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<th>Title</th>
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</thead>
<tbody>
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The Fekete-Szegö problem for strongly close-to-convex functions.

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INTRODUCTION

Denote by $S$ the class of normalized analytic univalent functions $f$ defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A classical theorem of Fekete and Szegö [2] states that for $f \in S$ given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0 \\ 1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\ 4 - 3\mu, & \text{if } \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $S$ such that equality holds. Recently Pfluger [8] has considered the problem when $\mu$ is complex. In the case of $C, S^*$ and $K$, the subclasses of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [5,6]. In particular for $f \in K$ and given by (1), Keogh and Merkes [5] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 1/3 \\ 1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3 \\ 1, & \text{if } 2/3 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Again, for each $\mu$, there is a function in $K$ such that equality holds. In this paper we extend this result to the class $K(\beta)$ of strongly close-to-convex functions of order $\beta$ in the sense of Pommerenke [9]. Thus
\( f \in K(\beta) \) if, and only if, \( f \), given by (1), is analytic in \( D \) and is such that there exists \( g \in S^* \) satisfying

\[
\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi\beta}{2},
\]

for \( z \in D \) and \( \beta \geq 0 \). Clearly \( K(0) = C \), \( K(1) = K \) and when \( 0 \leq \beta \leq 1 \), \( K(\beta) \) is a subset of \( K \) and hence contains only univalent functions. However in [4], Goodman showed that \( K(\beta) \) can contain functions with unbounded valence for \( \beta > 1 \).

Recently, Koepf [7] has considered the Fekete-Szegö problem for \( K(\beta) \) and obtained sharp results for some particular values of \( \mu \), all of which, with the exception of the case \( \mu = 1 \) and \( \beta \geq 1 \), are contained in the following result.

**RESULTS**

**THEOREM.** Let \( f \in K(\beta) \) and be given by (1), then for \( 0 \leq \beta \leq 1 \),

\[
|a_3 - \mu a_2^2| \leq 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3}, \quad \text{if } \mu \leq \frac{2\beta}{3(\beta + 1)},
\]

\[
\leq 1 - \mu + \frac{2\beta}{3} + \frac{\beta(2 - 3\mu)^2}{3[2 - \beta(2 - 3\mu)]}, \quad \text{if } \frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3},
\]

\[
\leq \frac{2\beta + 1}{3}, \quad \text{if } \frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)},
\]

\[
\leq \mu - 1 + \frac{\beta(3\mu - 2)(\beta + 2)}{3}, \quad \text{if } \mu \geq \frac{2(\beta + 2)}{3(\beta + 1)},
\]

and for \( \beta > 1 \), the first two inequalities hold. For each \( \mu \), there is a function in \( K(\beta) \) such that equality holds.

We shall require the following:

**LEMMA 1** ([10], p. 166). Let \( h \in P \), i.e., let \( h \) be analytic in \( D \) and satisfy \( \text{Re} \ h(z) > 0 \) for \( z \in D \), with \( h(z) = 1 + c_1 z + c_2 z^2 + \ldots \), then

\[
\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.
\]
**Lemma 2** ([6], Lemma 3). Let \( g \in S^* \) with \( g(z) = z + b_2z^2 + b_3z^3 + \ldots \), then for \( \mu \) real,

\[
|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.
\]

We note that Lemma 2 above can easily be extended to the wider class \( S^*(\alpha) \) of strongly starlike functions of order \( \alpha \geq 0 \), i.e., \( g \) analytic and normalized in \( D \) and satisfying

\[
\left| \arg \frac{zg'(z)}{g(z)} \right| \leq \frac{\alpha \pi}{2},
\]

see e.g. [1]. In this case, one obtains the sharp inequality

\[
|b_3 - \mu b_2^2| \leq \max\{\alpha, \alpha^2|3 - 4\mu|\},
\]

for \( \mu \) real.

**Proof of Theorem:** It follows from (2) that we can write

\[
zf'(z) = g(z)h(z)^\beta
\]

for \( g \in S^* \) and \( h \in P \). Equating coefficients in (3) we obtain

\[
2a_2 = \beta c_1 + b_2
\]

and

\[
3a_3 = \frac{\beta(\beta - 1)}{2} c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,
\]

so that

\[
a_3 - \mu a_2^2 = \frac{1}{3} \left( b_3 - \frac{3}{4} \mu b_2^2 \right) + \frac{\beta}{3} \left( c_2 + \left( \frac{\beta(2 - 3\mu)}{4} - \frac{1}{2} \right) c_1^2 \right)
\]

\[
+ \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) c_1 b_2.
\]

