A(T) \), and therefore \( \overline{T^{n+2}} = I^n\overline{T^2} \).

As we remarked in [4, Proposition 3],

(\( ** \) if \( \overline{T^{n+2}} = I^n\overline{T^2} \) for every \( n \geq 0 \), then \( G = R'/t^{1}R' \) is Cohen-Macaulay.

Since

\[
\text{length}_A(A/I^n+1) = \text{length}_A(A/I^{n+1}) - \sum_{0 \leq r \leq n} \text{length}_A(I^{n-r}I^{r+1}/I^{n+1}I^r)
\]

and

\[
\text{length}_A(I^{n-r}I^{r+1}/I^{n-r+1}I^r)
\leq \text{length}_A((I^{r+1}/I^r) \otimes (I^{n-r}/I^{n-r+1}))
= \text{length}_A(I^{r+1}/I^r)(n - r + d - 1)
= \text{length}_A(I^{r+1}/I^r)(n + d - 1) - r(n - r + d - 2) + \text{lower degree terms},
\]

we have

\[
\overline{e}_1(I) \leq \sum_{r \geq 0} \text{length}_A(I^{r+1}/I^r) \text{ i.e.,}
\]

\[
\overline{g}_s(I) = \overline{e}_1(I) - \text{length}_A(I/I) \leq \sum_{r \geq 1} \text{length}_A(I^{r+1}/I^r).
\]

**Proposition 8.** (1) \( \overline{g}_s(I) \leq \sum_{r \geq 1} \text{length}_A(I^{r+1}/I^r) \).
If $\text{depth}_MR' \geq d$, then $\overline{y}_s(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r})$ and $\overline{e}_2(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r})r$, where $M = (t^{-1}, It)R$.

3. $\overline{e}_2(I)$.

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

**Proposition 9.** $\overline{e}_2(I) = \overline{g}_s(I)$ if and only if $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

If part of the above proposition clearly holds by (*). Only if part follows from the following proposition.

**Proposition 10.** Assume that $d \geq 3$, and choose a system of generators $x_1, \ldots, x_d$ satisfying the conditions of Theorem 1, and put $z = \sum x_i T_i$, $C = A(T)/z A(T)$ and $J = IC$. If $J^{n+2} = J^n\overline{I^2}$ for every $n \geq 0$, then $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

**Proposition 11.** Let $r$ be either 1 or 2. Then the following assertions are equivalent.

1. $\overline{I^{n+r}} = I^n\overline{I^r}$ for every $n \geq 0$.
2. $[H_{N'}^{i}(R')]_j = 0$ for $i + j \geq r + 1$.
3. $[H_{N'}^{i}(R')]_j = 0$ for $i + j = r + 1$.

**Proof of Proposition 10:** Assume that $d \geq 3$, and choose a system of generators $x_1, \ldots, x_d$ satisfying the conditions of Theorem 3, and put $z = \sum x_i T_i$, $C = A(T)/z A(T)$, $J = IC$, as in Theorem 3. $S = A(T) \otimes R(= R(A(T), IA(T)))$, $S' = A(T) \otimes R'(= R'(A(T), IA(T)))$, $N' = ItS, M' = (t^{-1}, It)S' F = S/ztS(= R(C, J))$ and $F' = \sum \overline{I^n C t^n}(= R'(C, J))$. Suppose that $[H_{N'}^{i}(F')]_j = 0$ for $i + j = 3$.

We shall prove $[H_{N'}^{i}(R')]_j = 0$ for $i + j = 3$. Since $A \rightarrow A(T)$ is faithfully flat, $H_{N'}^{i}(R') = H_{N'}^{i}(S')$; thus it is sufficient to prove that $[H_{N'}^{i}(S')]_j = 0$ for $i + j = 3$. We first prove that $[H_{N'}^{i-1}(S'/ztS')]_{j+1} = 0$ for $3 \leq i \leq d$ and $j \geq 3 - i$. If this is proved, then $[H_{N'}^{i}(S')]_j = [H_{N'}^{i}(S'(-1))]_{j+1} \otimes F' [H_{N'}^{i}(S')]_j$ is injective; since every element of $H_{N'}^{i}(S')$ is annihilated by some power of $zt$, $[H_{N'}^{i}(S')]_j$ must be 0. Since $\dim F'(S'/ztS') = 0$, we have $H_{N'}^{i}(S'/ztS') = F'(S'/ztS')$ and $H_{N'}^{i}(S'/ztS') = H_{N'}^{i}(F')$ for $i \geq 2$. Therefore, for $3 \leq i \leq d$, $H_{N'}^{i-1}(S'/ztS')_{j+1} = H_{N'}^{i-1}(F')_{j+1} = 0$, since $F'$ is a Cohen-Macaulay ring. We next prove that $H_{N'}^{2}(S')_1 = 0$. It is known that $H_{N'}^{2}(S')_0 = 0$; therefore
$0 = H^2_{N'}(S'_0) \xrightarrow{zt} H^2_{N'}(S'_1) \rightarrow H^2_{N'}(S'/ztS')_1 = 0$; hence $H^2_{N'}(S')_1 = 0$. It is also known that $H^0_{M'}(S') = H^1_{M'}(S') = 0$ and $H^i_{M'}(S') = H^i_{N'}(S')$ for $i = 0, 1$. Therefore $H^i_{N'}(S') = 0$ for $i = 0, 1$.

