STARLIKENESS OF CERTAIN INTEGRAL

Mamoru Nunokawa (群馬大学教育・布川 譙）

1. Introduction.
Let A be the class of functions \( f(z) \) which are analytic in \( E=\{ z : |z|<1 \} \), with \( f(0)=f'(0)-1=0 \). A function \( f(z) \in A \) is said to be starlike iff

\[
\text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.
\]

We denote by \( S^* \) the subclass of \( A \) consisting of functions which are univalently starlike in \( E \).

R. Singh and S. Singh [3] have proved that if \( f(z) \in A \) and \( \text{Re} f'(z) > 0 \) in \( E \), then \( F(z) \in S^* \), where

\[
F(z) = \int_0^z \frac{f(t)}{t} \, dt.
\]

In this paper, we will improve the above result.

2. Preliminaries.

In this paper, we need the following lemmata.

**Lemma 1.** Let \( p(z) \) be analytic in \( E \), \( p(0)=1 \) and suppose that

\[
\text{Re} \left( p(z) + zp'(z) \right) > -\frac{\log(4/e)}{(2\log(e/2))} \quad \text{in } E,
\]

where \(-\log(4/e)/(2\log(e/2)) = 0.6294\ldots\).

Then we have

\[
\text{Re} p(z) > 0 \quad \text{in } E.
\]

We owe this lemma to [1].

**Lemma 2.** Let \( p(z) \) be analytic in \( E \), \( p(0)=1 \) and suppose that

\[
\text{Re} \left( p(z) + zp'(z) \right) > 0 \quad \text{in } E.
\]

Then we have

\[
\arg p(z) < \alpha^* - \frac{\pi}{2} \quad \text{in } E
\]

where

\[
1 = \alpha^* + \frac{2}{\pi} \tan^{-1} \alpha^*
\]

and

\[
0.6383 < \alpha^* < 0.6384.
\]
We owe this lemma to [2, Lemma 3].

**Lemma 3.** Let p(z) be analytic, p(0)=1 and suppose that
\[ \Re(p(z) +zp'(z)) > 0 \quad \text{in } E. \]

If g(z) is analytic in E, g(0)=1 and if
\[ \Re(p(z)[zg'(z)+g^2(z)+g(z)]) > \frac{\log(4/e)}{6} \left( \tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E, \]

then we have
\[ \Re(g(z)) > 0 \quad \text{in } E. \]

We owe this lemma to [2, Lemma 4].

3. Main theorem.

**Main Theorem.** Let f(z)∈ A and suppose that

\[ (1) \quad \Re(f'(z)) > \frac{\log(4/e)}{6} \left( \tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E, \]

where
\[ -0.03518 < \frac{1}{6} \log(4/e) \left( \tan^2 \alpha^* \frac{\pi}{2} - 3 \right) < -0.03502. \]

Then F(z)∈ S^*, where

\[ (2) \quad F(z) = \int_0^z \frac{f(t)}{t} \, dt. \]

**Proof.** From (2), we have

\[ (3) \quad f'(0)=1, \quad F'(z)=f(z)/z \quad \text{and} \quad F''(z)=(zf'(z)-f(z))/z^2. \]

Then we have

\[ (4) \quad \Re(zF''(z) + F'(z)) = \Re(f'(z)) > \frac{\log(4/e)}{6} \left( \tan^2 \alpha^* \frac{\pi}{2} - 3 \right) \quad \text{in } E. \]

From the assumption (1) and from Lemma 1, we have

\[ (5) \quad \Re(f'(z)) > 0 \quad \text{in } E. \]

Let us put
\[ p(z) = \frac{F(z)}{z} \]

and
\[ g(z) = \frac{zF'(z)}{F(z)}. \]
Since \( p(0) = 1 \) and
\[
\text{Re}(zp'(z) + p(z)) = \text{Re}F'(z) > 0 \quad \text{in } E,
\]
by LEMMA 2, we have
\[
\arg p(z) \leq \frac{\alpha}{2} \quad \text{in } E.
\]
On the other hand, by an easy calculation, and from (3) and (5), we have
\[
\begin{align*}
\text{Re} p(z) [zg'(z) + g^2(z) + g(z)] &= \text{Re}[zF'(z) + 2F'(z)] = \text{Re}[f'(z) + \frac{f(z)}{z}] \\
&> \text{Re}f'(z) \geq \frac{1}{6}(\tan^2 \alpha \frac{\pi}{2} - 3)(\log(4/e)) \quad \text{in } E.
\end{align*}
\]
Therefore, from LEMMA 3, we have
\[
\text{Reg}(z) > 0 \quad \text{in } E.
\]
This shows that
\[
\text{Re} \frac{zF'(z)}{F(z)} > 0 \quad \text{in } E.
\]
This completes our proof.

References

