

## SOME TOPICS ON MULTIVALENT FUNCTIONS

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### I. INTRODUCTION

Let  $A(p,n)$  denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}; n \in \mathbb{N})$$

which are analytic in the unit disk  $U = \{z: |z| < 1\}$ .

A function  $f(z)$  belonging to the class  $A(p,n)$  is said to be  $p$ -valently starlike of order  $\alpha$  if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$$

for some  $\alpha$  ( $0 \leq \alpha < p$ ) and for all  $z \in U$ . We denote by  $S(p,n,\alpha)$  the subclass of  $A(p,n)$  consisting of functions which are  $p$ -valently starlike of order  $\alpha$ .

A function  $f(z)$  in the class  $A(p,n)$  is said to be  $p$ -valently convex of order  $\alpha$  if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for some  $\alpha$  ( $0 \leq \alpha < p$ ) and for all  $z \in U$ . We denote by  $K(p,n,\alpha)$  the subclass of  $A(p,n)$  consisting of all such functions. Note that  $f(z) \in K(p,n,\alpha)$  if and only if  $zf'(z) \in S(p,n,\alpha)$ .

Further, a function  $f(z)$  belonging to the class  $A(p,n)$  is said to be  $p$ -valently close-to-convex of order  $\alpha$  if there exists a function  $g(z) \in K(p,n,0)$  such that

$$(1.4) \quad \operatorname{Re} \left\{ \frac{f'(z)}{g'(z)} \right\} > \alpha$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ) and for all  $z \in U$ . Also we denote by  $C(p, n, \alpha)$  the subclass of  $A(p, n)$  consisting of functions which are  $p$ -valently close-to-convex of order  $\alpha$ .

## 2. $p$ -VALENTLY $\alpha$ -CONVEX FUNCTIONS OF ORDER $\beta$

A function  $f(z)$  in the class  $A(p, n)$  is said to be  $p$ -valently  $\alpha$ -convex of order  $\beta$  if it satisfies

$$(2.1) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \beta$$

for some  $\alpha$  ( $\alpha \geq 0$ ),  $\beta$  ( $0 \leq \beta < p$ ), and for all  $z \in U$ . Denoting by  $M(p, n, \alpha, \beta)$  the subclass of  $A(p, n)$  consisting of all such functions, we see that  $M(p, n, 0, \beta) = S(p, n, \beta)$  and  $M(p, n, 1, \beta) = K(p, n, \beta)$ .

In particular,  $M(1, 1, \alpha, \beta)$  when  $p = 1$  and  $n = 1$  is the class which was studied by Zmorovich and Pokhilevich [6], and  $M(p, 1, \alpha, 0)$  when  $n = 1$  and  $\beta = 0$  is the class which was studied by Owa and Ren [4], and Ren and Owa [5].

LEMMA 2.1 (Miller and Mocanu [1]). Let  $\phi(u, v)$  be a complex valued function,

$$\phi: D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C}^2 \quad (\mathbb{C} \text{ is the complex plane}),$$

and let  $u = u_1 + iu_2$ ,  $v = v_1 + iv_2$ . Suppose that the function  $\phi(u, v)$  satisfies

- (i)  $\phi(u, v)$  is continuous in  $D$ ;
- (ii)  $(1, 0) \in D$  and  $\operatorname{Re}\{\phi(1, 0)\} > 0$ ;
- (iii)  $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$  for all  $(iu_2, v_1) \in D$  such that  $v_1 \leq -n(1 + u_2^2)/2$ .

Let  $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$  be regular in the

unit disk  $\mathbb{U}$  such that  $(q(z), zq'(z)) \in \mathbb{D}$  for all  $z \in \mathbb{U}$ . If

$$\operatorname{Re}\{\phi(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Using the above lemma, we derive

**THEOREM 2.1.** If  $f(z) \in \mathcal{M}(p, n, \alpha, \beta)$  with  $0 \leq (p - \alpha)/2 \leq \beta < p$ , then

$$(2.2) \quad \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma(p, n, \alpha, \beta) \quad (z \in \mathbb{U}),$$

where

$$(2.3) \quad \gamma(p, n, \alpha, \beta) = \frac{2\beta - \alpha n + \sqrt{(2\beta - \alpha n)^2 + 8\alpha p n}}{4}.$$

Therefore,  $f(z)$  is in the class  $\mathcal{S}(p, n, \gamma(p, n, \alpha, \beta))$ .

