数理解析研究所講究録 18
第 713 巻 1990 年 18-23

On unramified coverings of the affine line in positive characteristics*

Tatsuji Kambayashi (上林 達治)
Faculty of Science and Engineering
Tokyo Denki University
Hiki-gun, Saitama-ken, 350-03 / J A P A N

October 10, 1989

In 1957 a fundamental paper by Abhyankar [A] dealing with covering of algebraic curves in the 'modular' (*i.e.*, positive-characteristic) case appeared. In that paper, among other things, he made a conjecture that "any finite group generated by its p-Sylow subgroups can occur as Galois group of an étale Galois covering of the affine line in characteristic p". The conjecture, if proven true, would imply that any finite simple group whose order is divisible by p should occur as Galois group of such a covering.

In the early 1980's, after a long period of inaction, certain problems related to Abhyankar's Conjecture began to receive some attention (see, for instance, Harbater [H], Kambayashi-Srinivas [KS]). More recently, Jean-Pierre Serre appears to have taken considerable interest in the conjecture, and Ram Abhyankar has rejoined the ranks to resolve his old problem. They have already made significant progress, though as far as known to this writer none of their results have been published yet.

The purpose of this note is to explain the situation underlying Abhyankar's Conjecture and to review what has been done in this area so far, including very recent results due to Abhyankar and Serre as gathered from their letters to the writer.

*A summary of the writer's talk given at the "Symposium on Frobenius Maps in Commutative Algebra" held at Research Institute for Mathematical Sciences, Kyoto University in September 1989

1 Étale coverings of the affine space.

We shall work over a fixed algebraically closed ground field k whose characteristic p *may* be 0 for the time being, and shall consider only normal varieties over k (*i.e.*, normal, integral schemes of finite type over k). A morphism $Y \longrightarrow X$ of such varieties will be called an *étale covering of X by Y* if it is étale *and* finite. Such an étale covering is by definition *Galoisian*, or is a *Galois covering*, if the fields of rational functions $k(Y) \supseteq k(X)$ on Y and X form a Galois extension. In case $p = char(k) > 0$, a Galois covering $Y \longrightarrow X$ is said to be *tame* if its degree $[k(Y) : k(X)]$ is *not* divisible by p.

The starting point of all is the following well-known fact:

Theorem 1 *(a) The projective space $\mathbf{P}^n$ has no nontrivial étale coverings.*

(b) In characteristic $p = 0$, the affine space $\mathbf{A}^n$ has no nontrivial étale coverings.

(c) In case $p > 0$, the affine space $\mathbf{A}^n$ has no nontrivial étale tame Galois *coverings.*

The *proof*, *via* Bertini's First Theorem, comes down to the $n=1$ level, and one then shows using Hurwitz' Theorem that $\mathbf{P}^1$ has no nontrivial coverings at worst tamely ramified over one point on the line (*cf.* [KS] for instance).

From now on we confine ourselves to the characteristic $p > 0$ case.

It is important to note in part (c) above that *both* 'tame' *and* 'Galoisian' conditions are required to assure the non-existence of said coverings.

Example 1. (a) Drop the condition of tameness (or p not dividing the degree); then one gets the classical *Artin-Schreier* coverings:

$$Y = (y^p - y - f(x) = 0) \longrightarrow X = \mathbf{A}^1 = (\text{the } x\text{-axis})$$

with $f(x) \in k[x], f(x) \neq g(x)^p - g(x)$ for any $g(x) \in k[x]$, which is an étale Galois covering of degree p of the affine line.

(b) Drop the condition of being a Galois covering; then one gets Abhyankar's example (and many others like it):

$$Y = (y^{p+1} - xy^p + 1 = 0) \longrightarrow X = \mathbf{A}^1 = (\text{the } x\text{-axis}),$$

which is an étale, non-Galois covering of the affine line of degree $p+1$. See [A; Th. 1, p. 830].

