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THE SYZYGIES OF M-FULL IDEALS

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Introduction

The concept of $\underline{m}$-full ideals was introduced and studied first by D.Rees (unpublished). In 1983, after having considered and discussed the concept with Prof. Rees, I went on to show some of their properties in [10]. Some other authors also have obtained a considerable amount of results related to those ideals. (cf. [5], [7].) The purpose of this paper is to seek s zygies of $\underline{m}$-full ideals and try to analyze their structure. Let $\underline{a}$ be an $\underline{m}$-full ideal, and $\overline{\underline{a}}$ the reduction by a general element. Then it is possible to determine the number of basic syzygies of $\underline{a}$ in terms of $\overline{\underline{a}}$. As my argument shows, this means that a method can be found for obtaining a set of basic syzygies of $\underline{a}$ provided that that of $\overline{\underline{a}}$ is known. (Theorem 6.) Moreover the entire structure of the syzygy module is known when it is reduced by a general element. It turns out that $\underline{a}/z\underline{a}$ is the direct sum of $\overline{\underline{a}}$ and copies of the residue field. (Corollary 7.)

Thus we are naturally lead to define a new class of ideals which we call "completely $\underline{m}$-full." (Definition 2.) The meaning of this is that they provide us with an inductive set up. For those ideals we may calculate their Betti numbers using certain numerals $\ell_1, \ell_2, \ldots, \ell_n$, as will be shown in Corollary 9.

Finally we relate our results to the theory of Gröbner bases. Several authors have proved that the initial monomials of a Gröbner basis of a homogeneous ideal in a polynomial ring, with respect to generic variables, form a Borel stable ideal. (See [2], [6], [9].) One finds easily that in characteristic 0 a Borel stable ideal is completely $\underline{m}$-full. Since basic syzygies can be obtained through the reduction process of a Gröbner basis, we have the fact that the Betti numbers of a homogeneous ideal $\underline{a}$ do not exceed those of $in(\underline{a})$, which is the ideal generated by the initial monomials of a Gröbner basis. When $in(\underline{a})$ is completely $\underline{m}$-full, we can apply Theorem 6 to it recursively to express the Betti numbers using the numerals $\ell_1, \ell_2, \ldots, \ell_n$. If $\underline{a}$ is $\underline{m}$-primary, $\ell_1, \ldots, \ell_n$ are defined and calculated without referring to Gröbner bases. This is stated in Theorem 11.

The basic idea of this paper grew out of many discussions that I had with C.Huneke and W.Heinzer while I was in Purdue University in 1987. I would like to express my thanks to them.

§ 1. Definitions, notation and some examples

Let $(R, \underline{m}, k)$ be a local ring. We use the words "general elements" of R in the sense of D.Rees, which is explained as follows: Let $\underline{m} = (x_1, x_2, \ldots, x_n)$. Let $y_1, y_2, \ldots, y_n$ be a set of indeterminates and let $z = y_1x_1 + y_2x_2 + \ldots + y_nx_n$. Then z is called a general element of R. Strictly speaking, it is an element of $R^* := R(y_1, y_2, \ldots, y_n)$, which is the polynomial ring $R[y_1, y_2, \ldots, y_n]$ localized at

$\underline{m}R[y_1, y_2, \ldots, y_n]$, but, by abuse of language, we treat it as an element of R. For one thing it is easy to pass to R^* without affecting the situation involved, and for another, in most cases it is possible to find in R elements sufficiently general in some sense needed. Sometimes it is necessary for us to choose generators of $\underline{m}$ consisting of general elements. In this case we introduce indeterminates y_{ij} and let $z_i = \sum y_{ij}x_j$ and $\underline{m} = (z_1, z_2, \ldots, z_n)$. It should be understood that we either pass to R^* or substitute y_{ij} by suitable elements in R, if they exist, for the particular purpose. We note that a general element is in $\underline{m} \setminus \underline{m}^2$.

