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The Fekete-Szegö problem for strongly close-to-convex functions.

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INTRODUCTION

Denote by $S$ the class of normalized analytic univalent functions $f$ defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A classical theorem of Fekete and Szegö [2] states that for $f \in S$ given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu, & \text{if } \mu \leq 0 \\
1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\
4 - 3\mu, & \text{if } \mu \geq 1.
\end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $S$ such that equality holds. Recently Pfluger [8] has considered the problem when $\mu$ is complex. In the case of $C$, $S^*$ and $K$, the subclasses of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [5,6]. In particular for $f \in K$ and given by (1), Keogh and Merkes [5] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu, & \text{if } \mu \leq 1/3, \\
1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3, \\
1, & \text{if } 2/3 \leq \mu \leq 1, \\
4\mu - 3, & \text{if } \mu \geq 1.
\end{cases}$$

Again, for each $\mu$, there is a function in $K$ such that equality holds. In this paper we extend this result to the class $K(\beta)$ of strongly close-to-convex functions of order $\beta$ in the sense of Pommerenke [9]. Thus
$f \in K(\beta)$ if, and only if, $f$, given by (1), is analytic in $D$ and is such that there exists $g \in S^*$ satisfying

$$\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi \beta}{2},$$

for $z \in D$ and $\beta \geq 0$. Clearly $K(0) = C$, $K(1) = K$ and when $0 \leq \beta \leq 1$, $K(\beta)$ is a subset of $K$ and hence contains only univalent functions. However in [4], Goodman showed that $K(\beta)$ can contain functions with unbounded valence for $\beta > 1$.

Recently, Koepf [7] has considered the Fekete-Szegő problem for $K(\beta)$ and obtained sharp results for some particular values of $\mu$, all of which, with the exception of the case $\mu = 1$ and $\beta \geq 1$, are contained in the following result.

**RESULTS**

**THEOREM.** Let $f \in K(\beta)$ and be given by (1), then for $0 \leq \beta \leq 1$,

$$|a_3 - \mu a_2^2| \leq 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3},$$

if $\mu \leq \frac{2\beta}{3(\beta + 1)}$,

$$\leq 1 - \mu + \frac{2\beta}{3} + \frac{\beta(2 - 3\mu)^2}{3[2 - \beta(2 - 3\mu)]},$$

if $\frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3}$,

$$\leq \frac{2\beta + 1}{3},$$

if $\frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)}$,

$$\leq \mu - 1 + \frac{\beta(3\mu - 2)(\beta + 2)}{3},$$

if $\mu \geq \frac{2(\beta + 2)}{3(\beta + 1)}$,

and for $\beta > 1$, the first two inequalities hold. For each $\mu$, there is a function in $K(\beta)$ such that equality holds.

We shall require the following:

**LEMMA 1** ([10], p. 166). Let $h \in P$, i.e., let $h$ be analytic in $D$ and satisfy $\text{Re} \, h(z) > 0$ for $z \in D$, with $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}.$$
Lemma 2 ([6], Lemma 3). Let $g \in S^*$ with $g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$, then for $\mu$ real,

$$|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}.$$ 

We note that Lemma 2 above can easily be extended to the wider class $S^*(\alpha)$ of strongly starlike functions of order $\alpha \geq 0$, i.e., $g$ analytic and normalized in $D$ and satisfying

$$\left| \arg \frac{z g'(z)}{g(z)} \right| \leq \frac{\alpha \pi}{2},$$

see e.g. [1]. In this case, one obtains the sharp inequality

$$|b_3 - \mu b_2^2| \leq \max\{\alpha, \alpha^2 |3 - 4\mu|\},$$

for $\mu$ real.

Proof of Theorem: It follows from (2) that we can write

$$zf'(z) = g(z)h(z) \beta$$

(3)

for $g \in S^*$ and $h \in P$. Equating coefficients in (3) we obtain

$$2a_2 = \beta c_1 + b_2$$

and

$$3a_3 = \frac{\beta(\beta - 1)}{2} c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,$$

so that

$$a_3 - \mu a_2^2 = \frac{1}{3} \left( b_3 - \frac{3}{4} \mu b_2^2 \right) + \frac{\beta}{3} \left( c_2 + \left( \frac{\beta(2 - 3\mu)}{4} - \frac{1}{2} \right) c_1^2 \right) + \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) c_1 b_2.$$  

(4)

We consider first the case $\frac{2\beta}{3(\beta + 1)} \leq \mu \leq \frac{2}{3}$. Equation (4) gives
\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} \left| b_3 - \frac{3}{4} \mu b_2^2 \right| + \frac{\beta}{3} \left| c_2 - \frac{1}{2} c_1^2 \right| + \frac{\beta^2(2 - 3\mu)}{12} |c_1^2|
\]
\[
+ \beta \left( \frac{1}{3} - \frac{\mu}{2} \right) |c_1||b_2|,
\]
\[
\leq 1 - \mu + \frac{\beta}{3} \left( 2 - \frac{1}{2} |c_1^2| \right) + \frac{\beta^2(2 - 3\mu)}{12} |c_1^2|
\]
\[
+ \frac{\beta(2 - 3\mu)}{3} |c_1|,
\]
\[
= \Phi(x) \text{ say, with } x = |c_1|,
\]
where we have used Lemmas 1 and 2 and the fact that \( |b_2| \leq 2 \) for \( g \in S^* \). An elementary argument shows that the function \( \Phi \) attains a maximum at \( x_0 = 2(2 - 3\mu)/(2 - \beta(2 - 3\mu)) \), and so
\[
|a_3 - \mu a_2^2| \leq \Phi(x_0),
\]
which proves the Theorem if \( \mu \leq 2/3 \) and \( \beta \geq 0 \). Choosing
\[
c_1 = \frac{2(2 - 3\mu)}{2 - \beta(2 - 3\mu)}, \quad c_2 = 2, \quad b_2 = 2 \text{ and } b_3 = 3,
\]
in (4) shows that the result is sharp. We note that \( |c_1| \leq 2 \), i.e., \( \mu \geq 2\beta/(\beta+1) \).