For further investigation of étale coverings of the affine n-space $\mathbf{A}^n$ and, in particular, of the affine line $\mathbf{A}^1$, one should obviously start with coverings of degree p. There is a basic fact about this, apparently known to experts already:

Theorem 2 *Let $X = \operatorname{Spec} R$ be an affine variety, and let $Y \longrightarrow X$ be an étale* Galois *covering of degree p. Then, Y is an* Artin-Schreier *covering:*

$$Y = \operatorname{Spec}(R[T]/(T^p - T - f)), \quad f \in R.$$

The *proof* of this theorem is achieved by viewing Y as a $\mathbf{Z}/p$-torsor over X with respect to the étale topology and by showing $H^1_{\acute{e}t}(X, \mathbf{Z}/p) \simeq R/(F-1)R$, where F denotes the Frobenius map $t \mapsto t^p$. See [KS; Th. 2.1] for details.

Then, a natural question one might ask next is the following:

> QUESTION: Is every étale covering of *degree p* of an affine variety (or, at any rate, of the affine space $\mathbf{A}^n$) Galoisian, and therefore an Artin-Schreier covering?

Srinivas and I addressed this question in [KS] and have verified that:

> *The answer is 'YES' for characteristic $p = 2$ or 3.*

2 Nori's Theorem and other recent results

The question we asked as to whether or not all degree-p étale coverings of the affine space in characteristic p are Galoisian has been more than answered by the following theorem due to Madhav Nori (see [K] for proofs, *etc.*):

Theorem 3 *Let X be an irreducible affine scheme of positive dimension and of finite type over a field k of characteristic $p > 0$; let G be a connected affine algebraic group defined over a finite field $\mathbf{F}_q$ of $q = p^e$ elements. Assume that G,* modulo *its unipotent radical, is semisimple and simply connected. Then, there exist étale Galois coverings $Y \longrightarrow X$ of X with Galois group $G(Y/X) \simeq G(\mathbf{F}_q) =$* (the group of $\mathbf{F}_q$-rational points on G).

As an immediate corollary to this theorem we get examples showing that the answer to the question above is *negative* for the characteristic $p = 5$ or 7. Namely, writing S_n and A_n respectively for the symmetric and the alternating groups on n letters, we find:

Example 2. Consider the groups

$$PSL(2,5) \simeq A_5 \text{ and } PSL(2,7) \simeq SL(3,2).$$

The former contains a subgroup $\simeq A_4$ of index 5, while the latter has a subgroup $\simeq S_4$ of index 7, neither subgroups being normal. By pulling these back, we can build within $SL_2(\mathbf{F}_5)$ and $SL_2(\mathbf{F}_7)$ non-normal subgroups of index 5 and 7, respectively. Now apply Nori's theorem to these to obtain étale Galois coverings, over *any* affine variety, of degrees 120 and 336 in characteristics 5 and 7, respectively. Finally take the intermediate coverings corresponding to the non-normal subgroups found already. We have étale, non-Galois coverings of degree p in characteristics $p = 5,\ 7$.

With the apparition of Nori's Theorem and consequent EXAMPLE 2, it looked to me as though the matter had been put to rest insofar as unramified coverings of $\mathbf{A}^1$ were concerned—*until* I began a year ago to receive letters from Jean-Pierre Serre and then from Ram Abhyankar. Serre seems to feel that the

> *Problem* of determining which finite group can occur as Galois groups of an unramified covering of the affine line

is rather akin to its famous counterpart concerning Galois extensions of $\mathbf{Q}$, and he seems to be hot in pursuit of what one might regard his old baby [S]. Abhyankar, 'inspired' (his own word) by correspondence with Serre, has reëntered the scene and has already pushed through some tremendous calculations aided in part by *Macsyma.*

To conclude this report I shall attempt to list up some salient points that they make in their letters to me. What follows should be considered my free quotations. Barest indications of proofs will be offered as in their letters.[1]

Let us adopt a *shorthand*: In place of saying "a finite group G occurs as Galois group of an étale Galois covering of the affine line $\mathbf{A}^1$", we simply state that "*G occurs*".

[1] I have been able to attach my own proofs to certain of their results, while others remain unverified by me.

