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NORMAL HILBERT POLYNOMIALS

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1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughout this note $(A, m)$ will be a Cohen-Macaulay local ring of dimension $d \geq 2$, and $I$ will be a parameter ideal for $A$, i.e. $I$ is an $m$-primary ideal generated by $d$ elements. Assume that $A$ is analytically unramified and $A/m$ is infinite. For an ideal $J$ in $A$, $\overline{J}$ denotes the integral closure of $J$, i.e., $\overline{J} = \{x \in A | x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in J^i\}$.

It is well known that there exist uniquely determined integers $\overline{e}_0(I), \cdots, \overline{e}_d(I)$ such that

$$length_A(A/I^{n+1}) = \overline{e}_0(I)\left(\frac{n+d}{d}\right) - \overline{e}_1(I)\left(\frac{n+d-1}{d-1}\right) + \cdots + (-1)^d \overline{e}_d(I)$$

for all large $n$. We say that

$$P(I, n) = \overline{e}_0(I)\left(\frac{n+d}{d}\right) - \overline{e}_1(I)\left(\frac{n+d-1}{d-1}\right) + \cdots + (-1)^d \overline{e}_d(I)$$

is the normal Hilbert polynomial of $I$. $\overline{e}_0(I)$ is a well-known number called the multiplicity of $I$ i.e., $\overline{e}_0(I) = e(I) = length_A(A/I)$. Our purpose of this note is to report some properties of $\overline{e}_1(I)$, $\overline{e}_2(I)$ and $\overline{e}_3(I)$. Our results are contained in the following two theorems.

**Theorem 1.** (1) $\overline{e}_1(I) - length_A(I/I)$ $\geq$ $length_A(I^2/I)$, and the equality holds if and only if $I^{n+2} = I^n I^2$ for every $n \geq 0$.

(2) $\overline{e}_2(I) \geq \overline{e}_1(I) - length_A(I/I)$, and the equality holds if and only if $I^{n+2} = I^n I^2$ for every $n \geq 0$.

**Theorem 2.** (1) $\overline{e}_3(I) \geq 0$, and if $\overline{e}_3(I) = 0$, then $I^{n+2}$ is contained in $I^n$ for every $n \geq 0$.

(2) Assume that $A$ is Gorenstein and $I = m$. Then $\overline{e}_3(I) = 0$ if and only if $I^{n+2} = I^n I^2$ for every $n \geq 0$. 

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Ooishi called the number $\overline{e}_1(I) - length_A(\overline{I}/I)$ the normal sectional genus of $I$ and denoted it by $\overline{g}_s(I)$.

In two dimensional case, Huneke remarked in his paper that

$$length_A(A/\overline{I^{n+1}}) = length_A(A/I) \binom{n+2}{2} - \left( \sum_{r \geq 0} length_A(\overline{I^{r+1}}/I\overline{I^r}) \right) \binom{n+1}{1} + \sum_{r \geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r}) r$$

for all large $n$ ([3]). Therefore

$$\sum_{r \geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r}) r \geq \overline{g}_s(I) = \sum_{r \geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r}) \geq length_A(\overline{I^2}/I\overline{I})$$

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

2. Key Theorem.

Detailed studies in $\overline{e}_i(I)$'s are based on the following theorem and lemma.

THEOREM 3. There exists a system of generators $x_1, \ldots, x_d$ of $I$ such that, if we put $C = A(T)/(\sum_i x_i T_i)$ and $J = IC$, where $A(T) = A[T][T]$ with $T = (T_1, \ldots, T_d)$ indeterminates, then

(1) $\overline{J^n} \cap A = \overline{I^n}$ for every $n \geq 0$;

(2) $\overline{J} = \overline{I}C$;

(3) $\overline{J^n} = \overline{I^n}C \cong \overline{I^n} A(T)/(\sum_i x_i T_i) \overline{I^{n-1}} A(T)$ for all large $n$;

(4) $C$ is normal if $A$ is analytically normal and $\dim A \geq 3$.

LEMMA 4. Choose a system of minimal generators $x_1, \ldots, x_d$ of $I$, and put $B = A[x_1/x_2], R = R(A, I) = A[It, t^{-1}]$ (the Rees ring of $I$), $D = R_P$, where $P = (t^{-1}, m)R$, and $nD$ is the maximal ideal of $D$. Let $h : B \rightarrow D$ be a canonical homomorphism which maps $x_1/x_2$ to $x_1t/x_2t$. Then

(0) $nD \cap B = m[x_1/x_2]$.