Next consider the case \( \mu \leq \frac{2\beta}{3(\beta+1)} \). Since \( K(0) = C \), we may assume that \( \beta > 0 \). Again (4) gives
\[
|a_3 - \mu a_2^2| \leq \frac{3\mu(\beta + 1)}{2\beta} \left| a_3 - \frac{2\beta}{3(\beta + 1)} a_2^2 \right| + \left( 1 - \frac{3\mu(\beta + 1)}{2\beta} \right) |a_3|,
\]
\[
\leq \frac{3\mu(\beta + 1)}{2\beta} \left( 1 + \frac{2\beta}{3} \right) + \left( 1 - \frac{3\mu(\beta + 1)}{2\beta} \right) \left( \frac{2\beta(\beta + 2)}{3} + 1 \right),
\]
\[
= 1 - \mu + \frac{\beta(2 - 3\mu)(\beta + 2)}{3},
\]
for \( \beta > 0 \), where we have used the result already proved in the case \( \mu = \frac{2\beta}{3(\beta + 1)} \), and the fact that for \( f \in K(\beta) \), the inequality \( |a_3| \leq 1 + 2\beta(\beta + 2)/3 \) holds [3]. Equality is attained on choosing \( c_1 = c_2 = b_2 = 2 \) and \( b_3 = 3 \).

Suppose now that \( \frac{2}{3} \leq \mu \leq \frac{2(\beta + 2)}{3(\beta + 1)} \). Since \( g \in S^* \) we can write \( zg'(z) = g(z)p(z) \) for \( p \in P \), with \( p(z) = 1 + p_1z + p_2z^2 + \ldots \), and so equating coefficients we have that \( b_2 = p_1 \) and \( 2b_3 = p_1^2 + p_2 \).

We deal first with the case \( \mu = 2(\beta + 2)/(3(\beta + 1)) \). Thus (4) gives

\[
\frac{a_3 - 2(\beta + 2)}{3(\beta + 1)}a_2^2 = \frac{1}{6}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{\beta}{3}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\beta - 1}{12(\beta + 1)}p_1^2 - \frac{\beta^2c_1^2}{6(\beta + 1)} - \frac{\beta p_1 c_1}{3(\beta + 1)},
\]

and so if \( \beta \leq 1 \),

\[
\left|a_3 - \frac{2(\beta + 2)}{3(\beta + 1)}a_2^2\right| \leq \frac{1}{6}\left|p_2 - \frac{p_1^2}{2}\right| + \frac{\beta}{3}\left|c_2 - \frac{c_1^2}{2}\right| + \frac{(1 - \beta)}{12(\beta + 1)}|p_1^2| + \frac{\beta^2|c_1^2|}{6(\beta + 1)} + \frac{\beta|p_1 c_1|}{3(\beta + 1)},
\]

\[
\leq \frac{1}{6}\left(2 - \frac{|p_1^2|}{2}\right) + \frac{\beta}{3}\left(2 - \frac{|c_1^2|}{2}\right) + \frac{1 - \beta}{12(\beta + 1)}|p_1^2| + \frac{\beta^2|c_1^2|}{6(\beta + 1)} + \frac{\beta|p_1 c_1|}{3(\beta + 1)},
\]

\[
= \frac{2\beta + 1}{3} - \frac{\beta}{6(\beta + 1)}(|c_1| - |p_1|)^2,
\]

\[
\leq \frac{2\beta + 1}{3},
\]

where we have used Lemma 1.

Now write

\[
a_3 - \mu a_2^2 = \frac{(\beta + 1)(3\mu - 2)}{2} - \frac{2(\beta + 2)}{3(\beta + 1)}a_2^2 + \frac{3(\beta + 1)}{2}\left(2\frac{\beta + 2}{3(\beta + 1) - \mu}\right) \left(\frac{2}{3}a_2^2\right),
\]
and the result follows at once on using the Theorem already proved in the cases $\mu = 2/3$ and $\mu = 2(\beta + 2)/(3(\beta + 1))$ for $\beta \leq 1$. Equality is attained when $f$ is given by

$$f'(z) = \frac{(1 + z^2)^\beta}{(1 - z^2)^{\beta+1}}.$$ 

We finally assume that $\mu \geq \frac{2(\beta+2)}{3(\beta+1)}$. Write

$$a_3 - \mu a_2^2 = \left( a_3 - \frac{2(\beta+2)}{3(\beta+1)} a_2^2 \right) + \left( \frac{2(\beta+2)}{3(\beta+1)} - \mu \right) a_2^2,$$

and the result follows at once on using the Theorem already proved for $\mu = 2(\beta+2)/3(\beta+1)$ in the case $\beta \leq 1$ and the inequality $|a_2| \leq \beta + 1$, which was proved in [3]. Equality is attained in this last case on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$ in (4).

We remark that the methods used in [5] and [6], together with equation (4), suggest that in order to obtain sharp results for $\beta > 1$ and $\mu > 2/3$, an extension to the "area principle" may be required. Since $K(\beta)$ contains functions of unbounded valence for $\beta > 1$ establishing sharp estimates in this case may require deeper methods.

**References**


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