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ON THE STRUCTURE OF THE TORELLI GROUP AND THE CASSON INVARIANT

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Dedicated to Professor A. Hattori on his sixtieth birthday

1. Introduction

In our previous paper [14], we have investigated the relationship between the Casson invariant of oriented homology 3-spheres and the algebraic structure of certain subgroups of the mapping class group of orientable surfaces. More precisely let Σ_g be an oriented closed surface of genus $g \geq 2$ and let $\mathcal{M}_g$ be its mapping class group. We denote $\mathcal{I}_g$ for the Torelli group of Σ_g, which is the subgroup of $\mathcal{M}_g$ consisting of all the elements which act on the homology of Σ_g trivially. Also we denote $\mathcal{K}_g$ for the subgroup of $\mathcal{I}_g$ generated by all the Dehn twists on separating simple closed curves on Σ_g. Now fix a Heegaard embedding $f : \Sigma_g \to S^3$. Then for each element $\varphi \in \mathcal{K}_g$, the manifold S^3_φ which is obtained from S^3 by first cutting along $f(\Sigma_g)$ and then reglueing the resulting two pieces by the map φ, is an oriented homology 3-sphere. Hence we can define a mapping $\lambda^* : \mathcal{K}_g \to \mathbf{Z}$ by setting $\lambda^*(\varphi) = \lambda(S^3_\varphi)$. On the other hand, by making use of the theory of characteristic classes of surface bundles developed in [11,12,13], we defined a homomorphism $d : \mathcal{K}_{g,1} \to \mathbf{Z}$ where $\mathcal{K}_{g,1}$ is the analogue of the group $\mathcal{K}_g$ relative to an embedded disc $D^2 \subset \Sigma_g$ (see § 2 for the precise definition). It is the secondary invariant associated with the fact that the first characteristic class e_1 of surface bundles vanishes on $\mathcal{K}_{g,1}$. Now the main theorem of [14] (see also [15]) can be summarized that the two integer valued invariants λ^* and d are essentially equal each other. The precise formulation of this result, which is rather complicated, as well as its proof was given in the framework of a certain combination of Johnson's theory on the structure of the Torelli group [7,8,9,10] with ours [op. cit.]. Now the purpose of the present paper is to continue these lines of investigations. As a result we find further close links between the Casson invariant and the structure of (various subgroups of) the mapping class group.

Now we describe the contents of this paper briefly. In § 2 we generalize the main theorem of [14] mentioned above as follows. Namely we consider the general situation where there is given an embedding $f : \Sigma_g \to M$ of Σ_g into an oriented homology 3-sphere M which need not be Heegaard. We define a mapping $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$ by $\lambda_f(\varphi) = \lambda(M_\varphi) - \lambda(M)$ ($\varphi \in \mathcal{I}_g$) where M_φ is the homology 3-sphere obtained by cutting M along $f(\Sigma_g)$ and then reglueing the resulting two pieces by the map φ. We reformulate the main result of [14] in this more general situation. In § 3 we consider a certain quotient group of the Torelli group which exactly measures how the elements of the Torelli group act on the fourth nilpotent quotient of the fundamental group of Σ_g. It turns out that this quotient group is a central extension of a certain free abelian group by another one which arise naturally in the work of Johnson and its extension [7,8,14]. We determine the Euler class of this central extension (Theorem 3.1). Using this result, in § 4 we describe how the mapping $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$ differs from a homomorphism. We will give a complete answer in the case where f is a Heegaard embedding (Theorem 4.3). As a byproduct of this description, we show that the mod 2 reduction of λ_f is always a homomorphism, thus generalizing an earlier work by Birman-Craggs [2] (see Corollary 4.4 and Remark 4.7). In the final section (§ 5), we determine the set of all homomorphisms $\mathcal{K}_{g,1} \to \mathbf{Z}$ which are invariant under taking conjugates in the whole mapping class group. It turns out that there are essentially two such homomorphisms (one of which is the homomorphism d mentioned above, see Theorem 5.4). We also settle the same problem for the group $\mathcal{K}_g$ (Theorem 5.7). These results will play an important role in our future paper [16] where we will interpret the invariant $d : \mathcal{K}_{g,1} \to \mathbf{Z}$, which is the core of the Casson invariant in the context of our approach, as Hirzebruch's signature defect (see [4]) of certain framed 3-manifolds.

2. Reformulation of the previous results

In this section we recall our formula proved in [14] which expresses the Casson invariant of oriented homology 3-spheres in terms of the pasting maps of their Heegaard splittings. In doing so, instead of just summarizing the results of [14] we prefer to exhibit the formula somewhat differently in a more general setting.

Suppose then that we are given an embedding $f : \Sigma_g \to M$ where Σ_g is an oriented closed surface of genus $g \geq 2$ and M is an oriented homology 3-sphere (although the most important example is the case where M is the 3-sphere S^3 and f is a Heegaard embedding, we do not assume these conditions). Since M is a homology sphere, the embedded surface $f(\Sigma_g)$ splits M into two parts V_+ and V_-, where V_+ is the one such that the induced orientation on the boundary $\partial V_+ = f(\Sigma_g)$ coincides with the given one on Σ_g via f. We may call V_+ (resp. V_-) the positive (resp. negative) piece of M with respect to the embedding f. We can write $M = V_+ \cup_\iota V_-$ where $\iota : \partial V_+ \to \partial V_-$ is the "identity". Now let $\mathcal{I}_g$ be the Torelli group of Σ_g. Namely it is the subgroup of the mapping class group $\mathcal{M}_g$ of Σ_g consisting of all the elements which act on the homology of Σ_g trivially. Also let $\mathcal{K}_g$ be the subgroup of $\mathcal{I}_g$ generated by all the Dehn twists on separating simple closed curves on Σ_g. For each element $\varphi \in \mathcal{I}_g$, we consider the manifold $M_\varphi = V_+ \cup_{\iota\varphi} V_-$ which is the one obtained by first cutting M along the embedded surface $f(\Sigma_g)$ and then reglueing the two pieces V_+ and V_- along their boundaries by the map $\iota\varphi$. It is easy to see that M_φ is an oriented homology 3-sphere so that we have the Casson invariant $\lambda(M_\varphi)$.

Now define a mapping

$$\lambda_f : \mathcal{I}_g \to \mathbf{Z}$$

by $\lambda_f(\varphi) = \lambda(M_\varphi) - \lambda(M)$ ($\varphi \in \mathcal{I}_g$). We also consider the restricted map $\lambda_f : \mathcal{K}_g \to \mathbf{Z}$. (Recall here that any homology 3-sphere can be expressed as S^3_φ for some $\varphi \in \mathcal{K}_g$ with $f : \Sigma_g \to S^3$ a Heegaard embedding, see [14]). For each separating simple closed curve ω on Σ_g, we denote $D_\omega \in \mathcal{K}_g$ for the right handed Dehn twist on ω. By definition the group $\mathcal{K}_g$ is generated by such elements. Now almost the same proof as that of Proposition 3.5 of [14] yields

Proposition 2.1. *The mapping $\lambda_f : \mathcal{K}_g \to \mathbf{Z}$ is a homomorphism. Moreover for each generator $D_\omega \in \mathcal{K}_g$ we have $\lambda_f(D_\omega) = -\lambda'(f(\omega))$, where $\lambda'(f(\omega))$ is the Casson invariant of the knot $f(\omega)$ in M.*