**PROOF.** Define the function  $q(z)$  by

$$(2.4) \quad \frac{zf'(z)}{f(z)} = p(\gamma_1 + (1 - \gamma_1)q(z))$$

with  $\gamma_1 = \gamma(p, n, \alpha, \beta)/p$ . Then  $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$

is regular in the unit disk  $\mathbb{U}$ . Making use of the logarithmic differentiations of both sides in (2.4), we obtain

$$(2.5) \quad 1 + \frac{zf''(z)}{f'(z)} = p(\gamma_1 + (1 - \gamma_1)q(z)) + \frac{(1 - \gamma_1)zq'(z)}{\gamma_1 + (1 - \gamma_1)q(z)}$$

It follows from (2.4) and (2.5) that

$$(2.6) \quad \operatorname{Re}\left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \beta \right\}$$

$$= \operatorname{Re} \left\{ p\gamma_1 - \beta + p(1 - \gamma_1)q(z) + \frac{\alpha(1 - \gamma_1)zq'(z)}{\gamma_1 + (1 - \gamma_1)q(z)} \right\} \\ > 0.$$

Letting  $u = u_1 + iu_2$ ,  $v = v_1 + iv_2$ , and

$$(2.7) \quad \phi(u, v) = p\gamma_1 - \beta + p(1 - \gamma_1)u + \frac{\alpha(1 - \gamma_1)v}{\gamma_1 + (1 - \gamma_1)u},$$

we know that

$$(i) \quad \phi(u, v) \text{ is continuous in } D = \left( \mathbb{C} - \left\{ \frac{\gamma_1}{\gamma_1 - 1} \right\} \right) \times \mathbb{C};$$

$$(ii) \quad (1, 0) \in D \text{ and } \operatorname{Re}\{\phi(1, 0)\} = p - \beta > 0;$$

$$(iii) \quad \text{for all } (iu_2, v_1) \in D \text{ such that } v_1 \leq -n(1 + u_2^2)/2,$$

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= p\gamma_1 - \beta + \frac{\alpha\gamma_1(1 - \gamma_1)v_1}{\gamma_1^2 + (1 - \gamma_1)^2 u_2^2} \\ &\leq p\gamma_1 - \beta - \frac{\alpha\gamma_1(1 - \gamma_1)n(1 + u_2^2)}{2\{\gamma_1^2 + (1 - \gamma_1)^2 u_2^2\}} \\ &\leq 0 \end{aligned}$$

because  $0 \leq (p - \alpha n)/2 \leq \beta < p$  and  $0 < \gamma_1 < 1$ .

This implies that the function  $\phi(u, v)$  satisfies the conditions in Lemma 2.1. Thus, applying Lemma 2.1, we have

$$(2.8) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > p\gamma_1 = \gamma(p, n, \alpha, \beta) \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.1.

Making  $p = 1$ , Theorem 2.1 leads to

COROLLARY 2.1. If  $f(z) \in M(1, n, \alpha, \beta)$  with  $0 \leq (1 - \alpha n)/2 \leq \beta < 1$ , then

$$(2.9) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{2\beta - \alpha n + \sqrt{(2\beta - \alpha n)^2 + 8\alpha n}}{4} \quad (z \in U).$$

Taking  $\alpha = 1$  in Theorem 2.1, we have

COROLLARY 2.2. If  $f(z) \in K(p, n, \beta)$  with  $0 \leq (p - n)/2 \leq \beta < p$ , then

$$(2.10) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{2\beta - n + \sqrt{(2\beta - n)^2 + 8pn}}{4} \quad (z \in U).$$

Letting  $\beta = \alpha n/2$ , we have

COROLLARY 2.3. If  $f(z) \in M(p, n, \alpha, \alpha n/2)$  with  $\alpha n > 2p$ , then

$$(2.11) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \sqrt{\frac{\alpha pn}{2}} \quad (z \in U).$$

Further, using the same technique as in the proof of Theorem 2.1, we prove

THEOREM 2.2. If  $f(z) \in A(p, n)$  satisfies

$$(2.12) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} < \beta \quad (z \in U)$$

for some  $\alpha$  ( $\alpha \geq 0$ ) and  $\beta$  ( $\beta > p$ ), then

$$(2.13) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \gamma(p, n, \alpha, \beta) \quad (z \in U),$$

where  $\gamma(p, n, \alpha, \beta)$  is given by (2.3).

3. p-VALENTLY CLOSE-TO-CONVEX OF ORDER  $\delta$ 

In order to derive our next result, we need the following lemma.

LEMMA 3.1. If  $f(z) \in S(p, n, \alpha)$ , then

$$(3.1) \quad \operatorname{Re} \left\{ \frac{f(z)}{z^p} \right\}^\beta > \frac{n}{2\beta(p - \alpha) + n} \quad (z \in U),$$

where  $0 < \beta \leq n/2(p - \alpha)$ .