We here put $C = Bm[x_1/x_2]$ and $J = IC$. Then

(1) $\overline{J^n} \cap A = \overline{I^n}$ for every $n \geq 0$,

(2) $\overline{J} = \overline{IC}$ and

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(3) $I^{n+r} \subseteq I^n$ for every $n \geq 0$ if and only if $J^{n+r} \subseteq J^n$ for every $n \geq 0$.

Applying the theorem (and the lemma), we have the following results.

**Proposition 5.** With the same notation as in Theorem 3, we have the following assertions.

1. $\bar{e}_i(I) = \bar{e}_i(J)$ for every $i \leq d - 1$;
2. $\bar{g}_s(I) = \bar{g}_s(J)$;
3. $\overline{I^{2}}A(T) / I\overline{I}A(T)$ is a submodule of $\overline{J^{2}} / J\overline{J}$, in particular $\text{length}_{A}(\overline{I^{2}}/I\overline{I}) \geq \text{length}_{C}(\overline{J^{2}}/J\overline{J})$.
4. $I^{n+r}$ is contained in $I^n$ for every $n \geq 0$ if and only if $J^{n+r}$ is contained in $J^n$ for every $n \geq 0$.

**Proof.** We put $z = \sum x_i T_i$ for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that $\overline{I^{2}}A(T) \cap (I\overline{I}, z)A(T) = I\overline{I}A(T)$. $\overline{I^{2}}A(T) \cap (I\overline{I}, z)A(T) = I\overline{I}A(T) + \overline{I^{2}}A(T) \cap zA(T) = I\overline{I}A(T) + I\overline{I}A(T)z = I\overline{I}A(T)$. (4) follows from Lemma 4(3).

It is known that $\bar{e}_2(I) \geq \bar{g}_s(I)$ if $\text{dim}A = 2$; therefore by the induction on $d = \text{dim}A$, we have

**Corollary 6.** $\bar{e}_2(I) \geq \bar{g}_s(I)$.

As proved in [4, Proposition 10],

\[(*) \text{length}_{A}(A/I^{n+1}) \geq \text{length}_{A}(A/I^{n+1}\overline{I^{2}}) \geq \text{length}_{A}(A/I) \left( \frac{n+d}{d} \right) - \left( \text{length}_{A}(I/I) + \text{length}_{A}(\overline{I^{2}}/I\overline{I}) \right) \left( \frac{n+d-1}{d-1} \right) + \text{length}_{A}(\overline{I^{2}}/I\overline{I}) \left( \frac{n+d-2}{d-2} \right)\]

for all $n$. Therefore

$\bar{e}_1(I) \geq \text{length}_{A}(I/I) + \text{length}_{A}(\overline{I^{2}}/I\overline{I})$ i.e.,

$\bar{g}_s(I) = \bar{e}_1(I) - \text{length}_{A}(I/I) \geq \text{length}_{A}(\overline{I^{2}}/I\overline{I})$.

We give here the proof of (1) in Theorem 1.

**Proposition 7.** (1) $\bar{g}_s(I) \geq \text{length}_{A}(\overline{I^{2}}/I\overline{I})$. 3
(2) \( \overline{g}_s(I) = \text{length}_A(\overline{I^2}/I\overline{I}) \) if and only if \( \overline{I^{n+2}} = I^n\overline{I^2} \) for every \( n \geq 0 \).

**Proof.** (2) We may assume that \( A/m \) is infinite. We use the induction on \( d = \text{dim} A \). If \( d = 2 \), the assertion clearly holds. So assume that \( d > 2 \). If part follows from (*). only if part: Choose a system of generators \( x_1, \ldots, x_d \) of \( I \) satisfying the conditions of Theorem 3, and put \( z = \sum_i x_iT_i, C = A(T)/zA(T) \) and \( J = IC \).

Since \( \text{length}_A(\overline{I^2}/I\overline{I}) \leq \text{length}_C(\overline{J^2}/J\overline{J}) \leq \overline{g}_s(J) = \overline{g}_s(I) \) by Proposition 5, we have \( \text{length}_C(\overline{J^2}/J\overline{J}) = \overline{g}_s(J) \) and \( \overline{J^2} = \overline{I^2}C \). Thus \( \overline{I^{n+2}} = \overline{J^{n+2}} \) for every \( n \geq 0 \). Then \( \overline{I^{n+2}}A(T) = \overline{I^{n+2}A(T)} \) is contained in \( I^n\overline{I^2}A(T) + zA(T) \), and hence \( \overline{I^{n+2}}A(T) = \overline{I^{n+2}}A(T) \cap (I^n\overline{I^2}A(T) + zA(T)) = I^n\overline{I^2}A(T) + z\overline{I^{n+1}}A(T) \).