Motivated by the fact that Casson's invariant of a knot in an oriented homology 3-sphere can be expressed as a polynomial of degree two on the linking numbers among various homology classes of its Seifert surface (see Proposition 3.2 of [14]), we define a commutative algebra $\mathcal{A}$ over $\mathbf{Z}$ as follows. Let $H = H_1(\Sigma_g;\mathbf{Z})$ be the first integral homology group of Σ_g. Then $\mathcal{A}$ is defined to be the one generated by the symbol $\ell(u,v)$ for any two elements $u, v \in H$ with the relations

$$(i)\ \ell(v,u) = \ell(u,v) + u \cdot v$$

$$(ii)\ \ell(n_1u_1 + n_2u_2, v) = n_1\ell(u_1,v) + n_2\ell(u_2,v)\ (n_1, n_2 \in \mathbf{Z})$$

where $u \cdot v$ is the inersection number of u and v. In some sense $\mathcal{A}$ is the universal model for the linking pairing on H. More precisely, given an embedding $f : \Sigma_g \to M$ as before, we have the "evaluation map"

$$\varepsilon_f : \mathcal{A} \longrightarrow \mathbf{Z}$$

defined by $\varepsilon_f(\ell(u,v)) = \ell k(f_*(u), f_*(v)^+)$ where $f_*(v)^+$ is the homology class in $M \setminus f(\Sigma_g)$ obtained by pushing the homology class $f_*(v)$ to the positive direction (so that $f_*(v)^+$ is supported in V_-). Now for a technical reason we choose an embedded disc $D^2 \subset \Sigma_g$ and let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to D^2. We have a natural surjective homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ and its kernel can be canonically identified with $\pi_1(T_1\Sigma_g)$ where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g. We consider the Torelli group $\mathcal{I}_{g,1}$ which is the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on H trivially and also let $\mathcal{K}_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ generated by all the Dehn twists on bounding simple closed curves on $\Sigma_g \setminus D^2$. For each bounding simple closed curve ω on $\Sigma_g \setminus D^2$, let D_ω denote the Dehn twist on ω

and let $u_1, \cdots, u_h, v_1, \cdots, v_h$ be a symplectic basis of the homology of the subsurface of $\Sigma_g \setminus D^2$ which ω bounds (hereafter, following Johnson, we call such element a BSCC map of genus h). Then we can reformulate one of the main results of [14] as

Theorem 2.2. *The assignment*

$$\begin{aligned}\mathcal{K}_{g,1} \ni D_\omega \longmapsto & -\sum_{i=1}^{h} \{\ell(u_i,u_i)\ell(v_i,v_i) - \ell(u_i,v_i)\ell(v_i,u_i)\} \\ & -2 \sum_{i<j\leq h} \{\ell(u_i,u_j)\ell(v_i,v_j) - \ell(u_i,v_j)\ell(u_j,v_i)\}\end{aligned}$$

for each generator $D_\omega \in \mathcal{K}_{g,1}$ extends to a well defined homomorphism $\rho : \mathcal{K}_{g,1} \to \mathcal{A}$ and the following diagram is commutative

$$\begin{array}{ccc} \mathcal{K}_{g,1} & \xrightarrow{\rho} & \mathcal{A} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_f} \\ \mathcal{K}_g & \xrightarrow{\lambda_f} & \mathbf{Z} \end{array}$$

where $\mathcal{K}_{g,1} \to \mathcal{K}_g$ is the natural surjection.

Roughly speaking, for each generator $D_\omega \in \mathcal{K}_{g,1}$ the element $\rho(D_\omega) \in \mathcal{A}$ can be considered as the "universal model" for the Casson invariant of the bounding simple closed curve ω on $\Sigma_g \setminus D^2$ (up to signs). Namely if there is given an embedding $f : \Sigma_g \to M$, then we can consider $f(\omega)$ as a knot in M and we have $\lambda'(f(\omega)) = -\varepsilon_f(\rho(D_\omega))$. The following is an immediate corollary to the above theorem which we present here for later use.

Corollary 2.3. *(i) $\lambda_f(\varphi\psi\varphi^{-1}) = \lambda_f(\psi)$ for any $\varphi \in \mathcal{I}_g$ and $\psi \in \mathcal{K}_g$.*
(ii) $\lambda_f([\varphi,\psi]) = 0$ for any $\varphi \in \mathcal{I}_g$ and $\psi \in \mathcal{K}_g$.

Now if there were an algorithm to express any given element $\varphi \in \mathcal{K}_g$ as a product of Dehn twists on separating simple closed curves on Σ_g, then we can explicitly calculate the element $\rho(\varphi) \in \mathcal{A}$ and hence the desired value $\lambda_f(\varphi)$ by using Theorem 2.2. However unfortunately there is no such one and our next main result of [14] is that the homomorphism $\rho : \mathcal{K}_{g,1} \to \mathcal{A}$ can be decomposed as a sum of two more computable ones: $\rho = \frac{1}{24}d + \overline{\rho}$, where $d : \mathcal{K}_{g,1} \to \mathbf{Z}$ is a certain homomorphism and $\overline{\rho} : \mathcal{K}_{g,1} \to \mathcal{A} \otimes \mathbf{Q}$ is a modification of ρ. The point here is that the homomorphism $\overline{\rho}$ can be read off from Johnson's homomorphism $\tau_3 : \mathcal{K}_{g,1} \to \overline{T}$ (which will be recalled below) and the homomorphism d is the secondary invariant associated with the first characteristic class $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbf{Z})$ of surface bundles introduced in [14]. This homomorphism d should be considered as the core of the Casson invariant from our point of view.

Now, in order to define the homomorphim $\overline{\rho} : \mathcal{K}_{g,1} \to \mathcal{A} \otimes \mathbf{Q}$, we briefly recall the definition of Johnson's homomorpisms τ_k, which will be also needed in later sections (see [7,8,14] for details).

We denote Γ_1 for $\pi_1(\Sigma_g \setminus \operatorname{Int} D^2)$ and let $\{\Gamma_k\}_{k\geq 1}$ be the lower central series of Γ_1; $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ $(k \geq 2)$. The mapping class group $\mathcal{M}_{g,1}$ acts on the nilpotent quotient group $N_k = \Gamma_1/\Gamma_k$ naturally and we define $\mathcal{M}(k) = \{\varphi \in \mathcal{M}_{g,1}; \varphi$ acts on N_k trivially $\}$. Let $\mathcal{L} = \oplus_{k\geq 1}\mathcal{L}_k$ be the free graded Lie algebra on H (over $\mathbf{Z}$). Then as is well known there is a natural isomorphism $\Gamma_k/\Gamma_{k+1} \cong \mathcal{L}_k$. Now Johnson's homomorphism

$$\tau_k : \mathcal{M}_{g,1} \longrightarrow \operatorname{Hom}(H, \mathcal{L}_k)$$

is defined as $\tau_k(\varphi)([\gamma]) = [\varphi_*(\gamma)\gamma^{-1}]$ $(\varphi \in \mathcal{M}(k), \gamma \in \Gamma_1)$, where $[\gamma] \in H$ is the homology class of γ and $[\varphi_*(\gamma)\gamma^{-1}]$ denotes the image in $\mathcal{L}_k$ of the element $\varphi_*(\gamma)\gamma^{-1}$ which is contained in Γ_k by the assumption $\varphi \in \mathcal{M}(k)$. The intersection pairing on H defines a natural isomorphism $H \cong H^*$ so that we can write $\tau_k : \mathcal{M}(k) \to \mathcal{L}_k \otimes H$. If we choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of H, then explicitly we have

$$\tau_k(\varphi) = \sum_{i=1}^{g} \{\tau_k(\varphi)(x_i) \otimes y_i - \tau_k(\varphi)(y_i) \otimes x_i\} \in \mathcal{L}_k \otimes H$$

where τ_k's in the right hand side are the old ones. Now for $k = 2$, $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$ and $\mathcal{L}_2$ is naturally isomorphic to $\Lambda^2 H$ so that we can write $\tau_2 : \mathcal{I}_{g,1} \to \Lambda^2 H \otimes H$. Johnson proved that Im τ_2 is equal to $\Lambda^3 H \subset \Lambda^2 H \otimes H$, where $a \wedge b \wedge c \in \Lambda^3 H$ $(a, b, c \in H)$ is identified with $(a \wedge b) \otimes c + (b \wedge c) \otimes a + (c \wedge a) \otimes b \in \Lambda^2 H \otimes H$.