We consider first the case \( \frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3} \). Equation (4) gives
\[ |a_3 - \mu a_2^2| \leq \frac{1}{3} \left| b_3 - \frac{3}{4} \mu b_2^2 \right| + \frac{\beta}{3} \left| c_2 - \frac{1}{2} c_1^2 \right| + \frac{\beta^2 (2 - 3\mu)}{12} |c_1^2| \\
+ \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) |c_1| |b_2|, \]
\[ \leq 1 - \mu + \frac{\beta}{3} \left( 2 - \frac{1}{2} |c_1^2| \right) + \frac{\beta^2 (2 - 3\mu)}{12} |c_1^2| \\\n+ \frac{\beta (2 - 3\mu)}{3} |c_1|, \]
\[ = \Phi(x) \text{ say, with } x = |c_1|, \]

where we have used Lemmas 1 and 2 and the fact that \(|b_2| \leq 2\) for \(g \in S^*\). An elementary argument shows that the function \(\Phi\) attains a maximum at \(x_0 = 2(2 - 3\mu)/(2 - \beta(2 - 3\mu))\), and so

\[ |a_3 - \mu a_2^2| \leq \Phi(x_0), \]

which proves the Theorem if \(\mu \leq 2/3\) and \(\beta \geq 0\). Choosing

\[ c_1 = \frac{2(2 - 3\mu)}{2 - \beta(2 - 3\mu)}, \quad c_2 = 2, \quad b_2 = 2 \text{ and } b_3 = 3, \]

in (4) shows that the result is sharp. We note that \(|c_1| \leq 2\), i.e., \(\mu \geq 2\beta/(3(\beta + 1))\).

Next consider the case \(\mu \leq \frac{2\beta}{3(\beta + 1)}\). Since \(K(0) = C\), we may assume that \(\beta > 0\). Again (4) gives

\[ |a_3 - \mu a_2^2| \leq \frac{3\mu(\beta + 1)}{2\beta} \left| a_3 - \frac{2\beta}{3(\beta + 1)} a_2^2 \right| + \left( 1 - \frac{3\mu(\beta + 1)}{2\beta} \right) |a_3|, \]
\[ \leq \frac{3\mu(\beta + 1)}{2\beta} \left( 1 + \frac{2\beta}{3} \right) \left( 2\beta(\beta + 2) + 1 \right), \]
\[ = 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3}, \]
for $\beta > 0$, where we have used the result already proved in the case $\mu = 2\beta/3(\beta + 1)$, and the fact that for $f \in K(\beta)$, the inequality $|a_3| \leq 1 + 2\beta(\beta + 2)/3$ holds [3]. Equality is attained on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$.

Suppose now that $\frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)}$. Since $g \in S^*$ we can write $zg'(z) = g(z)p(z)$ for $p \in P$, with $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, and so equating coefficients we have that $b_2 = p_1$ and $2b_3 = p_1^2 + p_2$.

We deal first with the case $\mu = 2(\beta + 2)/(3(\beta + 1))$. Thus (4) gives

$$a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 = \frac{1}{6} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{\beta}{3} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{\beta - 1}{12(\beta + 1)} p_1^2 - \frac{\beta^2 c_1^2}{6(\beta + 1)} - \frac{\beta p_1 c_1}{3(\beta + 1)},$$

and so if $\beta \leq 1$,

$$\left| a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 \right| \leq \frac{1}{6} \left| p_2 - \frac{p_1^2}{2} \right| + \frac{\beta}{3} \left| c_2 - \frac{c_1^2}{2} \right| + \frac{1}{12(\beta + 1)} |p_1^2| \left( \frac{\beta^2 |c_1^2|}{6(\beta + 1)} + \frac{\beta |p_1 c_1|}{3(\beta + 1)} \right)^2,$$

where we have used Lemma 1.

Now write

$$a_3 - \mu a_2^2 = \frac{\beta + 1}{2} \left( a_3 - \frac{2(\beta + 2)}{3(\beta + 1)} a_2^2 \right) + \frac{3(\beta + 1)}{2} \left( \frac{2(\beta + 2)}{3(\beta + 1)} - \mu \right) \left( a_3 - \frac{2}{3} a_2^2 \right),$$
and the result follows at once on using the Theorem already proved in the cases $\mu = 2/3$ and $\mu = 2(\beta + 2)/(3(\beta + 1))$ for $\beta \leq 1$. Equality is attained when $f$ is given by

$$f'(z) = \frac{(1 + z^2)\beta}{(1 - z^2)^{\beta+1}}.$$  

We finally assume that $\mu \geq \frac{2(\beta + 2)}{3(\beta + 1)}$. Write

$$a_{3} - \mu a_{2}^{2} = \left( a_{3} - \frac{2(\beta + 2)}{3(\beta + 1)}a_{2}^{2} \right) + \left( \frac{2(\beta + 2)}{3(\beta + 1)} - \mu \right) a_{2}^{2},$$

and the result follows at once on using the Theorem already proved for $\mu = 2(\beta + 2)/3(\beta + 1)$ in the case $\beta \leq 1$ and the inequality $|a_{2}| \leq \beta + 1$, which was proved in [3]. Equality is attained in this last case on choosing $c_{1} = c_{2} = b_{2} = 2$ and $b_{3} = 3$ in (4).

We remark that the methods used in [5] and [6], together with equation (4), suggest that in order to obtain sharp results for $\beta > 1$ and $\mu > 2/3$, an extension to the "area principle" may be required. Since $K(\beta)$ contains functions of unbounded valence for $\beta > 1$ establishing sharp estimates in this case may require deeper methods.

**References**


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