By the induction on \( n \), \( \overline{I^{n+2}}A(T) = I^n\overline{I^2}A(T) \), and therefore \( \overline{I^{n+2}} = I^n\overline{I^2} \).

As we remarked in [4, Proposition 3],

\[ (** \text{ if } \overline{I^{n+2}} = I^n\overline{I^2} \text{ for every } n \geq 0, \text{ then } G = R'/t^{-1}R' \text{ is Cohen-Macaulay.} \]

Since

\[
\text{length}_A(A/\overline{I^{n+1}}) = \text{length}_A(A/I^{n+1}) - \sum_{0 \leq r \leq n} \text{length}_A(I^{n-r}I^{r+1}/I^{n-r+1}I^r)
\]

and

\[
\text{length}_A(I^{n-r}I^{r+1}/I^{n-r+1}I^r)
\]

\[
\leq \text{length}_A((\overline{I^{r+1}}/I\overline{I^r}) \otimes (I^{n-r}/I^{n-r+1}))
\]

\[
= \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}) \binom{n-r+d-1}{d-1}
\]

\[
= \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}) \binom{n+d-1}{d-1} - r \binom{n-r+d-2}{d-2} + \text{lower degree terms},
\]

we have

\[
\overline{e}_1(I) \leq \sum_{r \geq 0} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}) \text{ i.e.,}
\]

\[
\overline{g}_s(I) = \overline{e}_1(I) - \text{length}_A(\overline{I}/I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}).
\]

**Proposition 8.** (1) \( \overline{g}_s(I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\overline{I^r}). \)
If $\text{depth}_M R' \geq d$, then $\overline{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\overline{I}^{r+1}/\overline{I}^r)$ and $\overline{e}_2(I) = \sum_{r \geq 1} \text{length}_A(\overline{I}^{r+1}/\overline{I}^r)r_f$, where $M = (t^{-1}, It)R$.

3. $\overline{e}_2(I)$.

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

**Proposition 9.** $\overline{e}_2(I) = \overline{g}_s(I)$ if and only if $\overline{I}^{n+2} = I^n\overline{I}^2$ for every $n \geq 0$.

If part of the above proposition clearly holds by (*). Only if part follows from the following proposition.

**Proposition 10.** Assume that $d \geq 3$, and choose a system of generators $x_1, \ldots, x_d$ satisfying the conditions of Theorem 1, and put $z = \sum_i x_i T_i, C = A(T)/zA(T)$ and $J = IC$. If $\overline{J}^{n+2} = J^n\overline{J}^2$ for every $n \geq 0$, then $\overline{I}^{n+2} = I^n\overline{I}^2$ for every $n \geq 0$.

**Proposition 11.** Let $r$ be either 1 or 2. Then the following assertions are equivalent.

1. $\overline{I}^{n+r} = I^n\overline{I}^r$ for every $n \geq 0$.
2. $[H^i_{N'}(R')]_j = 0$ for $i + j \geq r + 1$.
3. $[H^i_{N'}(R')]_j = 0$ for $i + j = r + 1$.

Proof of Proposition 10: Assume that $d \geq 3$, and choose a system of generators $x_1, \ldots, x_d$ satisfying the conditions of Theorem 3, and put $z = \sum_i x_i T_i, C = A(T)/zA(T), J = IC$, as in Theorem 3. $S = A(T) \otimes R(= R'(A(T), IA(T))), S' = A(T) \otimes R'(= R'(A(T), IA(T))), N' = ItS, M' = (t^{-1}, It)S' F = S/ztS(= R(C, J))$ and $F' = \sum \overline{I}^n C t^n (= R'(C, J))$. Suppose that $[H^i_{N'}(F')]_j = 0$ for $i + j = 3$. We shall prove $[H^i_{N'}(R')]_j = 0$ for $i + j = 3$. Since $A \longrightarrow A(T)$ is faithfully flat, $H^i_{N'}(R') = H^i_{N'}(S')$; thus it is sufficient to prove that $[H^i_{N'}(S')]_j = 0$ for $i + j = 3$. We first prove that $[H^i_{N'}(S'/ztS')]_{j+1} = 0$ for $3 \leq i \leq d$ and $j \geq 3-i$. If this is proved, then $([H^i_{N'}(S')]_j = [H^i_{N'}(S'(-1))][j+1] \longrightarrow [H^i_{N'}(S')]_j+1$ is injective; since every element of $H^i_{N'}(S')$ is annihilated by some power of $zt$, $[H^i_{N'}(S')]_j$ must be 0. Since $\dim F'/F' = 0$, we have $H^i_{N'}(S'/ztS') = F'/S'/ztS'$ and $H^i_{N'}(S'/ztS') = H^i_{N'}(F')$ for $i \geq 2$. Therefore, for $3 \leq i \leq d$, $H^i_{N'}(S'/ztS')_{j+1} = H^i_{N'}(F')_{j+1} = 0$, since $F'$ is a Cohen-Macaulay ring. We next prove that $H^2_{N'}(S')_1 = 0$. It is known that $H^2_{N'}(S')_0 = 0$; therefore
0 = H^2_{N'}(S')_0 \xrightarrow{z_t} H^2_{N'}(S'/z_tS')_1 = 0; \text{ hence } H^2_{N'}(S')_1 = 0. \text{ It is also known that } H^0_{M'}(S') = H^1_{M'}(S') = 0 \text{ and } H^i_{M'}(S') = H^i_{N'}(S') = 0, i = 0, 1. \text{ Therefore } H^i_{N'}(S') = 0 \text{ for } i = 0, 1.