For $k = 3$, according to Johnson [9], $\mathcal{M}(3)$ is equal to $\mathcal{K}_{g,1}$ and in [14] we have determined the image of the homomorphism $\tau_3 : \mathcal{K}_{g,1} \to \mathcal{L}_3 \otimes H$ as follows. There is a natural isomorphism $\mathcal{L}_3 \cong \Lambda^2 H \otimes H / \Lambda^3 H$ so that $\mathcal{L}_3 \otimes H \cong \Lambda^2 H \otimes H^2 / \Lambda^3 H \otimes H$. We define a module T to be the submodule of $\Lambda^2 H \otimes \Lambda^2 H \subset \Lambda^2 H \otimes H^2$ generated by all the elements of the forms $(a \wedge b) \otimes (a \wedge b)$ and $(a \wedge b) \otimes (c \wedge d) + (c \wedge d) \otimes (a \wedge b)$ $(a, b, c, d \in H)$ (henceforth these elements will be denoted by $(a \wedge b)^{\otimes 2}$ and $a \wedge b \leftrightarrow c \wedge d$ respectively). By the definition of T, we have a natural homomorphism $T \to \mathcal{L}_3 \otimes H$ and it can be shown that the kernel T_0 of this homomorphism is the submodule of T generated by all the elements of the form $a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c$. Hence if we write $\overline{T}$ for T/T_0, then $\overline{T}$ can be considered as a submodule of $\mathcal{L}_3 \otimes H$. In these terminologies we have proved in [14] that Im τ_3 is a subgroup of $\overline{T}$ of index a power of two.

Now there is a uniquely defined homomorphism $\theta : T \to \mathcal{A}$ such that for any symplectic subbasis $u_1, \cdots, u_h, v_1, \cdots, v_h$ of H, the equality

$$\begin{aligned}\theta((u_1 \wedge v_1 + \cdots u_h \wedge v_h)^{\otimes 2}) = & \sum_{i=1}^{h} \{\ell(u_i, u_i)\ell(v_i, v_i) - \ell(u_i, v_i)\ell(v_i, u_i)\} \\ & +2 \sum_{i<j\leq h} \{\ell(u_i, u_j)\ell(v_i, v_j) - \ell(u_i, v_j)\ell(u_j, v_i)\}\end{aligned}$$

holds. Let $D_\omega \in \mathcal{K}_{g,1}$ be a BSCC map of genus h where ω is a bounding simple closed curve on $\Sigma_g \setminus D^2$. Choose a symplectic basis $u_1, \cdots, u_h, v_1, \cdots, v_h$ of the homology of the subsurface which ω bounds. If we define an element $t \in T$ by $t = -(u_1 \wedge v_1 + \cdots + u_h \wedge v_h)^{\otimes 2}$, then we have $\rho(D_\omega) = \theta(t)$. Hence the element t contains all the information of the "universal Casson invariant" $\rho(D_\omega)$. However Johnson's homomorphism $\tau_3 : \mathcal{K}_{g,1} \to \overline{T}$ computes only the image of t in $\overline{T}$: $\tau_3(D_\omega) = \bar{t} \in \overline{T}$. Now the definition of the homomorphism $\bar{\rho} : \mathcal{K}_{g,1} \to \mathcal{A} \otimes \mathbf{Q}$ is defined as follows.

There exists a certain homomorphism $\overline{d} : T \to \mathbf{Z}$, which is a formal counterpart of the homomorphism d, and for each generator $D_\omega \in \mathcal{K}_{g,1}$ as above, choose any element $t' \in T \otimes \mathbf{Q}$ such that the image of t' in $\overline{T}$ is equal to $\tau_3(D_\omega)$. Now set $\overline{\rho}(D_\omega) = \theta(t') + \frac{1}{3}\overline{d}(t') \in \mathcal{A} \otimes \mathbf{Q}$, which turns out to be independent of the choice of t'. This is the definition of the homomorphism $\overline{\rho} : \mathcal{K}_{g,1} \to \mathcal{A} \otimes \mathbf{Q}$. Now we summarize the above as

Theorem 2.4. *We have the equality*

$$\rho(\varphi) = \frac{1}{24} d(\varphi) + \overline{\rho}(\varphi)$$

for any $\varphi \in \mathcal{K}_{g,1}$.

3. A quotient group of the Torelli group

In this section we determine the structure of a certain quotient group of the Torelli group $\mathcal{I}_{g,1}$. Roughly speaking it is the group which contains exactly the informations carried by the homomorphisms τ_2 and τ_3. More precisely recall that we have a short exact sequence

$$1 \longrightarrow \mathcal{K}_{g,1} \longrightarrow \mathcal{I}_{g,1} \xrightarrow{\tau_2} \Lambda^3 H \longrightarrow 1$$

and the homomorphism $\tau_3 : \mathcal{K}_{g,1} \to \overline{T}$. Hence if we define $\overline{\mathcal{I}}_{g,1}$ to be the quotient group $\mathcal{I}_{g,1} \,/\, \mathrm{Ker}\, \tau_3$, then we have an extension $1 \to \mathrm{Im}\, \tau_3 \to \overline{\mathcal{I}}_{g,1} \to \Lambda^3 H \to 1$.

Theorem 3.1. *The extension*

$$1 \to \mathrm{Im}\, \tau_3 \to \overline{\mathcal{I}}_{g,1} \to \Lambda^3 H \to 1$$

is a central extension and its Euler class $\chi \in H^2(\Lambda^3 H; \operatorname{Im}\tau_3)$ *is given by*

$$\begin{aligned}
&\chi(\xi,\eta) = \\
&-(a\cdot d)b\wedge c \leftrightarrow e\wedge f - (a\cdot e)b\wedge c \leftrightarrow f\wedge d - (a\cdot f)b\wedge c \leftrightarrow d\wedge e \\
&-(b\cdot d)c\wedge a \leftrightarrow e\wedge f - (b\cdot e)c\wedge a \leftrightarrow f\wedge d - (b\cdot f)c\wedge a \leftrightarrow d\wedge e \ \in \operatorname{Im}\tau_3 \subset \overline{T}. \\
&-(c\cdot d)a\wedge b \leftrightarrow e\wedge f - (c\cdot e)a\wedge b \leftrightarrow f\wedge d - (c\cdot f)a\wedge b \leftrightarrow d\wedge e
\end{aligned}$$

where $\xi = a\wedge b\wedge c, \eta = d\wedge e\wedge f \in \Lambda^3 H$ $(a,b,c,d,e,f \in H)$ *and* $\xi\wedge\eta \in H_2(\Lambda^3 H; \mathbf{Z})$.

Before proving the above theorem, we prepare several technical results. Recall that we write Γ_1 for $\pi_1(\Sigma_g \setminus \operatorname{Int} D^2)$, $\{\Gamma_k\}_{k\geq 1}$ for its lower central series and $\mathcal{M}(k)$ for the subgroup of the mapping class group $\mathcal{M}_{g,1}$ consisting of all the elements which act on the nilpotent quotient $N_k = \Gamma_1/\Gamma_k$ trivially.

Lemma 3.2. *(i) For any element* $\varphi \in \mathcal{M}(k)$ *and* $\gamma \in \Gamma_\ell$*, we have* $\varphi(\gamma)\gamma^{-1} \in \Gamma_{k+\ell-1}$.
(ii) For any elements $\varphi \in \mathcal{I}_{g,1}$ *,* $\psi \in \mathcal{M}(k)$ *and* $\gamma \in \Gamma_\ell$*, we have the equality*

$$[\varphi\psi\varphi^{-1}(\gamma)\gamma^{-1}] = [\psi(\gamma)\gamma^{-1}]$$

in $\mathcal{L}_{k+\ell-1}$.

