<table>
<thead>
<tr>
<th>Title</th>
<th>The Fekete-Szego problem for strongly close-to-convex functions (Topics in Univalent Functions and Its Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>THOMAS, D.K.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1990-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/101741">http://hdl.handle.net/2433/101741</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
The Fekete-Szegő problem for strongly close-to-convex functions.

D.K. Thomas (ウェールズ大学)

INTRODUCTION

Denote by $S$ the class of normalized analytic univalent functions $f$ defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A classical theorem of Fekete and Szegő [2] states that for $f \in S$ given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu, & \text{if } \mu \leq 0 \\
1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\
4 - 3\mu, & \text{if } \mu \geq 1.
\end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $S$ such that equality holds. Recently Pfluger [8] has considered the problem when $\mu$ is complex. In the case of $C$, $S^*$ and $K$, the subclasses of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [5,6]. In particular for $f \in K$ and given by (1), Keogh and Merkes [5] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu, & \text{if } \mu \leq 1/3, \\
1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3, \\
1, & \text{if } 2/3 \leq \mu \leq 1, \\
4\mu - 3, & \text{if } \mu \geq 1.
\end{cases}$$

Again, for each $\mu$, there is a function in $K$ such that equality holds. In this paper we extend this result to the class $K(\beta)$ of strongly close-to-convex functions of order $\beta$ in the sense of Pommerenke [9]. Thus

Typeset by AMS-TEX
$f \in K(\beta)$ if, and only if, $f$, given by (1), is analytic in $D$ and is such that there exists $g \in S^*$ satisfying

$$\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi\beta}{2},$$

for $z \in D$ and $\beta \geq 0$. Clearly $K(0) = C$, $K(1) = K$ and when $0 \leq \beta \leq 1$, $K(\beta)$ is a subset of $K$ and hence contains only univalent functions. However in [4], Goodman showed that $K(\beta)$ can contain functions with unbounded valence for $\beta > 1$.

Recently, Koepf [7] has considered the Fekete-Szegö problem for $K(\beta)$ and obtained sharp results for some particular values of $\mu$, all of which, with the exception of the case $\mu = 1$ and $\beta \geq 1$, are contained in the following result.

**Results**

**Theorem.** Let $f \in K(\beta)$ and be given by (1), then for $0 \leq \beta \leq 1$,

$$|a_3 - \mu a_2^2| \leq 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3},$$

if $\mu \leq \frac{2\beta}{3(\beta + 1)}$,

$$\leq 1 - \mu + \frac{2\beta}{3} + \frac{\beta(2 - 3\mu)^2}{3[2 - \beta(2 - 3\mu)]},$$

if $\frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3}$,

$$\leq \frac{2\beta + 1}{3},$$

if $\frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)}$,

$$\leq \mu - 1 + \frac{\beta(3\mu - 2)(\beta + 2)}{3},$$

if $\mu \geq \frac{2(\beta + 2)}{3(\beta + 1)}$,

and for $\beta > 1$, the first two inequalities hold. For each $\mu$, there is a function in $K(\beta)$ such that equality holds.

We shall require the following:

**Lemma 1** ([10], p. 166). Let $h \in P$, i.e., let $h$ be analytic in $D$ and satisfy $\Re h(z) > 0$ for $z \in D$, with $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$
Lemma 2 ([6], Lemma 3). Let $g \in S^*$ with $g(z) = z + b_2z^2 + b_3z^3 + \ldots$, then for $\mu$ real,

$$|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.$$ 

We note that Lemma 2 above can easily be extended to the wider class $S^*(\alpha)$ of strongly starlike functions of order $\alpha \geq 0$, i.e., $g$ analytic and normalized in $D$ and satisfying

$$\left| \arg \frac{zg'(z)}{g(z)} \right| \leq \frac{\alpha \pi}{2},$$

see e.g. [1]. In this case, one obtains the sharp inequality

$$|b_3 - \mu b_2^2| \leq \max\{\alpha, \alpha^2|3 - 4\mu|\},$$

for $\mu$ real.

Proof of Theorem: It follows from (2) that we can write

$$zf'(z) = g(z)h(z)^\beta$$

for $g \in S^*$ and $h \in P$. Equating coefficients in (3) we obtain

$$2a_2 = \beta c_1 + b_2$$

and

$$3a_3 = \frac{\beta(\beta - 1)}{2} c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,$$

so that

$$a_3 - \mu a_2^2 = \frac{1}{3} \left( b_3 - \frac{3}{4} \mu b_2^2 \right) + \frac{\beta}{3} \left( c_2 + \left( \frac{\beta(2 - 3\mu)}{4} - \frac{1}{2} \right) c_1^2 \right)$$

$$+ \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) c_1 b_2.$$  

(4)

We consider first the case $\frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3}$. Equation (4) gives
\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} \left| b_3 - \frac{3}{4} \mu b_2^2 \right| + \frac{\beta}{3} \left| c_2 - \frac{1}{2} c_1^2 \right| + \frac{\beta^2 (2 - 3 \mu)}{12} |c_1^2|
\]
\[
+ \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) |c_1||b_2|,
\]
\[
\leq 1 - \mu + \frac{\beta}{3} \left( 2 - \frac{1}{2} |c_1^2| \right) + \frac{\beta^2 (2 - 3 \mu)}{12} |c_1^2|
\]
\[
+ \frac{\beta (2 - 3 \mu)}{3} |c_1|,
\]
\[
= \Phi(x) \text{ say, with } x = |c_1|,
\]

where we have used Lemmas 1 and 2 and the fact that \(|b_2| \leq 2\) for \(g \in S^*\). An elementary argument shows that the function \(\Phi\) attains a maximum at \(x_0 = 2(2 - 3 \mu)/(2 - \beta(2 - 3 \mu))\), and so
\[
|a_3 - \mu a_2^2| \leq \Phi(x_0),
\]

which proves the Theorem if \(\mu \leq 2/3\) and \(\beta \geq 0\). Choosing
\[
c_1 = \frac{2(2 - 3 \mu)}{2 - \beta(2 - 3 \mu)},\ c_2 = 2,\ b_2 = 2 \text{ and } b_3 = 3,
\]
in (4) shows that the result is sharp. We note that \(|c_1| \leq 2\), i.e., \(\mu \geq 2\beta/(3(\beta + 1))\).

