

# A NOTE ON CERTAIN SUBORDINATIONS

Seiichi FUKUI (和歌山大教育・福井誠一)

## 1. INTRODUCTION

Mocanu([1], Lemma 3) showed the following Lemma 1 in 1986:

LEMMA 1. *If  $p(z)$  is an analytic function in  $|z| < 1$ , with  $p(0) = 1$ , and if  $\operatorname{Re}\{p(z) + zp'(z)\} > 0$  in  $|z| < 1$ , then*

$$(1) \quad \left| \arg p(z) \right| < \frac{\pi}{3}$$

in  $|z| < 1$ .

Here we recall the definition of subordination. Let  $p(z)$  and  $q(z)$  be analytic functions in  $|z| < 1$  and  $p(0) = q(0)$ .  $p(z)$  is said to be subordinate to  $q(z)$  (written  $p(z) \prec q(z)$ ) if  $p(z) = q(w(z))$ ,  $|z| < 1$  for some analytic function  $w(z)$  with  $|w(z)| \leq |z|$  (See Duren [2], p-190).

Since a function  $q(z) = \frac{1-z}{1+z}$  satisfies  $q(0) = 1$  and  $\left| \arg q(z) \right| < \frac{\pi}{2}$ , the function  $p(z)$  in Lemma 1 is represented by the subordination,

$$(2) \quad p(z) \prec \left( \frac{1-z}{1+z} \right)^{2/3}$$

In 1987 Miller and Mocanu ([3], Theorem 5) proved the following Lemma 2 by considering a differential subordinate system.

LEMMA 2. Let  $\beta_0 = 1.218\cdots$  be the solution of

$$(3) \quad \beta \pi \frac{3\pi}{2} - \tan^{-1} \beta$$

and let

$$(4) \quad \alpha = \alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \beta$$

for  $0 < \beta \leq \beta_0$ . If  $p(z)$  is an analytic function in  $|z| < 1$ , with  $p(0) = 1$ , then

$$(5) \quad p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \implies p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta.$$

When  $\alpha = 1$  in the equality (4), we have

$$(6) \quad 1 = \beta + \frac{2}{\pi} \tan^{-1} \beta \quad (0 < \beta \leq \beta_0),$$

$\beta^* = 0.638\cdots$ , and  $\beta^*$  is the solution of (6). This shows that

$$(7) \quad \operatorname{Re}\{p(z) + zp'(z)\} > 0, \quad |z| < 1 \implies p(z) \prec \left(\frac{1+z}{1-z}\right)^\beta$$

and  $\beta^* < \frac{2}{3}$ . Hence Lemma 1 can be improved by such as the following

LEMMA 3. If  $p(z)$  is an analytic function in  $|z| < 1$ , with  $p(0) = 1$  and if  $\operatorname{Re}\{p(z) + zp'(z)\} > 0$ ,  $|z| < 1$ , then we have

$$(7) \quad \left| \arg p(z) \right| < \frac{\pi}{2} \beta^*,$$

$|z| < 1$ , where  $\beta^* = 0.638\cdots$ , and  $\beta^*$  is the solution of (6).

REMARK. It seems that an extremal function in Lemma 3 is

$$(8) \quad p(z) = \frac{2}{z} \log(1+z) - 1.$$

This function  $p(z) = \frac{2}{z} \log(1+z) - 1$  satisfies

$$(i) \quad \operatorname{Re}\{p(z)\} > 2 \log 2 - 1.$$

$$(ii) \quad p(z) \prec \left(\frac{1-z}{1+z}\right)^{\beta_1},$$

where  $\beta_1 = 0.503 \dots$ . But we can not prove these facts at present.

## 2. THEOREMS AND PROOFS

From Lemma 3 we can show the following

**THEOREM 1.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be analytic in  $|z| < 1$  and

$\operatorname{Re}\{f'(z)\} > 0$  in  $|z| < 1$ , then

$$(9) \quad \left| \arg \frac{f(z)}{z} \right| = \left| \arg \int_0^r f'(\rho e^{i\theta}) d\rho \right| < \frac{\pi}{2} \beta^*,$$

where  $z = re^{i\theta}$ ,  $0 < r < 1$ , and  $\beta^*$  is given in Lemma 3.