Recall that an ideal $\underline{a}$ of a local ring $(R, \underline{m}, k)$ is called $\underline{m}$-full if there exists an element z such that $\underline{ma}:z = \underline{a}$. (Such z may exist only in a faithfully flat extension of R.) Note that $\underline{ma}:z=\underline{a}$ for some z implies $\underline{ma}:z=\underline{a}$ for a general element z. $\underline{m}$-Primary $\underline{m}$-full ideals were treated in [10]. As to non $\underline{m}$-primary ideals, it should be noted that if depth $R/\underline{a} > 0$ then $\underline{a}$ is $\underline{m}$-full. This follows immediately from the general inclusions $\underline{a}:z \supset \underline{ma}:z \supset \underline{a}$. Also note that if $\underline{a}$ and $\underline{b}$ are $\underline{m}$-full then $\underline{a} \cap \underline{b}$ is $\underline{m}$-full. In fact $\underline{m}(\underline{a} \cap \underline{b}) \subset \underline{ma} \cap \underline{ma}$. It follows that $\underline{m}(\underline{a} \cap \underline{b}):z \subset (\underline{ma} \cap \underline{mb}):z = \underline{ma}:z \cap \underline{mb}:z = \underline{a} \cap \underline{b}$. Now we get $\underline{m}(\underline{a} \cap b):z = \underline{a} \cap \underline{b}$, since the other inclusion is obvious. So the intersections of $\underline{m}$-primary $\underline{m}$-full ideals with ideals $\underline{a}$ such that depth $R/\underline{a} > 0$ give us abundant examples of non $\underline{m}$-primary $\underline{m}$-full ideals. Here is another example.

EXAMPLE 1. Suppose that $R = k[x_1, x_2, \ldots, x_n]$ is the polynomial ring over a field k of characteristic 0. Consider the group of automorphisms of R induced by the linear transformations

$$\begin{cases} x_n \longrightarrow a_1x_1 + a_2x_2 + \ldots + a_nx_n, & a_n \neq 0, \\ x_i \longrightarrow x_i \quad , & i < n. \end{cases}$$

In the matrix notation, this group corresponds to the following subgroup in GL(n).

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & * \\ 0 & 1 & 0 & 0 & \ldots & 0 & * \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & \ldots & 1 & * \\ 0 & 0 & 0 & & \ldots & 0 & * \end{pmatrix} \right\}$$

Then an ideal is $\underline{m}$-full if it is stable under the action of this group.

Proof. Call the group above G. It is easy to see that a G-stable ideal is characterized by saying that (1) it is generated by monomials in x_n, and (2) is closed under the Euler derivations, $x_i\partial/\partial x_n$, $i = 1, 2, \ldots, n-1$. Here a monomial in x_n means an element of the form $f'x_n{}^e$, where $f' \in R' :=$ $k[x_1, x_2, \ldots , x_{n-1}]$, and e an integer. By (1) we assume $\underline{a}$ is generated by $h_i = h_i^! x_n{}^{e_i}$, $i = 1, 2, \ldots, m$, $h_i^! \in R'$. Then $\underline{ma}$ is generated by x_jh_i, $j = 1, 2, \ldots, n$,

$i = 1, 2, \ldots, m$. We want to show that $\underline{ma}:x_n = \underline{a}$. That $\underline{ma}:x_n \supset \underline{a}$ is obvious. Assume $f \in \underline{ma}:x_n$. Then $x_n f \in \underline{ma}$. Since $\underline{ma}$ is also G-stable, we may assme $x_n f$ (hence f) is a monomial in x_n. Write

$$x_n f = \sum A_{ij} x_j h_i = \sum A_{ij} x_j h_i' x_n^{e_i} \quad \ldots\ldots\ldots\ldots\ldots \quad (*)$$

where $A_{ij} \in R$. We express each A_{ij} as a polynomial in x_n with coefficients in R', expand the right hand side of (*), and keeping in mind the fact $x_j h_i' x_n^{e_i}$ are all monomials in x_n collect the terms whose x_n-degree is the same as that of $x_n f$. Then it should be equal to $x_n f$, as it is a monomial in x_n. Thus we may assume all A_{ij} in (*) are monomials in x_n. Now we divide the right hand side of (*) by x_n, term by term. Note that if $x_j \neq x_n$ and $e_i = 0$, A_{ij} should be divisible by x_n. Now notice that if $e_i > 0$, then $x_j h_i' x_n^{e_i - 1}$ differs from $x_j \partial/\partial x_n h_i$ only by a non-zero constant multiple. By (2) we conclude $f \in \underline{a}$.

DEFINITION 2. Let $(R, \underline{m}, k)$ be a local ring with emb.dim R = n. We define the "completely $\underline{m}$-full" ideals recursively as follows.

(a) If emb.dim R = 0 (i.e., R is a field), then the 0 ideal is completely $\underline{m}$-full.

