

NOTES ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS  
DEFINED BY RUSCHEWEYH DERIVATIVES

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1. INTRODUCTION

Let  $A_p$  denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk  $U = \{z: |z| < 1\}$ . For functions

$$(1.2) \quad f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \quad (j = 1, 2)$$

belonging to  $\Lambda$ , we denote by  $f_1 * f_2(z)$  the convolution of  $f_1(z)$  and  $f_2(z)$ , that is,

$$(1.3) \quad f_1 * f_2(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n.$$

Using the convolution, we define the Ruscheweyh derivative  $D^{\alpha+p-1}f(z)$  by

$$(1.4) \quad D^{\alpha+p-1}f(z) = \frac{z^p}{(1-z)^{\alpha+p}} * f(z) \quad (\alpha > -p)$$

for  $f(z) \in A_p$ . A function  $f(z)$  belonging to  $A_p$  is said to be in the class  $R(\alpha+p-1)$  if it satisfies

$$(1.5) \quad \operatorname{Re} \left\{ \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} \right\} > \frac{\alpha+p-1}{\alpha+p} \quad (z \in U),$$

where  $\alpha > -p$ .

For an integer  $\alpha$  greater than  $-p$ , Soni [4] proved

**THEOREM A.**  $R(\alpha+p) \subset R(\alpha+p-1)$ .

for  $\alpha = n \in N_0 = \{0, 1, 2, \dots\}$  and  $p = 1$ , Singh and Singh [3] proved

**THEOREM B.**  $R(n+1) \subset R(n)$ .

In the present paper, we improve the above results.

## 2. PROPERTIES OF THE CLASS $R(\alpha+p-1)$

We begin with the statement of the following lemma due to Miller and

Mocanu [2].

**LEMMA 1.** Let  $\varphi(u, v)$  be a complex valued function,

$$\varphi: D \rightarrow \mathbb{C}, D \subset \mathbb{C}^2 \quad (\mathbb{C} \text{ is the complex plane}),$$

and let  $u = u_1 + iu_2$ ,  $v = v_1 + iv_2$ . Suppose that the function  $\varphi(u, v)$  satisfies

- (i)  $\varphi(u, v)$  is continuous in  $D$ ;
- (ii)  $(1, 0) \in D$  and  $\operatorname{Re}\{\varphi(1, 0)\} > 0$ ;
- (iii) for all  $(iu_2, v_1) \in D$  such that  $v_1 \leq -\frac{1+u_2^2}{2}$ ,  
 $\operatorname{Re}\{\varphi(iu_2, v_1)\} \leq 0$ .

Let  $q(z) = 1 + q_1z + q_2z^2 + \dots$  be regular in  $U$  such that  $(q(z), zq'(z)) \in D$  for all  $z \in U$ . If

$$\operatorname{Re}\{\varphi(q(z), zq'(z))\} > 0 \quad (z \in U),$$

then  $\operatorname{Re}\{q(z)\} > 0 \quad (z \in U)$ .

Applying the above lemma, we derive

**THEOREM 1.** If  $f(z) \in R(\alpha+p)$ ,  $\alpha \geq 1-p$ ,  $p \in \mathbb{N}$ , then

$$(2.1) \quad \operatorname{Re} \left\{ \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} \right\} > \beta(\alpha, p) \quad (z \in U),$$

where

$$(2.2) \quad \beta(\alpha, p) = \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)}.$$

**PROOF.** We define the function  $q(z)$  by

$$(2.3) \quad \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} = \beta + (1-\beta)q(z),$$

where  $\beta = \beta(\alpha, p)$ . Noting that

$$(2.4) \quad z(D^{\alpha+p-1} f(z))' = (\alpha+p)D^{\alpha+p} f(z) - \alpha(D^{\alpha+p-1} f(z)) \quad (\alpha > -p),$$

we have, from (2.3), that

$$(2.5) \quad \frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} = \frac{1}{\alpha+p-1} \left\{ 1 + (\alpha+p)(\beta+(1-\beta)q(z)) + \frac{(1-\beta)zq'(z)}{\beta+(1-\beta)q(z)} \right\},$$

or

$$(2.6) \quad \operatorname{Re} \left\{ \frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} - \frac{\alpha+p}{\alpha+p+1} \right\}$$

$$= \operatorname{Re} \frac{1}{\alpha+p+1} \left\{ (1-\alpha-p) + (\alpha+p)(\beta+(1-\beta)q(z)) + \frac{(1-\beta)zq'(z)}{\beta+(1-\beta)q(z)} \right\}$$

> 0.