Proof. To prove (i), we use the induction on ℓ. If $\ell = 1$, then the claim follows immediately from the definition of the group $\mathcal{M}(k)$. Now we assume that the claim holds up to $\ell-1$ and prove it for ℓ. Suppose that an element $\gamma \in \Gamma_\ell$ is expressed as $\gamma = [\gamma_1, \gamma_2]$ with $\gamma_1 \in \Gamma_{\ell-1}$ and $\gamma_2 \in \Gamma_1$. Then by the induction assumption, we have $\varphi(\gamma_1) = \gamma_1\alpha$ and $\varphi(\gamma_2) = \gamma_2\beta$ for some $\alpha \in \Gamma_{k+\ell-2}$ and $\beta \in \Gamma_k$. Then we compute

$$\begin{aligned}
\varphi(\gamma)\gamma^{-1} &= [\varphi(\gamma_1), \varphi(\gamma_2)][\gamma_2, \gamma_1] \\
&= \gamma_1\alpha\gamma_2\beta\alpha^{-1}\gamma_1^{-1}\beta^{-1}\gamma_1\gamma_2^{-1}\gamma_1^{-1} \\
&= \gamma_1[\alpha, \gamma_2\beta]\gamma_2[\beta, \gamma_1^{-1}]\gamma_2^{-1}\gamma_1^{-1}.
\end{aligned}$$

Now both of the elements $[\alpha, \gamma_2\beta]$ and $[\beta, \gamma_1^{-1}]$ are contained in $\Gamma_{k+\ell-1}$. Hence $\varphi(\gamma)\gamma^{-1} \in \Gamma_{k+\ell-1}$ as required. In general any element $\gamma \in \Gamma_\ell$ can be expressed as a product of elements of the form $[\gamma_1, \gamma_2]$ given above and an easy inductive argument shows that the claim also holds for γ. Next we prove (ii). If we apply (i) to the elements $\varphi \in \mathcal{I}_{g,1}, \psi \in \mathcal{M}(k)$ and $\gamma \in \Gamma_\ell$, we can write $\varphi(\gamma) = \gamma_1\gamma, \psi(\gamma) = \gamma_2\gamma$ for some $\gamma_1 \in \Gamma_{\ell+1}, \gamma_2 \in \Gamma_{k+\ell-1}$. It follows that $\varphi^{-1}(\gamma) = \varphi^{-1}(\gamma_1^{-1})\gamma$. Then we compute

$$\begin{aligned}
\varphi\psi\varphi^{-1}(\gamma)\gamma^{-1} &= \varphi\psi(\varphi^{-1}(\gamma_1^{-1})\gamma)\gamma^{-1} \\
&= \varphi(\psi\varphi^{-1}(\gamma_1^{-1})\gamma_2\gamma)\gamma^{-1} \\
&= \varphi\psi\varphi^{-1}(\gamma_1^{-1})\varphi(\gamma_2)\gamma_1.
\end{aligned}$$

Now again by (i), we can conclude that $\varphi\psi\varphi^{-1}(\gamma_1^{-1}) \equiv \gamma_1^{-1} \pmod{\Gamma_{k+\ell}}$ and $\varphi(\gamma_2) \equiv \gamma_2 \pmod{\Gamma_{k+\ell}}$. Hence $\varphi\psi\varphi^{-1}(\gamma)\gamma^{-1} \equiv \gamma_1^{-1}\gamma_2\gamma_1 \equiv \gamma_2 \pmod{\Gamma_{k+\ell}}$. Since $\gamma_2 = \psi(\gamma)\gamma^{-1}$, This finishes the proof.

As a corollary to Lemma 3.2,(i), we have

Corollary 3.3. *For any two elements $\varphi \in \mathcal{M}(k)$ and $\psi \in \mathcal{M}(\ell)$, the commutator $[\varphi, \psi]$ is contained in $\mathcal{M}(k + \ell - 1)$.*

Proof. Clearly it is enough to prove the assertion for the case $k \geq \ell$. Henceforth we assume this condition. Now let $\gamma \in \Gamma_1$ be any element.Then by the assumption we can write $\psi(\gamma) = \gamma_1\gamma$ and $\varphi(\gamma) = \gamma_2\gamma$ for some $\gamma_1 \in \Gamma_\ell$ and $\gamma_2 \in \Gamma_k$. Then we have $\psi^{-1}(\gamma) = \psi^{-1}(\gamma_1^{-1})\gamma$ and $\varphi^{-1}(\gamma) = \varphi^{-1}(\gamma_2^{-1})\gamma$. Now we compute

$$\begin{aligned}
[\varphi, \psi](\gamma) &= \varphi\psi\varphi^{-1}(\psi^{-1}(\gamma_1^{-1})\gamma) \\
&= \varphi\psi(\varphi^{-1}\psi^{-1}(\gamma_1^{-1})\varphi^{-1}(\gamma_2^{-1})\gamma) \\
&= [\varphi, \psi](\gamma_1^{-1})\varphi\psi\varphi^{-1}(\gamma_2^{-1})\varphi(\gamma_1)\gamma_2\gamma.
\end{aligned}$$

Hence

$$[\varphi,\psi](\gamma)\gamma^{-1} = [\varphi,\psi](\gamma_1^{-1})\gamma_1\cdot\gamma_1^{-1}\varphi\psi\varphi^{-1}(\gamma_2^{-1})\gamma_2\gamma_1\cdot\gamma_1^{-1}\gamma_2^{-1}\varphi(\gamma_1)\gamma_1^{-1}\gamma_2\gamma_1\cdot[\gamma_1^{-1},\gamma_2^{-1}].$$

Now by Lemma 3.2,(i), we have $[\varphi,\psi](\gamma_1^{-1})\gamma_1 \in \Gamma_{k+\ell-1}$ because certainly we have $[\varphi,\psi] \in \mathcal{M}(k)$. Similarly we have $\varphi\psi\varphi^{-1}(\gamma_2^{-1})\gamma_2 \in \Gamma_{k+\ell-1}$, $\varphi(\gamma_1)\gamma_1^{-1} \in \Gamma_{k+\ell-1}$ and of course $[\gamma_1^{-1},\gamma_2^{-1}] \in \Gamma_{k+\ell}$. Hence $[\varphi,\psi](\gamma)\gamma^{-1} \in \Gamma_{k+\ell-1}$ as required. This completes the proof.

Recall that for any element $\varphi \in \mathcal{M}(k)$ we have a homomorphism $\tau_k(\varphi) : H \to \mathcal{L}_k$. Now by virtue of Lemma 3.2,(i), for each element $\varphi \in \mathcal{M}(k)$ and a positive integer ℓ we can define a similar homomorphism $\varphi\{\ \} : \mathcal{L}_\ell \to \mathcal{L}_{k+\ell-1}$ by setting $\varphi\{[\gamma]\} = [\varphi(\gamma)\gamma^{-1}] \in \mathcal{L}_{k+\ell-1}$, where $[\gamma] \in \mathcal{L}_k$ is the image in $\mathcal{L}_k$ of any element $\gamma \in \Gamma_k$. It is easy to check that this correspondence in fact defines a well defined homomorphism.

Proposition 3.4. *Let $\varphi \in \mathcal{M}(k)$ and $\psi \in \mathcal{M}(\ell)$ so that we have $[\varphi,\psi] \in \mathcal{M}(k+\ell-1)$ (see Corollary 3.3). If k and ℓ are greater than one, then we have*

$$\tau_{k+\ell-1}([\varphi,\psi])(u) = \varphi\{\tau_\ell(\psi)(u)\} - \psi\{\tau_k(\varphi)(u)\}\ (u \in H).$$

Proof. Let $\gamma \in \Gamma_1$ be any element. Then we can write $\psi(\gamma) = \gamma_1\gamma, \varphi(\gamma) = \gamma_2\gamma$ for some $\gamma_1 \in \Gamma_\ell$ and $\gamma_2 \in \Gamma_k$. Now in the computation of $[\varphi,\psi](\gamma)\gamma^{-1}$ in the proof of Corollary 3.3, the term $[\varphi,\psi](\gamma_1^{-1})\gamma_1$ is contained in $\Gamma_{k+2\ell-2} \subset \Gamma_{k+\ell}$ (by Corollary 3.3 and the assumption $\ell \geq 2$). Hence we have

$$[\varphi,\psi](\gamma)\gamma^{-1} \equiv \varphi\psi\varphi^{-1}(\gamma_2^{-1})\gamma_2\cdot\varphi(\gamma_1)\gamma_1^{-1}\ (\mathrm{mod}\,\Gamma_{k+\ell}).$$

On the other hand by Lemma 3.2,(ii), we have

$$\begin{aligned}[\varphi\psi\varphi^{-1}(\gamma_2^{-1})\gamma_2] &= [\psi(\gamma_2^{-1})\gamma_2] \\ &= -\,\psi\{\tau_k(\varphi)([\gamma])\}\end{aligned}$$

as elements of $\mathcal{L}_{k+\ell-1}$. Also we have $[\varphi(\gamma_1)\gamma_1^{-1}] = \varphi\{\tau_\ell(\psi)[\gamma]\} \in \mathcal{L}_{k+\ell-1}$. Hence we can conclude that $\tau_{k+\ell-1}([\varphi,\psi])([\gamma]) = \varphi\{\tau_\ell(\psi)([\gamma])\} - \psi\{\tau_k(\varphi)([\gamma])\}$ as required. This completes the proof.