(b) If emb.dim R > 0, then $\underline{a}$ is completely $\underline{m}$-full if $\underline{am}:z = \underline{a}$ and $\underline{a} + zR/zR$ is completely $\underline{m}$-full as an ideal of R/zR, where z is a general element. (Since $z \in \underline{m} \backslash \underline{m}^2$, the definition makes sense by induction on emb.dim R.)

EXAMPLE 3. Let R be as in Example 1. Let B be the Borel subgroup of $GL(n)$. I.e.,

$$B = \left\{ \begin{pmatrix} * & * & * & * & \cdots & * & * \\ 0 & * & * & * & \cdots & * & * \\ 0 & 0 & * & * & \cdots & * & * \\ \cdots & & & & & & \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & 0 & 0 & \cdots & & 0 & * \end{pmatrix} \right\}.$$

Let B act on R in the same way as in Example 1. Then any Borel stable ideal is completely $\underline{m}$-full. (This should be clear in view of Example 1.)

We use μ, τ, ℓ to denote, respectively, the minimal number of generators, the type and the length. Let $\underline{a}$ be an $\underline{m}$-primary ideal of a local ring $(R, \underline{m}, k)$. Define $\Phi(\underline{a}) = \ell(R/\underline{a}+zR)$ for a general elemenet z. (cf. [10].) Let M be a finite R-module and let b_i be the Betti numbers of M, i.e., $b_i = \dim_k \mathrm{Tor}_i(M, k)$. In this case we write $b_i = b_i(M)$. Note that if $\underline{a}$ is an ideal, then $b_i(\underline{a}) = b_{i+1}(R/\underline{a})$, and $\mu(\underline{a}) = b_0(\underline{a}) = b_1(R/\underline{a})$. Note also that $b_1(\underline{a}) = b_2(R/\underline{a})$ is the number of basic syzygies. If R is a regular local ring of dimension n, then $\tau(\underline{a}) = b_n(R/\underline{a})$.

Let z be a general element of R, and let $\bar{\ }: R \longrightarrow R/zR$ denote the natural surjection. Then for an ideal $\underline{a}$ of R

the image $\overline{\underline{a}}$ is the ideal $\underline{a} + zR/zR$ considered as an ideal of R/zR. We have the following result. (For proof see [10] Theorem 2.)

PROPOSITION 4. An $\underline{m}$-primary ideal $\underline{a}$ is $\underline{m}$-full if and only if $\mu(\underline{a}) = \phi(\underline{m}\underline{a}) = \phi(\underline{a}) + \mu(\overline{\underline{a}})$. (The second equal holds generally.) In this case $\tau(\underline{a}) = \phi(\underline{a})$.

§2. The syzygies of $\underline{m}$-full ideals

PROPOSITION 5 (Huneke). Let $(R, \underline{m}, k)$ be a local ring and $\underline{a}$ an $\underline{m}$-full ideal. Let z be a general element of R, and let $\overline{} : R \longrightarrow \overline{R} = R/zR$ denote the natural map. Then any syzygy of $\overline{\underline{a}}$ lifts to a syzygy of $\underline{a}$.

Proof. Write $\underline{a} = (f_1, f_2, \ldots, f_r, zf_{r+1}, \ldots, zf_s)$, with $\mu(\overline{\underline{a}}) = r$ and $\mu(\underline{a}) = s$. Suppose $\sum_{i=1}^{r} \overline{a}_i\overline{f}_i = 0$. Then $\sum_{i=1}^{r} a_if_i = zh$ for some $h \in R$. Observe that $h \in \underline{ma}:z = \underline{a}$. So $h = \sum_{i=1}^{r} g_if_i + \sum_{j=r+1}^{s} g_j(zf_j)$. This gives us the syzygy

$$\textstyle\sum_i (a_i - zg_i)f_i + \sum_j(-zg_j)(zf_j) = 0,$$ as wanted. Q.E.D.

Temporarily we will call a syzygy obtained this way essential. Namely, an essential syzygy of $\underline{a}$ is a syzygy that reduces to a non-trivial syzygy of $\underline{a}$ mod z. (We understand that we fix a general element z in the beginning.) Obviously there are at least $b_1(\overline{\underline{a}})$ such independent syzygies. In the next paragraph we will find another kind of syzygies which we call superficial.