4. $\overline{e}_3(I)$.

In general, it is known that $(-1)^d \overline{e}_d(I) = F(0) = \sum (-1)^i \text{length}_A([H^i_N(R')]_0)$.

If $d = 3$, then $\overline{e}_3(I) = \text{length}_A([H^3_N(R')]_0) = \text{length}_A(H^2(X,O_X))$, because $[H^3_N(R')]_0 = [H^3_M(R')]_0 = 0$. Therefore by Proposition 5,

$\text{(***)} \overline{e}_3(I) \geq 0$.

It is then natural to ask when $\overline{e}_3 = 0$. It follows from [4, Appendix 2] that there exists a canonical graded homomorphism $\alpha : H^d_N(R') \rightarrow H^d_m(A)[t,t^{-1}]$. We denote by $\alpha_j$ the graded part of degree $j$ of $\alpha$. Then we have

**Lemma 12.** $\alpha_j = 0$ (i.e., $[H^d_M(R')]_j = [H^d_N(R')]_j$) if and only if $\overline{I^{n+d-1+j}} \subseteq I^n$ for every $n \geq 0$.

**Proof of Theorem 2.** (1): By Proposition 5, we may assume that $d = \text{dim} A = 3$. Then the assertion follows from (***) and Lemma 12. (2): This follows from the following proposition.

**Proposition 13.** Assume that $A$ is a Gorenstein local ring and $\overline{I^2} = I\overline{I}$. If $\overline{I^{n+2}}$ and $m\overline{I^{n+1}}$ are contained in $I^n$ for every $n \geq 0$, then $\text{length}_A(\overline{I^2}/\overline{I}) = 1$ and $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

**Proof.** Since $m\overline{I^{n+1}} \supseteq I^n$, we have $(I^n : m)/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n\overline{I}$, and hence $\text{length}_A(\overline{I^{n+1}}/I^n\overline{I}) \leq \text{length}_A(\overline{I^n/I^n}) = (n+1+d-1)$, because $A$ is Gorenstein. Therefore

\[
\text{length}_A(\overline{I^{n+1}}/I^n) = \text{length}_A(A/I^n) - \text{length}_A(\overline{I^n/I^{n+1}}) - \text{length}_A(\overline{I^{n+1}}/I^n) \\
\geq \text{length}_A(A/I) \binom{n+d}{d} - \text{length}_A(\overline{I/I}) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1} \\
= \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\overline{I/I}) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}.
\]
We have already proved in [4, Proposition 10] that $\text{length}_A((I^n\overline{I}/I^{n+1}) = \text{length}_A(\overline{I}/I)^{n+d-1}d$. Thus by (*), $\text{length}_A(\overline{I}/I) = 1$ and $\text{length}_A(A/I^{n+1}) = \text{length}_A(A/I)^{n+d-1}d - (\text{length}_A(\overline{I}/I) + 1)(^{n+d-1}d-1) + (^{n+d-2}d-2)$, and in particular, $\overline{I}^{n+1} = I^{n-1}\overline{I}^{2}$.

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

**Conjecture:** Assume that $A$ is Gorenstein and $d = \text{dim} A \geq 3$. Then $\bar{e}_3(I) = 0$ if and only if $\overline{I}^{n+2} = I^n\overline{I}^2$ for every $n \geq 0$.

Assume that $d = 3$ and $\bar{e}_3(I) = 0$: By Proposition 11, if $[\overline{H}_M(R')]_1 = [\overline{H}_N(R')]_1 = 0$, then $\overline{I}^{n+2} = I^n\overline{I}^2$ for every $n \geq 0$.

**References**