Proposition 3.5. *For each element $\varphi \in \mathcal{M}(k)$, the homomorphism $\varphi\{\ \} : \mathcal{L}_\ell \to \mathcal{L}_{k+\ell-1}$ is given explicitly as*

$$\varphi\{\xi\} = \sum_{i=1}^{\ell}[\cdots[u_1,u_2],\cdots],\tau_k(\varphi)(u_i)],\cdots],u_\ell]$$

where $\xi = [\cdots[u_1,u_2],\cdots],u_\ell] \in \mathcal{L}_\ell$ $(u_i \in H)$.

Proof. We assume that $k \geq 2$ because for $k = 1$ the assertion is empty. By the definition of the homomorphism $\varphi\{\ \}$, it suffices to prove the following statement

(*) For any element $\gamma = [\cdots[\gamma_1,\gamma_2],\cdots],\gamma_\ell] \in \Gamma_\ell$ $(\gamma_i \in \Gamma_1)$, we have $\varphi(\gamma)\gamma^{-1} \equiv \prod_{i=1}^{\ell}[\cdots[\gamma_1,\gamma_2],\cdots],\varphi(\gamma_i)\gamma_i^{-1}],\cdots],\gamma_\ell] \pmod{\Gamma_{k+\ell}}$.

We prove the above assertion by the induction on ℓ. If $\ell = 1$, then the assertion is clear. We assume that (*) holds up to $\ell - 1$ and prove it for ℓ. Write $\gamma = [\alpha,\gamma_\ell]$ where $\alpha = [\cdots[\gamma_1,\gamma_2],\cdots],\gamma_{\ell-1}] \in \Gamma_{\ell-1}$. By the induction assumption we can write

$$\begin{aligned}\varphi(\alpha) &\equiv \widetilde{\alpha}\alpha \\ &\equiv \alpha\widetilde{\alpha} \pmod{\Gamma_{k+\ell-1}}\end{aligned}$$

where $\widetilde{\alpha} = \prod_{i=1}^{\ell-1}[\cdots[\gamma_1,\gamma_2],\cdots],\varphi(\gamma_i)\gamma_i^{-1}],\cdots],\gamma_{\ell-1}] \in \Gamma_{k+\ell-2}$. Also we can write

$\varphi(\gamma_\ell) = \gamma_\ell \beta$ for some $\beta \in \Gamma_k$. Then we compute

$$\begin{aligned}
\varphi(\gamma)\gamma^{-1} &= [\varphi(\alpha), \varphi(\gamma_\ell)][\gamma_\ell, \alpha] \\
&= \alpha\widetilde{\alpha}\gamma_\ell\beta\widetilde{\alpha}^{-1}\alpha^{-1}\beta^{-1}\alpha\gamma_\ell^{-1}\alpha^{-1} \\
&= \alpha[\widetilde{\alpha}, \gamma_\ell]\alpha^{-1} \cdot \alpha\gamma_\ell[\widetilde{\alpha}, \beta]\gamma_\ell^{-1}\alpha^{-1} \cdot \alpha\gamma_\ell\alpha^{-1}[\alpha, \beta]\alpha\gamma_\ell^{-1}\alpha^{-1} \\
&\equiv [\widetilde{\alpha}, \gamma_\ell][\alpha, \beta] \pmod{\Gamma_{k+\ell}}
\end{aligned}$$

because $[\widetilde{\alpha}, \beta] \in \Gamma_{k+\ell}$. Now using the fact that $\beta \equiv \varphi(\gamma_\ell)\gamma_\ell^{-1} \pmod{\Gamma_{k+1}}$ it is easy to see that $[\alpha, \beta] \equiv [\alpha, \varphi(\gamma_\ell)\gamma_\ell^{-1}] \pmod{\Gamma_{k+\ell}}$. Also it can be easily shown by induction that $[\widetilde{\alpha}, \gamma_\ell] \equiv \prod_{i=1}^{\ell-1} [\cdots[\gamma_1, \gamma_2], \cdots], \varphi(\gamma_i)\gamma_i^{-1}], \cdots], \gamma_{\ell-1}], \gamma_\ell] \pmod{\Gamma_{k+\ell}}$. Hence

$$\varphi(\gamma)\gamma^{-1} \equiv \prod_{i=1}^{\ell} [\cdots[\gamma_1, \gamma_2], \cdots], \varphi(\gamma_i)\gamma_i^{-1}], \cdots], \gamma_\ell] \pmod{\Gamma_{k+\ell}}$$

as required. This completes the proof.

Now we are ready to prove the main result of this section.

Proof of Theorem 3.1. In general we have $\tau_3(\varphi\psi\varphi^{-1}) = \varphi_*\tau_3(\psi)$ ($\varphi \in \mathcal{M}_{g,1}$, $\psi \in \mathcal{K}_{g,1}$). Hence if φ is contained in $\mathcal{I}_{g,1}$, then $\tau_3(\varphi\psi\varphi^{-1}) = \tau_3(\psi)$. It follows immediately that the extension $1 \to \operatorname{Im}\tau_3 \to \overline{\mathcal{I}}_{g,1} \to \Lambda^3 H \to 1$ is a central extension. Next we determine the Euler class $\chi \in H^2(\Lambda^3 H; \operatorname{Im}\tau_3)$ of this central extension. To do this we briefly recall the definition of the "Euler class" of a general central extension $0 \to A \to G \to Q \to 1$ (see [3] for details). Choose a set map $s : Q \to G$ such that $s(1) = 1$ and set $c(f, g) = s(f)s(g)s(fg)^{-1} \in A$. It is easy to see that c is a 2-cocycle of the group Q with values in A. If we change the map s, then the cocycle c changes only by a coboundary so that the cohomology class $[c] \in H^2(Q; A)$ is well defined. This is the definition of the Euler class of central extensions. Now assume that Q is a free abelian group (as in our case). Then we have $H^2(Q; A) \cong \operatorname{Hom}(H_2(Q; \mathbf{Z}), A)$

and $H_2(Q;\mathbf{Z}) \cong \Lambda^2 Q$. For any element $f \wedge g \in \Lambda^2 Q$ $(f, g \in Q)$, we have

$$\begin{aligned} \chi(f \wedge g) &= c(f,g) - c(g,f) \\ &= [s(f), s(g)] \in A \end{aligned}$$

because $s(fg) = s(gf)$. With these preparations we compute the Euler class $\chi \in H^2(\Lambda^3 H; \operatorname{Im} \tau_3)$. Let $\xi = a \wedge b \wedge c$ and $\eta = d \wedge e \wedge f$ be any two elements of $\Lambda^3 H$ $(a, b, c, d, e, f \in H)$. To evaluate the value of χ on the cycle $\xi \wedge \eta \in \Lambda^2(\Lambda^3 H) \cong H_2(\Lambda^3 H; \mathbf{Z})$ we choose elements $\varphi, \psi \in \mathcal{I}_{g,1}$ such that $\tau_2(\varphi) = \xi$ and $\tau_2(\psi) = \eta$. It is easy to deduce from our identification of $\Lambda^3 H$ as a submodule of $\Lambda^2 H \otimes H \cong \operatorname{Hom}(H, \Lambda^2 H)$ (see §2) that the homomorphism $\tau_2(\varphi) : H \to \mathcal{L}_2 = \Lambda^2 H$ is given by

$$\tau_2(\varphi)(u) = (u \cdot a) b \wedge c + (u \cdot b) c \wedge a + (u \cdot c) a \wedge b \ (u \in H).$$