Now, we prove

THEOREM 3.1. If  $f(z) \in A(p, n)$  satisfies

$$(3.2) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha - \beta \quad (z \in U)$$

for some  $\alpha$  ( $\alpha > 0$ ) and  $\beta$  ( $0 < \beta \leq n/2(1 - \gamma)$ ), where  $\gamma = \alpha/(p + \beta)$ , then  $f(z) \in C(p, n, \delta(p, n, \alpha, \beta))$ , where

$$(3.3) \quad \delta(p, n, \alpha, \beta) = \frac{n(p + \beta)}{(p + \beta)(n + 2\beta) - 2\alpha\beta}.$$

PROOF. Note that  $f(z)$  satisfies

$$(3.4) \quad \operatorname{Re} \left\{ 1 + \beta + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U).$$

We define the function  $g(z)$  by

$$(3.5) \quad \frac{zg'(z)}{g(z)} = \frac{1}{p + \beta} \left( 1 + \beta + \frac{zf''(z)}{f'(z)} \right).$$

Then  $g(z)$  belongs to the class  $S(1, n, \gamma)$  with  $\gamma = \alpha/(p + \beta)$ .

Noting that

$$(3.6) \quad \frac{zf'(z)}{g(z)^p} = \left( \frac{g(z)}{z} \right)^\beta$$

and  $g(z)^p \in \mathcal{S}(p, n, p\gamma)$ , Lemma 3.1 leads to

$$(3.7) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{g(z)^p} \right\} = \operatorname{Re} \left( \frac{g(z)}{z} \right)^\beta$$

$$> \frac{n}{2\beta(1-\gamma) + n}$$

$$= \delta(p, n, \alpha, \beta).$$

This completes the assertion of Theorem 3.1.

Taking  $\alpha = 0$  in Theorem 3.1, we have

**COROLLARY 3.1.** If  $f(z) \in A(p, n)$  satisfies

$$(3.8) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\beta \quad (z \in U)$$

for some  $\beta$  ( $0 < \beta \leq n/2$ ), then  $f(z) \in \mathcal{C}(p, n, \delta(n, \beta))$ , where  $\delta(n, \beta) = n/(n+2\beta)$ .

Further, making  $\alpha = \beta$  in Theorem 3.1, we have

**COROLLARY 3.2.** If  $f(z) \in K(p, n, 0)$ , then  $f(z) \in \mathcal{C}(p, n, 1/2)$ .  
Therefore,  $K(p, n, 0)$  is the subclass of  $\mathcal{C}(p, n, 1/2)$ .

#### 4. CONVOLUTIONS FOR MULTIVALENT FUNCTIONS

For functions  $f_j(z)$ ,  $j = 1, 2$ , defined by

$$(4.1) \quad f_j(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,j} z^k \quad (p \in \mathbb{N}; n \in \mathbb{N}),$$

we denote by  $f_1 * f_2(z)$  the convolution (or Hadamard product) of two

functions  $f_1(z)$  and  $f_2(z)$ , that is,

$$(4.2) \quad f_1 * f_2(z) = z^p + \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k.$$

LEMMA 4.1 (Owa [2]), If  $f(z) \in S(1,1,\alpha)$  and  $g(z) \in K(1,1,\beta)$ , then  $f * g(z) \in S(1,1,\alpha)$ .

LEMMA 4.2 (Owa [2]), If  $f(z) \in K(1,1,\alpha)$  and  $g(z) \in K(1,1,\beta)$ , then  $f * g(z) \in K(1,1,\gamma)$ , where  $\gamma = \max(\alpha, \beta)$ .

LEMMA 4.3 (Owa [3]), If  $f(z) \in C(1,1,\alpha)$  and  $g(z) \in K(1,1,\beta)$ , then  $f * g(z) \in C(1,1,\gamma)$ , where  $\gamma = \max(\alpha, \beta)$ .

In view of the above lemmas, we have

- REMARK. (i)  $f(z) \in S(1,n,\alpha)$ ,  $g(z) \in K(1,n,\beta)$   
 $\implies f * g(z) \in S(1,n,\alpha)$ .  
(ii)  $f(z) \in K(1,n,\alpha)$ ,  $g(z) \in K(1,n,\beta)$   
 $\implies f * g(z) \in K(1,n,\gamma)$ ,  $\gamma = \max(\alpha, \beta)$ .  
(iii)  $f(z) \in C(1,n,\alpha)$ ,  $g(z) \in K(1,n,\beta)$   
 $\implies f * g(z) \in C(1,n,\gamma)$ ,  $\gamma = \max(\alpha, \beta)$ .

Finally, from the above remark, we give

- CONJECTURE. (i)  $f(z) \in S(p,n,\alpha)$ ,  $g(z) \in K(p,n,\beta)$   
 $\implies f * g(z) \in S(p,n,\alpha)$ .  
(ii)  $f(z) \in K(p,n,\alpha)$ ,  $g(z) \in K(p,n,\beta)$   
 $\implies f * g(z) \in K(p,n,\gamma)$ ,  $\gamma = \max(\alpha, \beta)$ .  
(iii)  $f(z) \in C(p,n,\alpha)$ ,  $g(z) \in K(p,n,\beta)$   
 $\implies f * g(z) \in C(p,n,\gamma)$ ,  $\gamma = \max(\alpha, \beta)$ .



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