Next consider the case \(\mu \leq \frac{2\beta}{3(\beta + 1)}\). Since \(K(0) = C\), we may assume that \(\beta > 0\). Again (4) gives
\[
|a_3 - \mu a_2^2| \leq \frac{3\mu(\beta + 1)}{2\beta} \left| a_3 - \frac{2\beta}{3(\beta + 1)} a_2^2 \right| + \left( 1 - \frac{3\mu(\beta + 1)}{2\beta} \right) |a_3|,
\]
\[
\leq \frac{3\mu(\beta + 1)}{2\beta} \left( 1 + \frac{2\beta}{3} \right) + \left( 1 - \frac{3\mu(\beta + 1)}{2\beta} \right) \left( \frac{2\beta(\beta + 2)}{3} + 1 \right),
\]
\[
= 1 - \mu + \frac{\beta(2 - 3 \mu)(\beta + 2)}{3},
\]
for $\beta > 0$, where we have used the result already proved in the case 
$\mu = 2\beta/3(\beta + 1)$, and the fact that for $f \in K(\beta)$, the inequality $|a_3| \leq 1 + 2\beta(\beta + 2)/3$ holds [3]. Equality is attained on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$.

Suppose now that $\frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)}$. Since $g \in S^*$ we can write $zg'(z) = g(z)p(z)$ for $p \in P$, with $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, and so equating coefficients we have that $b_2 = p_1$ and $2b_3 = p_1^2 + p_2$.

We deal first with the case $\mu = 2(\beta + 2)/(3(\beta + 1))$. Thus (4) gives

$$a_3 - \frac{2(\beta + 2)}{3(\beta + 1)}a_2^2 = \frac{1}{6} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{\beta}{3} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{\beta - 1}{6(\beta + 1)}p_1^2 - \frac{\beta^2 c_1^2}{6(\beta + 1)} - \frac{\beta p_1 c_1}{3(\beta + 1)},$$

and so if $\beta \leq 1$,

$$\left| a_3 - \frac{2(\beta + 2)}{3(\beta + 1)}a_2^2 \right| \leq \frac{1}{6} \left| p_2 - \frac{p_1^2}{2} \right| + \frac{\beta}{3} \left| c_2 - \frac{c_1^2}{2} \right| + \frac{1 - \beta}{12(\beta + 1)}|p_1^2| + \frac{\beta^2 |c_1^2|}{6(\beta + 1)} + \frac{\beta |p_1 c_1|}{3(\beta + 1)},$$

$$\leq \frac{1}{6} \left( 2 - \frac{|p_1^2|}{2} \right) + \frac{\beta}{3} \left( 2 - \frac{|c_1^2|}{2} \right) + \frac{1 - \beta}{12(\beta + 1)}|p_1^2| + \frac{\beta^2 |c_1^2|}{6(\beta + 1)} + \frac{\beta |p_1 c_1|}{3(\beta + 1)},$$

$$= \frac{2\beta + 1}{3} - \frac{\beta}{6(\beta + 1)}(|c_1| - |p_1|)^2,$$

$$\leq \frac{2\beta + 1}{3},$$

where we have used Lemma 1.

Now write

$$a_3 - \mu a_2^2 = \frac{(\beta + 1)(3\mu - 2)}{2} \left( a_3 - \frac{2(\beta + 2)}{3(\beta + 1)}a_2^2 \right) + \frac{3(\beta + 1)}{2} \left( \frac{2(\beta + 2)}{3(\beta + 1) - \mu} \right) \left( a_3 - \frac{2}{3}a_2^2 \right),$$

5
and the result follows at once on using the Theorem already proved in the cases $\mu = 2/3$ and $\mu = 2(\beta + 2)/(3(\beta + 1))$ for $\beta \leq 1$. Equality is attained when $f$ is given by

$$f'(z) = \frac{(1 + z^2)^{\beta}}{(1 - z^2)^{\beta+1}}.$$ 

We finally assume that $\mu \geq \frac{2(\beta+2)}{3(\beta+1)}$. Write

$$a_3 - \mu a_2^2 = \left(a_3 - \frac{2(\beta+2)}{3(\beta+1)} a_2^2 \right) + \left(\frac{2(\beta+2)}{3(\beta+1)} - \mu \right) a_2^2,$$

and the result follows at once on using the Theorem already proved for $\mu = 2(\beta+2)/3(\beta+1)$ in the case $\beta \leq 1$ and the inequality $|a_2| \leq \beta+1$, which was proved in [3]. Equality is attained in this last case on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$ in (4).

We remark that the methods used in [5] and [6], together with equation (4), suggest that in order to obtain sharp results for $\beta > 1$ and $\mu > 2/3$, an extension to the "area principle" may be required. Since $K(\beta)$ contains functions of unbounded valence for $\beta > 1$ establishing sharp estimates in this case may require deeper methods.

References


1980 *Mathematics subject classifications*: Primary 30C45

Department of Mathematics and Computer Science, University College Swansea, Swansea SA2 8PP, Wales.