Similarly we have

$$\tau_2(\psi)(u) = (u \cdot d) e \wedge f + (u \cdot e) f \wedge d + (u \cdot f) d \wedge e.$$

Hence if we apply Proposition 3.4 and Proposition 3.5 to $\varphi, \psi \in \mathcal{I}_{g,1} = \mathcal{M}(2)$, we obtain

$$
\begin{aligned}
\tau_3([\varphi,\psi]) = \varphi\{\tau_2(\psi)(u)\} - \psi\{\tau_2(\varphi)(u)\} \\
= (u\cdot d)\{&(f\cdot a)[e,[b,c]] + (f\cdot b)[e,[c,a]] + (f\cdot c)[e,[a,b]] \\
&+ (e\cdot a)[[b,c],f] + (e\cdot b)[[c,a],f] + (e\cdot c)[[a,b],f]\} \\
+(u\cdot e)\{&(d\cdot a)[f,[b,c]] + (d\cdot b)[f,[c,a]] + (d\cdot c)[f,[a,b]] \\
&+ (f\cdot a)[[b,c],d] + (f\cdot b)[[c,a],d] + (f\cdot c)[[a,b],d]\} \\
+(u\cdot f)\{&(e\cdot a)[d,[b,c]] + (e\cdot b)[d,[c,a]] + (e\cdot c)[d,[a,b]] \\
&+ (d\cdot a)[[b,c],e] + (d\cdot b)[[c,a],e] + (d\cdot c)[[a,b],e]\} \\
-(u\cdot a)\{&(c\cdot d)[b,[e,f]] + (c\cdot e)[b,[f,d]] + (c\cdot f)[b,[d,e]] \\
&+ (b\cdot d)[[e,f],c] + (b\cdot e)[[f,d],c] + (b\cdot f)[[d,e],c]\} \\
-(u\cdot b)\{&(a\cdot d)[c,[e,f]] + (a\cdot e)[c,[f,d]] + (a\cdot f)[c,[d,e]] \\
&+ (c\cdot d)[[e,f],a] + (c\cdot e)[[f,d],a] + (c\cdot f)[[d,e],a]\} \\
-(u\cdot c)\{&(b\cdot d)[a,[e,f]] + (b\cdot e)[a,[f,d]] + (b\cdot f)[a,[d,e]] \\
&+ (a\cdot d)[[e,f],b] + (a\cdot e)[[f,d],b] + (a\cdot f)[[d,e],b]\}.
\end{aligned}
$$

Now by the identification of $\overline{T}$ as a submodule of $\mathrm{Hom}(H,\mathcal{L}_3)$ (see §2), the element $t = b\wedge c \leftrightarrow e\wedge f \in \overline{T}$ can be written as

$$
\begin{aligned}
t(u) =&(u\cdot b)[c,[e,f]] - (u\cdot c)[b,[e,f]] \\
&-(u\cdot e)[[b,c],f] + (u\cdot f)[[b,c],e].
\end{aligned}
$$

It is now a routine matter to check that the formula given in the theorem is the correct one. This completes the proof.

Here is an example which will clarify the effectiveness of Theorem 3.1.

Example 3.6. We consider a compact surface of genus two with one boundary component as illustrated in Figure 1, where we fix a symplectic basis x_1, x_2, y_1, y_2 of the homology group. Also let $\varphi, \psi \in \mathcal{I}_{2,1}$ and $\zeta \in \mathcal{K}_{2,1}$ be the elements defined there, where the $+$ (resp. $-$) sign means that we take the right handed (resp. left handed) Dehn twist on the corresponding simple closed curve.

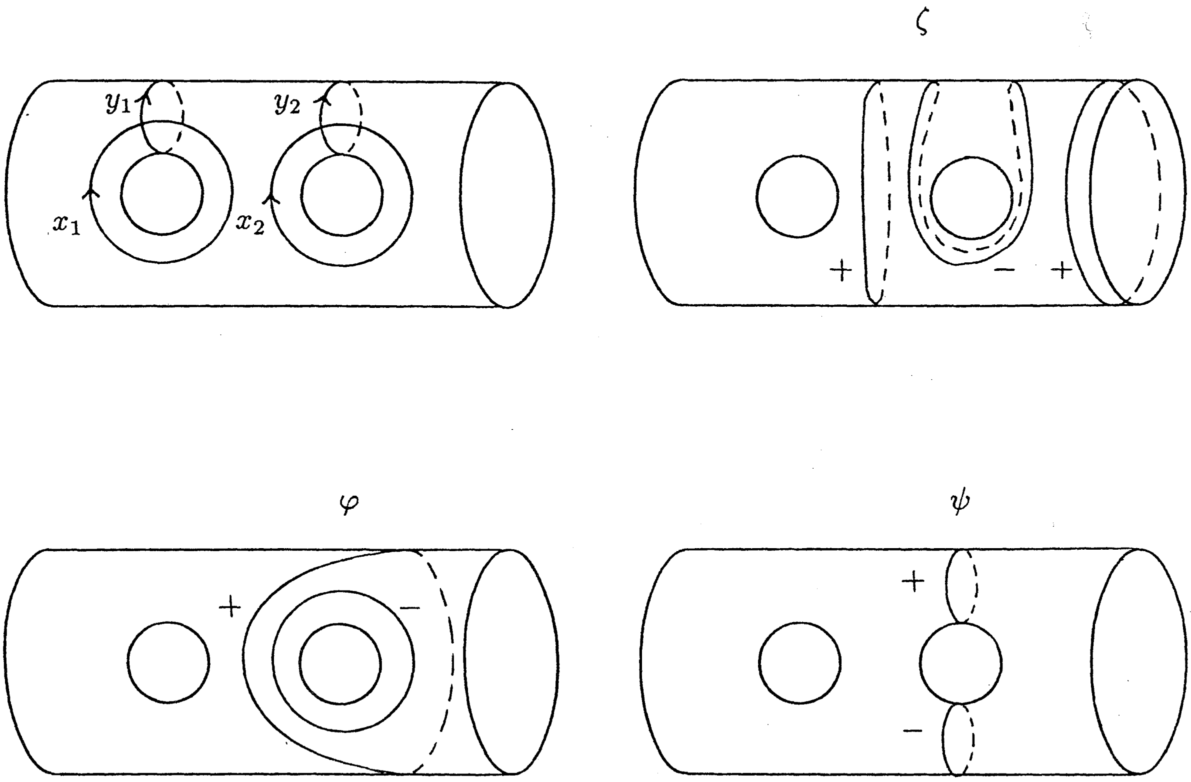

Figure 1

We can show by a direct computation that $[\varphi, \psi] = \zeta$. Now we compute (cf. [7,14])

$$\begin{aligned}
\tau_2(\varphi) =& x_1 \wedge y_1 \wedge x_2 \\
\tau_2(\psi) =& x_1 \wedge y_1 \wedge y_2 \\
\tau_3(\zeta) =& -(x_1 \wedge y_1)^{\otimes 2} + (x_2 \wedge y_2)^{\otimes 2} - (x_1 \wedge y_1 + x_2 \wedge y_2)^{\otimes 2} \\
=& - x_1 \wedge y_1 \leftrightarrow x_1 \wedge y_1 - x_1 \wedge y_1 \leftrightarrow x_2 \wedge y_2.
\end{aligned}$$

On the other hand, by Theorem 3.1 we have

$$\begin{aligned}
&\chi(x_1 \wedge y_1 \wedge x_2, x_1 \wedge y_1 \wedge y_2) \\
=& - y_1 \wedge x_2 \leftrightarrow y_2 \wedge x_1 + x_2 \wedge x_1 \leftrightarrow y_1 \wedge y_2 - x_1 \wedge y_1 \leftrightarrow x_1 \wedge y_1 \\
\equiv& - x_1 \wedge y_1 \leftrightarrow x_1 \wedge y_1 - x_1 \wedge y_1 \leftrightarrow x_2 \wedge y_2
\end{aligned}$$

as elements of $\overline{T}$, because $x_1 \wedge y_1 \leftrightarrow x_2 \wedge y_2 - y_1 \wedge x_2 \leftrightarrow y_2 \wedge x_1 + x_2 \wedge x_1 \leftrightarrow y_1 \wedge y_2$ is contained in T_0. This checks Theorem 3.1 in this case. Observe here that although the Euler class χ has a meaning with values in T (not just in $\overline{T}$), it does *not* give the correct answer. For the genus two case we can modify χ to obtain the correct one with values in T by adding a constant term, but here we omit it.

