ON CERTAIN P-VALENTLY STARLIKENESS CONDITIONS
FOR ANALYTIC FUNCTIONS

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1. Introduction.

Let $A(p)$ denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in the open unit disk $E = \{z: |z| < 1\}$. A function $f(z) \in A(p)$ is called $p$-valently starlike with respect to the origin if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad \text{in } E.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are $p$-valently starlike in $E$. Krzyz [1] showed by a counterexample that if $f(z) \in A(1)$, the condition $\text{Re} \ f'(z) > 0$ in $E$ does not imply $f(z) \in S(1)$. In [3], Mocanu proved the following theorem.

**Theorem.** If $f(z) \in A(1)$ and

$$|\arg f'(z)| < \alpha \frac{\pi}{2} = 0.968... \quad \text{in } E,$$

where $\alpha = 0.6165...$ is the unique root of the equation

$$2 \tan^{-1}(1 - \alpha) + \pi(1 - 2\alpha) = 0,$$

then $f(z) \in S(1)$.

**Definition.** Let $F(z)$ be analytic and univalent in $E$ and suppose that $F(E) = D$. If $f(z)$ is analytic in $E$, $f(0) = F(0)$, and $f(E) \subset D$, then we say that $f(z)$ is subordinate to $F(z)$ in $E$, and we write

$$f(z) \prec F(z).$$
2. Preliminaries.

We shall use the following lemma to prove our results.

Lemma 1. Let $\beta^* = 1.218...$ be the solution of $\pi \beta = 3\pi/2 - \tan^{-1} \beta$ and let $\alpha = \alpha(\beta) = \beta + (2/\pi) \tan^{-1} \beta$, for $0 < \beta < \beta^*$. If $p(z)$ is analytic in $\mathbb{E}$, with $p(0) = 1$, then

$$p(z) + zp'(z) < \left( \frac{1+z}{1-z} \right)^{\alpha} \quad \Rightarrow \quad p(z) < \left( \frac{1+z}{1-z} \right)^{\beta}.$$ 

We owe this lemma to [2, Theorem 5].

3. Main theorem.

Theorem 1. Let $p \geq 2$. If $f(z) \in A(p)$ and

\[(1) \quad |\arg f^{(p)}(z)| < \frac{\pi}{2} \alpha_1 = 1.249... \quad \text{in } \mathbb{E},\]

where $\alpha_1 = \frac{1}{2} + \frac{2}{\pi} \tan^{-1} \frac{1}{2} = 0.795...$, then $f(z) \in S(p)$ or $f(z)$ is $p$-valently star-like in $\mathbb{E}$.

Proof. If we put

$$p(z) = \frac{f^{(p-1)}(z)}{p! z},$$

then we have

$$p(z) + zp'(z) = \frac{1}{p!} f^{(p)}(z)$$

and $p(0) = 1$.

From the assumption (1), we have

$$|\arg(p(z) + zp'(z))| = |\arg f^{(p)}(z)| < \frac{\pi}{2} \alpha_1 \quad \text{in } \mathbb{E}.$$ 

Then, from Lemma 1, we have

\[(2) \quad |\arg \frac{f^{(p-1)}(z)}{p! z}| = |\arg \frac{f^{(p-1)}(z)}{z}| < \frac{\pi}{4} \quad \text{in } \mathbb{E}.\]

On the other hand, we easily have

\[(3) \quad \frac{f^{(p-2)}(z)}{z^2} = \frac{1}{z^2} \int_0^z f^{(p-1)}(t) \, dt = \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho \, d\rho\]

where $z = re^{i\theta}, \ 0 < r < 1, \ t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$. 
From (2), we easily have

$$|\arg \frac{f^{(p-1)}(z)}{z}| = |\arg \frac{f^{(p-1)}(z)}{z}| < \frac{\pi}{4} \quad \text{in E},$$

and the same is true of its integral mean of (3) (see e.g. [5, Lemma 1]). Therefore we have

$$|\arg \frac{f^{(p-2)}(z)}{z^2}| = |\arg \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho \, d\rho|$$

$$= |\arg \int_0^r \frac{f^{(p-1)}(t)}{t} \rho \, d\rho|$$

$$< \frac{\pi}{4} \quad \text{in E.}$$

From (2) and (4), we have

$$|\arg \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}| = |\arg \frac{f^{(p-1)}(z)}{f^{(p-2)}(z)}|$$

$$\leq |\arg \frac{f^{(p-1)}(z)}{z} + |\arg \frac{f^{(p-2)}(z)}{z^2}|$$

$$< \frac{\pi}{2} \quad \text{in E.}$$

This shows that

$$\text{Re} \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \quad \text{in E.}$$

Applying the same method as in the proof of [4, Theorem 5], we have

$$\text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in E.}$$

This shows that $f(z)$ is $p$-valently starlike in E.

Theorem 2. Let $p \geq 2$. If $f(z) \in A(p)$ and

$$\text{Re} f^{(p)}(z) > 0 \quad \text{in E},$$

then we have

$$|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} 2\alpha_2 \quad \text{in E},$$

where $\alpha_2 = 0.638...$ is the solution of the equation

$$1 = \beta + \frac{2}{\pi} \tan^{-1}\beta.$$
Proof. Applying the same method as in the proof of Theorem 1 and from the assumption (5), we have

$$|\arg \frac{f^{(p-1)}(z)}{z} < \frac{\pi}{2} \alpha_2 \quad \text{in E.}$$

(6)

Applying the same method as in the proof of Theorem 1 and from (6), we have

$$|\arg \frac{f^{(p-2)}(z)}{z^2} < \frac{\pi}{2} \alpha_2 \quad \text{in E.}$$

Repeating the same method as the above, we have

$$|\arg \frac{f'(z)}{z^{p-1}} < \frac{\pi}{2} \alpha_2 \quad \text{in E}$$

(7)

and

$$|\arg \frac{f(z)}{z^p} < \frac{\pi}{2} \alpha_2 \quad \text{in E.}$$

(8)

Then from (7) and (8), we have

$$|\arg \frac{zf'(z)}{f(z)}| \leq |\arg \frac{f'(z)}{z^{p-1}}| + |\arg \frac{f(z)}{z^p}|$$

$$< \frac{\pi}{2} 2\alpha_2 \quad \text{in E.}$$

This completes our proof.

References