Letting

$$(2.7) \quad \varphi(u, v) = \frac{1}{\alpha+p+1} \left\{ (1-\alpha-p) + (\alpha+p)(\beta+(1-\beta)u) + \frac{(1-\beta)v}{\beta+(1-\beta)u} \right\},$$

we see that

(i)  $\varphi(u, v)$  is continuous in  $D = (C - \{\frac{\beta}{\beta-1}\}) \times C$ ;

(ii)  $(1, 0) \in D$  and  $\operatorname{Re}\{\varphi(1, 0)\} = \frac{1}{\alpha+p+1} > 0$ ;

(iii) for all  $(iu_2, v_1) \in D$  such that  $v_1 \leq -\frac{1+u_2^2}{2}$ ,

$$\operatorname{Re}\{\varphi(iu_2, v_1)\} = \frac{1}{\alpha+p+1} \left\{ (1-\alpha-p) + \beta(\alpha+p) + \frac{\beta(1-\beta)v_1}{\beta^2+(1-\beta)^2u_2^2} \right\}$$

$$\leq \frac{1}{\alpha+p+1} \left\{ (1-\alpha-p) + \beta(\alpha+p) - \frac{\beta(1-\beta)(1+u_2^2)}{2(\beta^2+(1-\beta)^2u_2^2)} \right\}.$$

$$= \frac{1}{\alpha+p+1} \left\{ 2(\alpha+p)\beta^2 - [2(\alpha+p)-3]\beta-1 \right\} (\beta+(\beta-1)u_2^2)$$

$$+ (1-\beta)[2(\alpha+p)(\beta-1)+1]u_2^2\}$$

$$= \frac{(1-\beta)[2(\alpha+p)(\beta-1)+1]u_2^2}{\alpha+p+1}$$

$$\leq 0$$

because  $2(\alpha+p)\beta^2 - [2(\alpha+p)-3]\beta-1 = 0$ , and under the condition  $\alpha+p \geq 1$ , we have

$$0 < \beta \leq 1 - \frac{1}{2(\alpha+p)} < 1.$$

Therefore, the function  $\varphi(u, v)$  satisfies the conditions in Lemma 1. Since  $\operatorname{Re}\{q(z)\} > 0$  ( $z \in U$ ), we conclude that

$$(2.8) \quad \operatorname{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \beta = \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)}$$

REMARK 1. Since

$$(2.9) \quad \operatorname{Re}\left\{\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right\} > \frac{2(\alpha+p)-3+\sqrt{4(\alpha+p)^2-4(\alpha+p)+9}}{4(\alpha+p)} > \frac{\alpha+p-1}{\alpha+p}$$

for  $\alpha \geq 1-p$ , Theorem 1 is the improvement of Theorem A due to Soni [4].

REMARK 2. For  $\alpha = n \in N_0$  and  $p = 1$ , we have

$$(2.10) \quad \operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \frac{2n-1+\sqrt{4n^2+4n+9}}{4(n+1)} > \frac{n}{n+1}.$$

Therefore, Theorem 1 is also the improvement of Theorem B by Singh and Singh [3].

Taking  $\alpha = 2-p$  in Theorem 1, we have

**COROLLARY 1.** If  $f(z) \in R(2)$ , then

$$(2.11) \quad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{\sqrt{17} - 3}{4} \quad (z \in U),$$

that is,  $f(z)$  is  $p$ -valently convex of order  $\frac{\sqrt{17} - 3}{4}$ .

**PROOF.** For  $\alpha = 2-p$ , we have

$$(2.12) \quad \operatorname{Re}\left\{\frac{D^2f(z)}{Df(z)}\right\} = \operatorname{Re}\left\{1 + \frac{1}{2} \frac{zf''(z)}{f'(z)}\right\} > \frac{1 + \sqrt{17}}{8}.$$

Therefore, (2.12) leads to

$$(2.13) \quad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1 + \sqrt{17}}{4} - 1 = \frac{\sqrt{17} - 3}{4}.$$

### 3. AN APPLICATION OF JACK'S LEMMA

In order to prove our next result, we need the following lemma by Jack [1].

LEMMA 2. Let  $w(z)$  be regular in  $U$  with  $w(0) = 0$ . If  $|w(z)|$  attains its maximum value on the circle  $|z| = r$  at a point  $z_0 \in U$ , then

$$z_0 w'(z_0) = kw(z_0),$$

where  $k$  is real and  $k \geq 1$ .

THEOREM 2. If  $f(z) \in A_p$  satisfies

$$(3.1) \quad \operatorname{Re} \left\{ \frac{D^{\alpha+p+1} f(z)}{D^{\alpha+p} f(z)} \right\} > \frac{1 + (\alpha+p-1)^2}{\alpha+p+1} \quad (z \in U),$$

and  $\alpha + p \geq 1$  then

$$(3.2) \quad \operatorname{Re} \left[ \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} \right]^{\frac{1}{2}} > \frac{\alpha+p-1}{\alpha+p} \quad (z \in U).$$

PROOF. For  $f(z) \in A_p$  satisfying (3.1), we define the function  $w(z)$  by

$$(3.3) \quad \operatorname{Re} \left[ \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} \right]^{\frac{1}{2}} = \frac{\alpha+p-1}{\alpha+p} + \frac{1}{\alpha+p} \frac{1-w(z)}{1+w(z)} = \frac{(\alpha+p) + (\alpha+p-2)w(z)}{(\alpha+p)(1+w(z))}.$$

Then  $w(z)$  is regular in  $U$  with  $w(0) = 0$  and  $w(z) \neq -1$  ( $z \in U$ ).

