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ON AN INVARIANCE PROPERTY OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

The object of the present paper is to link MacGregor's theorem to Pommerenke's one, together.

1. Introduction

MacGregor [2] extended the theorem which was obtained by Sakaguchi [4] and Libera [1] as the following:

Suppose that the function f(z) and h(z) are analytic in $E=\{z \mid |z|<1\}$, $f(0)=h(0)=0$ and h(z) maps E onto a region which is starlike with respect to the origin. If

$$\mathrm{Re}\,\frac{f'(z)}{h'(z)} > \beta \qquad \text{in } E,$$

then

$$\mathrm{Re}\,\frac{f(z)}{h(z)} > \beta \qquad \text{in } E.$$

On the other hand, Pommerenke [3] proved that if f(z) is analytic in E, h(z) is convex in E and

$$\left|\arg \frac{f'(z)}{h'(z)}\right| \leqq \frac{\alpha\pi}{2} \qquad \text{in } E$$

where $0 \leqq \alpha \leqq 1$, then we have

$$\left|\arg \frac{f(z'')-f(z')}{h(z'')-h(z')}\right| \leqq \frac{\alpha\pi}{2} \qquad \text{in } E$$

where $|z'|<1$ and $|z''|<1$.

2. Main theorem

MAIN THEOREM. Let f(z) be analytic in E, h(z) be convex in E and suppose that

(1) $$\left|\arg \left(\frac{f'(z)}{h'(z)} - \beta\right)\right| \leqq \frac{\alpha\pi}{2} \qquad \text{in } E$$

where $0 \leqq \alpha \leqq 1$.

Then we have

$$| \arg \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) | \leqq \frac{\alpha\pi}{2} \qquad \text{in } E$$

where $|z'| < 1$ and $|z''| < 1$.

PROOF. We will use the same method as in the proof of [3, Lemma 1].

Let $z=h^{-1}(w)$ be the inverse function $w=h(z)$. Then $h^{-1}(w)$ is analytic in the convex domain $h(E)=\{w \mid w=h(z), |z|<1\}$.

$$(2) \qquad \frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta = \frac{f(h^{-1}(w'')) - f(h^{-1}(w'))}{w'' - w'} - \beta$$

$$= \int_0^1 [\, g'(\, w' + (w'' - w')t\,) - \beta \,]dt$$

where $g(w)=f(h^{-1}(w))$, $w'=h(z')$ and $w''=h(z'')$. On the other hand, we have

$$(3) \qquad g'(w) = \frac{dg(w)}{dw} = \left(\frac{dg(w)}{dz} \right) / \left(\frac{dw}{dz} \right)$$

$$= \left(\frac{df(z)}{dz} \right) / \left(\frac{dh(z)}{dz} \right) = f'(z)/h'(z).$$

From the assumption (1) and from (3), we have

$$(4) \qquad | \arg (\, g'(\, w' + (w'' - w')t) - \beta \,) | \leqq \frac{\alpha\pi}{2} \qquad \text{in } E.$$

From the property of integral mean value of (2) and from (4), (see e.g. [3, Lemma 1]), we have

$$| \arg \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) | \leqq \frac{\alpha\pi}{2} \qquad \text{in } E.$$

This completes our proof.

Putting $\alpha = 1$ in the main theorem, we have

$$\mathrm{Re} \left(\frac{f'(z)}{h'(z)} - \beta \right) > 0 \text{ in } E \quad \Longrightarrow \quad \mathrm{Re} \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) > 0$$

in E.

This is an extended MacGregor's theorem.

REFERENCES

[1] R.J.Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16(1965), 755-758.

[2] T.H. MacGregor, A subordination for convex functions of order α, J. London Math. Soc. (2), 9(1975), 530-536.

[3] Ch.Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc. 114(1965), 176-186.

[4] K.Sakaguchi, On a ceratin univalent mapping, J. Math. Soc. Japan. 11(1959), 72-75.