First notice that $\underline{ma}:z=\underline{a}$ implies $\underline{a}:\underline{m}=\underline{a}:z$. In fact $\underline{a}:\underline{m} = (\underline{ma}:z):\underline{m} = (\underline{ma}:\underline{m}):z \supset \underline{a}:z$. Thus $\underline{a}:z = \underline{a}:\underline{m}$, since the other inclusion is obvious. Again assume $\underline{ma}:z = \underline{a}$ with z a general element and write $\underline{a} = (f_1, \ldots, f_r, zf_{r+1}, \ldots, zf_s)$ as in the proof of Proposition 5. Now suppose x is any element in $\underline{m} \setminus zR$, and j_0 is an integer such that $r+1 \leqq j_0 \leqq s$. Since $\underline{a}:\underline{m} = \underline{a}:z$, $f_{j_0} \in \underline{a}:\underline{m}$. Hence $xf_{j_0} \in \underline{a}$. So we may write

$$xf_{j_0} = \sum_{i=1}^{r} a_i f_i + \sum_{j=r+1}^{s} a_j(zf_j).$$

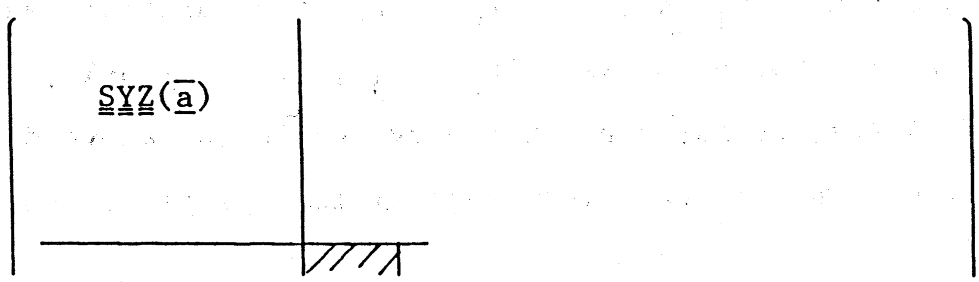
Multiply both sides by z. Then $x(zf_{j_0}) = \sum(za_i)f_i + \sum (za_j)(zf_j)$. This gives us the following syzygy:

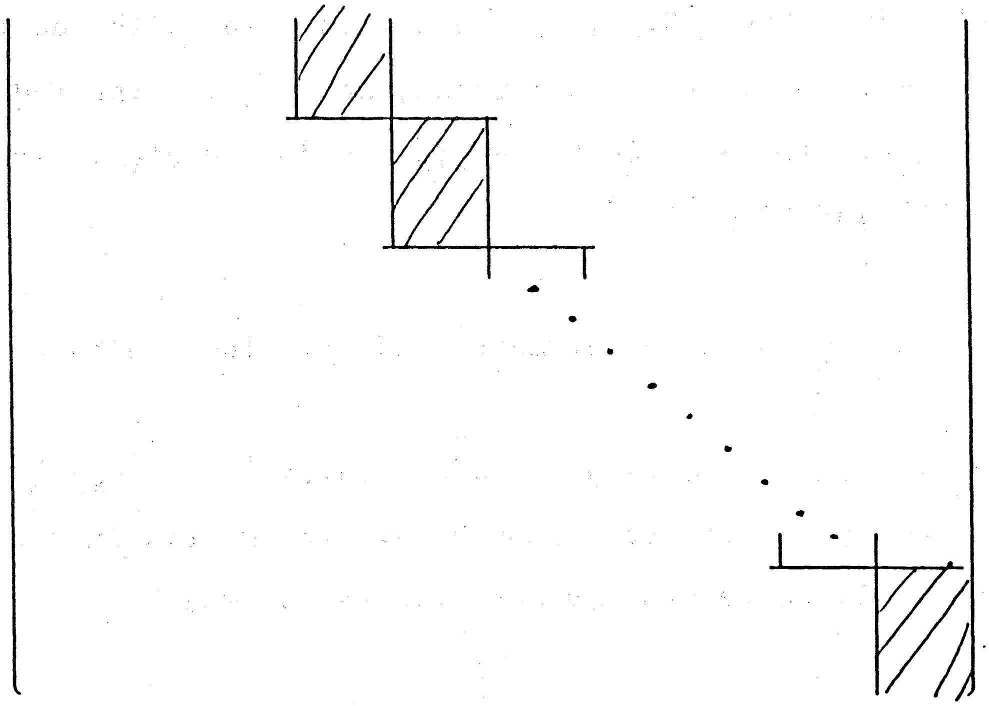
$$[-za_1 \ -za_2 \ \ldots \ -za_r \ -za_{r+1} \ \ldots\ldots\ldots$$
$$\ldots\ldots\ldots \ -za_{j_0-1} \ \ x-za_{j_0} \ -za_{j_0+1} \ \ldots \ -za_s].$$