Professor Serre's remarks:

(S1) He has calculated the Galois group of the Galois closure of Abhyankar's example, EXAMPLE 1(b) above, and after 'some extra work' has found that it is $PSL(2,p)$, which of course has no quotients of order p as soon as $p \geq 5$.

(S2) Towards Abhyankar's conjecture stated at the beginning of this article, he has shown that all Chevalley groups of rank 1 occur, *i.e.*, $PSL(2,q)$, $PSU(3,q^2)$, and the Suzuki and Ree groups in char. 2 and 3, where q is a power of p. The construction, according to him, uses 'the Deligne-Lusztig curves'.

(S3) *Apropos* of EXAMPLE 2 above, Serre offers a novel construction of an étale, non-Galoisian covering of $\mathbf{A}^1$ of *any* characteristic $p \geq 5$: Let E be a supersingular elliptic curve in char. p, and express its nowhere-vanishing differential ω of the first kind as $\omega = df$. A proper choice of such an f will produce an étale covering

$$f : E - \{\text{unique pole of} f\} \longrightarrow \mathbf{A}^1.$$

(S4) Every finite p-group occurs—not only for $\mathbf{A}^1$ but also for any irreducible affine scheme X of positive dimension in char. p. As a consequence, for any unipotent algebraic group G defined over a finite field, $G(\mathbf{F}_q)$ occurs for X. (*cf.* [K; intro., p.640].) The proof uses two propositions about the profinite group Γ which, by definition, is the Galois group of the maximal unramified p-extension of X: Namely, $H^2(\Gamma, \mathbf{Z}/p) = 0$ and $H^1(\Gamma, \mathbf{Z}/p)$ is infinite-dimensional.

Professor Abhyankar's remarks.

(A1) Pushing (S1) further, he has calculated Galois groups of the Galois closures of the following unramified coverings Y_t of the affine line:

$$Y_t := (y^{p+t} - xy^p + 1 = 0) \longrightarrow \mathbf{A}^1 = (\text{the } x\text{-axis}),$$

where t is a positive integer not divisible by p. This type of covering is a special case of Abhyankar's own example [A; Prop. 1, p.831], and Y_1 is precisely EXAMPLE 1(b) treated in (S1). It has been shown that, for $t > 1$, the Galois group for Y_t is the *alternating* group A_{p+t}.

(A2) In a similar vein, he has examined the following situation:

Fix the integers $n \geq p = \mathrm{char}(k)$, $0 \leq r < n, 0 \leq t < n$, s any integer, and choose nonzero $a \in k$. Let $Y = Y(n, r, s, t; a)$ be the morphism

$$Y := (y^n + y^r - ax^s y^t = 0) \longrightarrow \mathbf{A}^1 = (\text{the } x\text{-axis}).$$

With the aid of several group-theorists about multiply-transitive groups and that of *Macsyma*, he has found a large number of cases in which the Y gives an étale covering of the affine line, and proceeded to compute the corresponding Galois groups. Omitting to cite these cases individually, let me just state the consequences:

(a) For all $n \geq p > 2$, the alternating group A_n occurs.

(b) For all $n \geq p = 2$, the symmetric group S_n occurs.

REFERENCES

[A] S. S. Abhyankar, 'Coverings of algebraic curves", *American Journal of Mathematics*, 79(1957), 825–856.

[H] D. Harbater, 'Ordinary and supersingular covers in characteristic p", *Pacific Journal of Mathematics*, 113(1984), 349–363.

[KS] T. Kambayashi, V. Srinivas, 'On étale coverings of the affine space" IN: Algebraic Geometry—Proceedings of Ann Arbor Conference, *Lecture Notes in Math.*, vol. 1008(1983), Springer-Verlag, New York.

[K] T. Kambayashi, 'Nori's construction of Galois coverings in positive characteristics", IN: Algebraic and Topological Theories—*to the memory of Dr. Takehiko Miyata*, 640–647, 1985, Kinokuniya, Tokyo.

[S] J.-P. Serre, "Sur la topologie des variétés algébriques en caractéristique p" IN: Symposium Internacional de la Topología Algebraica, 24–53, 1956, Univ. Autonom. de Mexico.