4. \overline{e}_3(I).

In general, it is known that
\((-1)^d \overline{e}_d(I) = F(0) = \sum_i (-1)^i \text{length}_A([H^i_N(R')]_0).

If \(d = 3\), then \(\overline{e}_3(I) = \text{length}_A([H^3_N(R')]_0) = \text{length}_A(H^2(X, O_X)),\) because \([H^3_N(R')]_0 = [H^2_M(R')]_0 = 0. \text{ Therefore by Proposition 5,}\)

\(\text{(***)} \overline{e}_3(I) \geq 0.

It is then natural to ask when \(\overline{e}_3 = 0\). \text{ It follows from [4, Appendix 2] that there exists a canonical graded homomorphism } \alpha : H^d_N(R') \rightarrow H^d_m(A)[t, t^{-1}]. \text{ We denote by } \alpha_j \text{ the graded part of degree } j \text{ of } \alpha. \text{ Then we have}\)

\text{Lemma 12. } \alpha_j = 0 \text{ (i.e., } [H^d_M(R')]_j = [H^d_N(R')]_j) \text{ if and only if } \overline{I^{n+d-1+j}} \subseteq I^n \text{ for every } n \geq 0.

\text{Proof of Theorem 2. (1): By Proposition 5, we may assume that } d = \text{dim}A = 3. \text{ Then the assertion follows from (***) and Lemma 12. (2): This follows from the following proposition.}\)

\text{Proposition 13. Assume that } A \text{ is a Gorenstein local ring and } \overline{I^2} = I\overline{I}. \text{ If } \overline{I^{n+2}} \text{ and } m\overline{I^{n+1}} \text{ are contained in } I^n \text{ for every } n \geq 0, \text{ then } \text{length}_A(I^2/I\overline{I}) = 1 \text{ and } \overline{I^n+2} = \overline{I^nI^2} \text{ for every } n \geq 0.

\text{Proof. Since } m\overline{I^{n+1}} \subseteq I^n, \text{ we have } (I^n : m)/I^n \supseteq (I^{n+1} + I^n)/I^n = \overline{I^{n+1}I^n}, \text{ and hence length}_A((I^n : m)/I^n) = \binom{n+1+d-1}{d-1}, \text{ because } A \text{ is Gorenstein. Therefore}\)

\text{length}_A(A/I^n+1) = \text{length}_A(A/I^n) - \text{length}_A((I^n\overline{I}/I^{n+1}) - \text{length}_A(I^{n+1}/I^n\overline{I})

\geq \text{length}_A(A/I) \binom{n+d}{d} - \text{length}_A(I/I) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1}

= \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(I/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}.

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We have already proved in [4, Proposition 10] that $\text{length}_A((I^nI/I^{n+1}) = \text{length}_A(I^d/I)(n^{d-1}d-1).$ Thus by (*), $\text{length}_A(I^d/I^d) = 1$ and $\text{length}_A(A/I^{n+1}) = \text{length}_A(A/I)(n^{d-1}d-1) - (\text{length}_A(I/I) + 1)(n^{d-1}d-2)$, and in particular, $I^{n+1} = I^{n-1}I^d$.

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

**CONJECTURE:** Assume that $A$ is Gorenstein and $d = \dim A \geq 3$. Then $\overline{e}_3(I) = 0$ if and only if $I^{n+2} = I^nI^2$ for every $n \geq 0$.

Assume that $d = 3$ and $\overline{e}_3(I) = 0$: By Proposition 11, if $[H^2_M(R')]_1 = [H^2_N(R')]_1 = 0$, then $I^{n+2} = I^nI^2$ for every $n \geq 0$.

**REFERENCES**