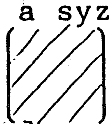
Suppose that $(x_1, x_2, \ldots, x_{n-1}, z)$ is a minimal generating set of $\underline{m}$. For each pair (x_i, zf_j), $1 \leqq i \leqq n-1$, $r+1 \leqq j \leqq s$, we may construct a sygygy in the above described fashion. We will call them superficial syzygies. Obviously there are $(\mu(\underline{m}) - 1) \times (s-r)$ such syzygies. They are, together with essential syzygies, all independent, since they are independent modulo z. We claim that we have obtained all basic syzygies of $\underline{a}$, provided that z is a non-zero-divisor. In fact we prove

THEOREM 6. Let $(R, \underline{m}, k)$ be a local ring with depth $R > 0$. Suppose that $\underline{a}$ is an ideal of R such that $\underline{ma}:z=\underline{a}$ for a general element z. Let $r = \mu(\bar{\underline{a}})$ and $s = \mu(\underline{a})$. Then $b_1(\underline{a}) = b_1(\bar{\underline{a}}) + (\mu(\underline{m})-1)\times(s-r)$. (Recall that b_1 of an ideal is the minimal number of basic syzygies.)

Proof. Since z is a general element and since depth $R > 0$, z is a non-zero-divisor. Let M be the submodule of R^s generated by all the syzygies, both essential and superficial, described above. Assume, contrary to the assertion, $b_1(\underline{a}) > b_1(\bar{\underline{a}}) + (\mu(\underline{m})-1)\times(s-r)$. This means that there is a basic syzygy of $\underline{a}$ which is not in M. Say $A = [a_1 \;....\; a_s]$ is such a syzygy. Then this will be a basic syzygy even after any element of M is added to it. Let $\underline{M}$ be the matrix consisting of the generators of M. Then $\underline{M}$ mod z looks like this.





where $\underline{\underline{SYZ}}(\bar{\underline{a}})$ is a syzygy matrix of $\bar{\underline{a}}$ (in the obvious sense), and each block  is the transpose of $[\bar{x}_1 \ \bar{x}_2 \ \ldots \ \bar{x}_{n-1}]$. Now by adding elements of M to A, we may assume that all a_i are in zR for $i \geq r+1$, since every one of them is in $(x_1, x_2, \ldots\ldots, x_{n-1})$ mod z. Then A is an essential syzygy. But all essential syzygies are already in M mod z. Thus we conclude that $A + M$ contains an element whose entries are all multiples of z. This is a contradiction since z is a non-zero-divisor and any element in $A + M$ is basic. Q.E.D.

Remark. (i) Note that $s-r = 0$ if depth $R/\underline{a} > 0$.

(ii) Suppose that $\underline{a}$ is $\underline{m}$-primary. Then $s-r = \phi(\underline{a})$ by Proposition 4.

COROLLARY 7. Let $(R, \underline{m}, k)$ be a local ring with depth $R > 0$. Suppose that $\underline{a}$ is an $\underline{m}$-full ideal, and $\overline{\underline{a}}$ is the reduction by a general element. Put $r = \mu(\overline{\underline{a}})$ and $s = \mu(\underline{a})$. Then $\underline{a} \otimes R/zR \cong \overline{\underline{a}} \oplus (R/\underline{m})^{(s-r)}$.

Proof. Let $\underline{\underline{M}}$ be a syzygy matrix of $\underline{a}$. Then we have the exact sequence

$$(R/zR)^{s'} \xrightarrow{\overline{\underline{\underline{M}}}} (R/zR)^{s} \longrightarrow \underline{a}/z\underline{a} \longrightarrow 0.$$

Since $\overline{\underline{\underline{M}}} = \underline{\underline{M}} \otimes R/zR$ is isomorphic to the matrix in the proof of Theorem 6, we get the isomorphism as asserted.