4. The mapping $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$

Recall that in § 2 we have defined a mapping $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$ by $\lambda_f(\varphi) = \lambda(M_\varphi) - \lambda(M)$ ($\varphi \in \mathcal{I}_g$) where M_φ is the homology 3-sphere obtained by cutting a given oriented homology 3-sphere M along an embedded oriented surface $f(\Sigma_g) \subset M$ and then reglueing the resulting two pieces V_+ and V_- by the map φ. Although the restriction of λ_f to the subgroup $\mathcal{K}_g$ is a homomorphism, λ_f itself is not a homomorphism (except the case $g = 2$). The purpose of this section is to determine completely the deviation of the mapping λ_f from the additivity in the case where f is a Heegaard embedding (see Remark 4.7 for the general case). In order to formulate the result, we define a mapping $\delta_f : \mathcal{I}_g \times \mathcal{I}_g \to \mathbf{Z}$ as follows. First as in §2 we fix an embedded disc $D^2 \subset \Sigma_g$ and consider the Torelli group $\mathcal{I}_{g,1}$ of Σ_g relative to D^2. Then we have the homomorphism $\tau_2 : \mathcal{I}_{g,1} \to \Lambda^3 H$. Next we denote H_x for the kernel of the homomorphism $H = H_1(\Sigma_g;\mathbf{Z}) \to H_1(V_-;\mathbf{Z})$ which is induced from the mapping $\Sigma_g \to f(\Sigma_g) \subset V_-$ and similarly we denote H_y for the kernel of the homomorphism $H \to H_1(V_+;\mathbf{Z})$. Then since M is assumed to be a homology sphere, it is easy to see that H is decomposed as a direct sum $H_x \oplus H_y$ of isotropic subspaces H_x and H_y each of which has maximal rank g. It follows that we can choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of H such that $x_i \in H_x$ and $y_i \in H_y$ for all $i = 1, \cdots, g$. With these preparations, we have

Definition 4.1. *We define a mapping*

$$\delta_f : \mathcal{I}_g \times \mathcal{I}_g \to \mathbf{Z}$$

as follows. For any two elements $\varphi, \psi \in \mathcal{I}_g$, choose any of their lifts $\widetilde{\varphi}, \widetilde{\psi} \in \mathcal{I}_{g,1}$ with respect to the natural surjection $\mathcal{I}_{g,1} \to \mathcal{I}_g$. Write

$$\tau_2(\widetilde{\varphi}) = \sum_{i<j<k} a_{ijk}\; y_i \wedge y_j \wedge y_k + \textit{other terms}$$
$$\tau_2(\widetilde{\psi}) = \sum_{i<j<k} b_{ijk}\; x_i \wedge x_j \wedge x_k + \textit{other terms}$$

in terms of the basis of $\Lambda^3 H$: $x_i \wedge x_j \wedge x_k$ $(i<j<k)$, $x_i \wedge x_j \wedge y_k$ $(i<j)$, $x_i \wedge y_j \wedge y_k$ $(j<k)$, and $y_i \wedge y_j \wedge y_k$ $(i<j<k)$ which is induced from the symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of H chosen above. Then we set

$$\delta_f(\varphi, \psi) = \sum_{i<j<k} a_{ijk} b_{ijk}.$$

Lemma 4.2. *The above definition is well defined. Namely the value $\delta_f(\varphi, \psi)$ does not depend on the various choices made.*

Proof. We have to prove that the value is independent of the

(i) choices of the lifts $\widetilde{\varphi}, \widetilde{\psi} \in \mathcal{I}_{g,1}$ and

(ii) choice of the symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$.

First we consider (i). Recall that $\mathrm{Ker}(\mathcal{I}_{g,1} \to \mathcal{I}_g)$ is naturally isomorphic to $\pi_1(T_1\Sigma_g)$ where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g. Also according to Johnson [7], $\tau_2(\mathrm{Ker}(\mathcal{I}_{g,1} \to \mathcal{I}_g)) \subset \Lambda^3 H$ can be identified as $\{\omega_0 \wedge u; u \in H\}$ where $\omega_0 = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$ is the "symplectic class" of H. It follows immediately that the value $\delta_f(\varphi, \psi)$ is independent of the choices of $\widetilde{\varphi}$ and $\widetilde{\psi}$.

Next we consider (ii). Since we are only considering those symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ such that $x_i \in H_x$ and $y_i \in H_y$ for all i, any two such bases are related by a matrix of the form

$$\begin{pmatrix} A & O \\ O & {}^t A^{-1} \end{pmatrix} \in \mathrm{Sp}(2g; \mathbf{Z})$$

for some $A \in \mathrm{GL}(g; \mathbf{Z})$. As is well known $\mathrm{GL}(g; \mathbf{Z})$ is generated by the following matrices

$$R = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad T_{\ell m} = \begin{array}{cc} & \begin{array}{cccc} & & m & \end{array} \\ \ell & \begin{pmatrix} 1 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{array} \quad (\ell \neq m).$$

Now it is easy to check the relevant invariance under the basis change corresponding to the matrix R. We check it for the matrix $T_{\ell m}$. The new symplectic basis $x'_1, \cdots, x'_g, y'_1, \cdots, y'_g$ which is obtained from the old one by applying the matrix $T_{\ell m}$ is expressed as

$$x'_k = x_k \ (k \neq m),\ x'_m = x_\ell + x_m \text{ and}$$
$$y'_k = y_k \ (k \neq \ell),\ y'_\ell = y_\ell - y_m.$$

Hence if we represent the given two elements $\tau_2(\widetilde{\varphi})$ and $\tau_2(\widetilde{\psi})$ with respect to the new basis as $\tau_2(\widetilde{\varphi}) = \sum a'_{ijk} y'_i \wedge y'_j \wedge y'_k + \textit{other terms}$ and $\tau_2(\widetilde{\psi}) = \sum b'_{ijk} x'_i \wedge x'_j \wedge x'_k + \textit{other terms}$, then $a'_{ijk} = a_{ijk}$ *unless* one of the indices i, j, k is equal to m and the other two are different from ℓ. Similarly $b'_{ijk} = b_{ijk}$ *unless* one of i, j, k is equal to ℓ and the other two are different from m. Moreover we have

$$a'_{\ell jk} = a_{\ell jk} + a_{mjk} \ (j, k \neq \ell, m)$$
$$b'_{mjk} = b_{mjk} - b_{\ell jk} \ (j, k \neq \ell, m).$$

It follows that $a'_{\ell jk} b'_{\ell jk} + a'_{mjk} b'_{mjk} = a_{\ell jk} b_{\ell jk} + a_{mjk} b_{mjk}$. Hence the sum $\sum a_{ijk} b_{ijk}$ is invariant under (ii). This completes the proof.

Now we are ready to state the main theorem of this section.

Theorem 4.3. *Let M be an oriented homology 3-sphere and let $f : \Sigma_g \to M$ be a Heegaard embedding. Also let $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$ be the mapping defined as $\lambda_f(\varphi) = \lambda(M_\varphi) - \lambda(M)$ ($\varphi \in \mathcal{I}_g$) where M_φ is the homology sphere obtained from M by cutting along $f(\Sigma_g)$ and then pasting back by the map $\varphi \in \mathcal{I}_g$. Then we have*

$$\lambda_f(\varphi\psi) = \lambda_f(\varphi) + \lambda_f(\psi) + 2\delta_f(\varphi, \psi)$$

for all $\varphi, \psi \in \mathcal{I}_g$.