After taking the logarithmic derivative of both sides of (3.3), using (2.4)



and (3.3) we obtain

$$(3.4) \quad \frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} = \frac{1}{\alpha+p+1} \left\{ 1 + \frac{(\alpha+p) + (\alpha+p-2)w(z)}{1+w(z)} \right\}^2 - \frac{4zw'(z)}{(1+w(z))((\alpha+p) + (\alpha+p-2)w(z))}.$$

If there exists a point  $z_0 \in U$  such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 2, we have

$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1).$$

It follows from

$$\operatorname{Re} \left\{ \frac{1}{1+w(z_0)} \right\} = \frac{1}{2}$$

and

$$\operatorname{Re} \left\{ \frac{w(z_0)}{1+w(z_0)} \right\} = \frac{1}{2}$$

that

$$(3.5) \quad \left\{ \frac{D^{\alpha+p+1} f(z_0)}{D^{\alpha+p} f(z_0)} \right\}$$

$$= \operatorname{Re} \frac{1}{\alpha+p+1} \left\{ 1 + \frac{(\alpha+p) + (\alpha+p-2)w(z_0)}{1+w(z_0)} \right\}^2 - \frac{4kw'(z_0)}{(1+w(z_0))((\alpha+p) + (\alpha+p-2)w(z_0))}.$$

Let  $w(z_0) = e^{i\theta}$ , we have

$$(1 + w(z_0))^2 = (1 + e^{i\theta})^2 = (e^{-\frac{i\theta}{2}} + e^{\frac{i\theta}{2}})^2 e^{i\theta}$$

$$= (2 \cos \frac{\theta}{2})^2 e^{i\theta} = 2(1 + \cos \theta)w(z_0).$$

It follows that

$$(3.6) \quad \operatorname{Re} \left[ \frac{(\alpha+p) + (\alpha+p-2)w(z_0)}{1+w(z_0)} \right]^2$$

$$= \operatorname{Re} \left\{ \frac{(\alpha+p)^2}{2(1+\cos \theta)e^{i\theta}} \right\} + \operatorname{Re} \left\{ \frac{2(\alpha+p)(\alpha+p-2)}{2(1+\cos \theta)} \right\} + \operatorname{Re} \left\{ \frac{(\alpha+p-2)^2 e^{i\theta}}{2(1+\cos \theta)} \right\}$$

$$= \frac{\{(\alpha+p)^2 + (\alpha+p-2)^2\} \cos \theta + 2(\alpha+p)(\alpha+p-2)}{2(1+\cos \theta)}$$

$$\leq \frac{(\alpha+p)^2 + (\alpha+p-2)^2 + 2(\alpha+p)(\alpha+p-2)}{4} = (\alpha+p-1)^2,$$

since  $(\alpha+p)^2 + (\alpha+p-2)^2 \geq 2(\alpha+p)(\alpha+p-2)$  and  $\frac{Ax+B}{1+x}$  is an increasing function of  $x$  if  $A \geq B$ .

Next we have

$$\begin{aligned}
 (3.7) \quad & \operatorname{Re} \left\{ \frac{w(z_0)}{(1+w(z_0))((\alpha+p) + (\alpha+p-2)w(z_0))} \right\} \\
 &= \frac{1}{|(\alpha+p) + (\alpha+p-2)w(z_0)|^2} \operatorname{Re} \left\{ \frac{w(z_0)[(\alpha+p) + (\alpha+p-2)\overline{w(z_0)}]}{1 + w(z_0)} \right\} \\
 &= \frac{1}{|(\alpha+p) + (\alpha+p-2)w(z_0)|^2} \left\{ (\alpha+p)\frac{1}{2} + (\alpha+p-2)\frac{1}{2} \right\} \geq 0.
 \end{aligned}$$

From (3.5), (3.6) and (3.7) we obtain

$$\operatorname{Re} \left\{ \frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} \right\} \leq \frac{1 + (\alpha+p-2)^2}{\alpha+p+1}.$$

This contradicts our condition (3.1). Therefore, we conclude that  $w(z)$  satisfies  $|w(z)| < 1$  for all  $z \in U$ , which completes the proof of our theorem.

Letting  $\alpha = 2-p$ , Theorem 2 gives

**COROLLARY.** If  $f(z) \in A_p$  satisfies

$$(3.8) \quad \operatorname{Re} \left\{ \frac{D^3f(z)}{D^2f(z)} \right\} > \frac{2}{3} \quad (z \in U),$$

then

$$(3.9) \quad \operatorname{Re} \sqrt{2 + \frac{z f'''(z)}{f'(z)}} > \frac{\sqrt{2}}{2} \quad (z \in U).$$

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