*Proof.* We put  $p(z) = \frac{f(z)}{z}$  in Lemma 3. then we have  $p(0) = 1$ , and

$$(10) \quad \operatorname{Re}\{p(z) + z p'(z)\} = \operatorname{Re}\{f'(z)\} > 0 \quad \text{in } |z| < 1.$$

Therefore, we obtain the following relations

$$(11) \quad \begin{aligned} \left| \arg p(z) \right| &= \left| \arg \frac{f(z)}{z} \right| \\ &= \left| \arg \frac{1}{z} \int_0^z f'(t) dt \right| = \left| \arg \frac{1}{re^{i\theta}} \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho \right| \\ &= \left| \arg \frac{1}{r} \int_0^r f'(\rho e^{i\theta}) d\rho \right| \end{aligned}$$

$$= \left| \arg \int_0^r f'(\rho e^{i\theta}) d\rho \right| < \frac{\pi}{2} \beta^*,$$

where  $z = r e^{i\theta}$ ,  $0 < r < 1$ ,  $t = \rho e^{i\theta}$  and  $0 \leq \rho \leq r$ .

First the function  $f(z)$  in Theorem 1 are to be said close-to-convex functions in  $|z| < 1$  and next Theorem 2 is a result for convex functions in  $|z| < 1$ .

**THEOREM 2.** Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  be analytic in  $|z| < 1$  and

$\operatorname{Re}\left\{1 + \frac{zf'(z)}{f'(z)}\right\} > 0$  in  $|z| < 1$  (i.e.,  $f(z)$  are a univalent and convex

functions) then we have

$$(12) \quad \left| \arg \int_0^r \left\{1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\right\} d\rho \right| < \frac{\pi}{2} \beta^*,$$

where  $z = r e^{i\theta}$ ,  $0 < r < 1$ , and  $\beta^*$  is given in Lemma 3.

**Proof.** We put  $q(z) = \frac{zf'(z)}{f'(z)}$  and

$$(13) \quad p(z) = \frac{1}{z} \int_0^z \left\{q(t) + \frac{t q'(t)}{q(t)}\right\} dt,$$

then we have

$$(14) \quad p(z) + z p'(z) = q(z) + \frac{z q'(z)}{q(z)} = 1 + \frac{zf'(z)}{f'(z)},$$

$p(0) = 1$ ,  $q(0) = 1$  and  $p(z)$ ,  $q(z)$  are analytic in  $|z| < 1$ .

Here we again use a method similar to that in Theorem 1. Since

$$(15) \quad \operatorname{Re}\{p(z) + z p'(z)\} = \operatorname{Re}\left\{1 + \frac{zf'(z)}{f'(z)}\right\} > 0, \quad |z| < 1$$

by the hypotheses, we have the following relations

$$(16) \quad \left| \arg p(z) \right| = \left| \arg \frac{1}{z} \int_0^z \left\{q(t) + \frac{t q'(t)}{q(t)}\right\} dt \right|$$

$$\begin{aligned}
&= \left| \arg \frac{1}{z} \int_0^z \left( 1 + \frac{tf'(t)}{f'(t)} \right) dt \right| \\
&= \left| \arg \frac{1}{r} \int_0^r \left( 1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \right) d\rho \right| \\
&= \left| \arg \int_0^r \left( 1 + \frac{\rho e^{i\theta} f'(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \right) d\rho \right| < \frac{\pi}{2} \beta^*,
\end{aligned}$$

where  $z = re^{i\theta}$ ,  $0 < r < 1$ ,  $t = \rho e^{i\theta}$ ,  $0 \leq \rho \leq r$ , and  $\beta^*$  is given in Lemma 3. This completes the proof.

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Department of Mathematics,  
Faculty of Education,  
Wakayama University,  
930 Sakaedani Wakayama-City  
Wakayama 640, Japan