COROLLARY 8. Let $(R, \underline{m}, k)$ a regular local ring with $n = \mu(\underline{m})$. Let $\underline{a}$ be an $\underline{m}$-full ideal and $\overline{\underline{a}}$ the reduction by a general element and $r = \mu(\overline{\underline{a}})$, $s = \mu(\underline{a})$ as above. Then

$$b_i(R/\underline{a}) = b_i(\overline{R}/\overline{\underline{a}}) + \binom{n-1}{i-1} \times (s-r).$$

Proof. Put $b_i = b_i(R/\underline{a})$. Then we have a minimal free resolution:

$$0 \longrightarrow R^{b_n} \longrightarrow R^{b_{n-1}} \longrightarrow \dots \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow \underline{a}\ .$$

Since depth$(R/\underline{a}) \geq 1$ and since $pd_R(R/zR) = 1$, we get a minimal free resolution of $\underline{a}/z\underline{a}$ by applying the tensor product $\otimes R/zR$ to it. Since a minimal free resolution of $\overline{R}/(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{n-1})$ over $\overline{R}$ is given by the exteria algebra on the generators of $\overline{\underline{m}}$, the assertion follows immediately.

COROLLARY 9. Let $(R, \underline{m}, k)$ be a regular local ring of dimension n. Let $\underline{a}$ be a completely $\underline{m}$-full ideal. Let $z_1, \dots, z_n$ be a set of generators of $\underline{m}$ consisting of general elements. (cf. §1.) Set

$$R^{(0)} = R,$$

$$R^{(i)} = R/(z_n, z_{n-1}, \dots, z_{n-i+1})R, \qquad i = 1, \dots, n,$$

$$\ell_i = \mu(\underline{a}R^{(i-1)}) - \mu(\underline{a}R^{(i)}), \qquad i = 1, \dots, n.$$

Then $b_i(R/\underline{a}) = \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \ell_1 + \begin{pmatrix} n-2 \\ i-1 \end{pmatrix} \ell_2 + \dots\dots + \begin{pmatrix} 1 \\ i-1 \end{pmatrix} \ell_{n-1} + \begin{pmatrix} 0 \\ i-1 \end{pmatrix} \ell_n.$

Here $\begin{pmatrix} p \\ q \end{pmatrix} = \dfrac{p!}{(p-q)!q!}$ for $0 \leq q \leq p$, and $\begin{pmatrix} p \\ q \end{pmatrix} = 0$ otherwise.

Proof. Immediate by induction.

Remark. In the corollary above, if $\underline{a}$ is $\underline{m}$-primary, then $\ell_i = \ell(R/\underline{a} + (z_n, z_{n-1}, \dots, z_{n-i+1}))$. (See Proposition 4.)

DEFINITION 10. Let $(R, \underline{m}, k)$ be a regular local ring. For an $\underline{m}$-primary ideal $\underline{a}$, we define $B_i(R/\underline{a})$ to be the right hand side of Corollary 9, with $\ell_i = \ell(R/\underline{a} + (z_n, \dots, z_{n-i+1}))$. In particular the same definition is used for $\underline{m}$-primary homogeneous ideals in a polynomial ring.

THEOREM 11. Let R be a polynomial ring over a field of characteristic 0. Let $\underline{a}$ be a homogeneous $\underline{m}$-primary ideal. Then $b_i(R/\underline{a}) \leq B_i(R/\underline{a})$ for all i.

Proof. We need the theory of Gröbner bases. The reader unfamiliar with it is referred to [3], [4] and [8]. We confine ourselves with the outline of proof. First fix a set of generic variables $z_1, z_2, \ldots, z_n$, and the graduated reverse lexicographic order on the set of monomials with $z_1 > z_2 > \ldots > z_n$. For $f \in R$ we denote by in(f) the initial monomial of f, and for an ideal $\underline{a}$ we denote by in($\underline{a}$) the ideal generated by all the monomials in(f), $f \in \underline{a}$. We say that $g_1, g_2, \ldots, g_s \in \underline{a}$ are a Gröbner basis of $\underline{a}$ if in(g_1) , ..., in(g_s) generate in($\underline{a}$). It is known that in($\underline{a}$) is Borel stable, hence completely $\underline{m}$-full. (See for example [2] Proposition 1.) It is easy to see that if $(g_1, \ldots, g_s)$ is a Gröbner basis of $\underline{a}$ then $(g_1, \ldots, g_s, z_n)$ is a Gröbner basis of $\underline{a}$ + zR. (Here we need to use the reverse lexicographic order. See [1] Lemma 2.2.) Hence $B_i(R/\underline{a}) = B_i(R/\text{in}(\underline{a}))$. Now by the general theory of Gröbner bases, a set of basic syzygis of $\underline{a}$ is obtained through the reduction process of syzygies (including higher syzygies) of initial monomials of its Gröbner basis. Therefore b_i does not exceed B_i for any i. For details see [8] Lemma 7.6 on p.157.

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