Corollary 4.4. *(i) The mapping $\lambda_f : \mathcal{I}_g \to \mathbf{Z}$ is a homomorphism for $g = 2$.*

(ii) If we define $\overline{\lambda}_f : \mathcal{I}_g \to \mathbf{Z}/2$ to be the mod 2 *reduction of λ_f, then $\overline{\lambda}_f$ is a homomorphism for all g.*

Remark 4.5. As is well known the mod 2 reduction of the Casson invariant is equal to the Rohlin invariant. Hence statement (ii) in the above corollary is due to Birman-Craggs [2] (see also Remark 4.7).

Before proving Theorem 4.3, we prepare a few preliminary results.

Proposition 4.5. *Let M be an oriented homology 3-sphere and let $f : \Sigma_g \to M$ be an embedding. Then for any two elements $\varphi, \psi \in \mathcal{I}_g$, we have*

$$\lambda_f([\varphi, \psi]) = 2\delta_f(\varphi, \psi) - 2\delta_f(\psi, \varphi)$$

Proof. Fix an embedded disc $D^2 \subset \Sigma_g$ and we consider everything at the level of the group $\mathcal{I}_{g,1}$ rather than $\mathcal{I}_g$. Now first of all we claim that the following two assertions hold:

(i) the value $\lambda_f([\varphi, \psi])$ depends only on $\tau_2(\varphi)$ and $\tau_2(\psi) \in \Lambda^3 H$ for any $\varphi, \psi \in \mathcal{I}_{g,1}$.

(ii) $\lambda_f([\varphi_1 \cdots \varphi_s, \psi_1 \cdots \psi_t]) = \sum_{i=1}^{s} \sum_{j=1}^{t} \lambda_f([\varphi_i, \psi_j])$ for any $\varphi_i, \psi_j \in \mathcal{I}_{g,1}$.

In view of the existence of the exact sequence $1 \to \mathcal{K}_{g,1} \to \mathcal{I}_{g,1} \to \Lambda^3 H \to 1$, we have only to prove that $\lambda_f([\varphi, \psi]) = \lambda_f([\varphi\varphi_1, \psi\psi_1])$ for any $\varphi_1, \psi_1 \in \mathcal{K}_{g,1}$. But we have $[\varphi\varphi_1, \psi] = \varphi[\varphi_1, \psi]\varphi^{-1}[\varphi, \psi]$. Hence we obtain $\lambda_f([\varphi\varphi_1, \psi]) = \lambda_f([\varphi, \psi])$ by Proposition 2.1 and Corollary 2.3. Similarly we have $\lambda_f([\varphi\varphi_1, \psi\psi_1]) = \lambda_f([\varphi\varphi_1, \psi])$ proving (i). To prove (ii), we first assume that $s = 2$ and $t = 1$. Then we have $[\varphi_1\varphi_2, \psi] = \varphi_1[\varphi_2, \psi]\varphi_1^{-1}[\varphi_1, \psi]$ and hence $\lambda_f([\varphi_1\varphi_2, \psi]) = \lambda_f([\varphi_1, \psi]) + \lambda_f([\varphi_2, \psi])$ by Proposition 2.1 and Corollary 2.3 again. The general case follows from this by an easy inductive argument.

In view of the above two assertions (i) and (ii), for any two elements $\xi, \eta \in \Lambda^3 H$ we may write $\lambda_f([\xi, \eta])$ instead of $\lambda_f([\varphi, \psi])$ where $\varphi, \psi \in \mathcal{I}_{g,1}$ are any elements such that $\tau_2(\varphi) = \xi$ and $\tau_2(\psi) = \eta$. Also the value $\lambda_f([\xi, \eta])$ is "bilinear" with respect to

ξ and η. Now we choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of H such that x_i is homologous to 0 in V_- and y_i is homologous to 0 in V_+ for all i. For any element $u \in H$, let u^+ be the cycle in V_- which is obtained by pushing u to the positive direction. Then by a standard argument we have

$$\ell k(x_i, y_j^+) = -\delta_{ij} \text{ and}$$
$$\ell k(x_i, x_j^+) = \ell k(y_i, y_j^+) = 0.$$

Now we compute $\lambda_f([\xi, \eta])$ where ξ and η run through any member of the basis of $\Lambda^3 H$: $x_i \wedge x_j \wedge x_k$ $(i < j < k)$, $x_i \wedge x_j \wedge y_k$ $(i < j)$, $x_i \wedge y_j \wedge y_k$ $(j < k)$ and $y_i \wedge y_j \wedge y_k$ $(i < j < k)$ which is associated to the symplectic basis of H chosen above. To determine the value $\lambda_f([\xi, \eta])$ we first use Theorem 3.1 to compute $\tau_3([\xi, \eta])$ and then apply Theorem 2.2 and Theorem 2.4. We give the results in the following list (we refer to [14] for the explicit evaluations of the invariants $\theta_0 = \varepsilon_f \circ \theta, \overline{d}$ and d). In the list we denote $r = r(\xi, \eta)$ for $\chi(\xi \wedge \eta) \in T$ which is a lift of the element $\tau_3([\xi, \eta]) \in \overline{T}$ to T (recall that χ has a meaning with values in T) and also recall that $\lambda_f = \frac{1}{24} d + \theta_0 + \frac{1}{3}\overline{d}$ (see § 2).

(I) The case where $\xi = x_i \wedge x_j \wedge x_k$, $\eta = x_\ell \wedge x_m \wedge x_n$.

In this case $r = r(\xi, \eta) = 0$ and hence $\theta_0(r) = \overline{d}(r) = 0$. Also we have $d([\xi, \eta]) = 0$. Therefore we can conclude that $\lambda_f([\xi, \eta]) = 0$.

(II) The case where $\xi = x_i \wedge x_j \wedge x_k$, $\eta = x_\ell \wedge x_m \wedge y_n$.

In this case we have

$$r(\xi, \eta) = -\delta_{in} x_j \wedge x_k \leftrightarrow x_\ell \wedge x_m - \delta_{jn} x_k \wedge x_i \leftrightarrow x_\ell \wedge x_m - \delta_{kn} x_i \wedge x_j \leftrightarrow x_\ell \wedge x_m$$

and hence $\theta_0(r) = \overline{d}(r) = 0$. Also we have $d([\xi, \eta]) = 0$. Therefore $\lambda_f([\xi, \eta]) = 0$.

(III) The case where $\xi = x_i \wedge x_j \wedge x_k$, $\eta = x_\ell \wedge y_m \wedge y_n$.

In this case we have

$$r(\xi,\eta) = -\,\delta_{im}x_j \wedge x_k \leftrightarrow y_n \wedge x_\ell - \delta_{in}x_j \wedge x_k \leftrightarrow x_\ell \wedge y_m$$
$$-\,\delta_{jm}x_k \wedge x_i \leftrightarrow y_n \wedge x_\ell - \delta_{jn}x_k \wedge x_i \leftrightarrow x_\ell \wedge y_m$$
$$-\,\delta_{km}x_i \wedge x_j \leftrightarrow y_n \wedge x_\ell - \delta_{kn}x_i \wedge x_j \leftrightarrow x_\ell \wedge y_m$$

and hence $\theta_0(r) = \overline{d}(r) = 0$. Also we have $d([\xi,\eta]) = 0$ and we conclude $\lambda_f([\xi,\eta]) = 0$.

(IV) The case where $\xi = x_i \wedge x_j \wedge x_k$, $\eta = y_\ell \wedge y_m \wedge y_n$.

In this case we have

$$r(\xi,\eta) = -\,\delta_{i\ell}x_j \wedge x_k \leftrightarrow y_m \wedge y_n - \delta_{im}x_j \wedge x_k \leftrightarrow y_n \wedge y_\ell - \delta_{in}x_j \wedge x_k \leftrightarrow y_\ell \wedge y_m$$
$$-\,\delta_{j\ell}x_k \wedge x_i \leftrightarrow y_m \wedge y_n - \delta_{jm}x_k \wedge x_i \leftrightarrow y_n \wedge y_\ell - \delta_{jn}x_k \wedge x_i \leftrightarrow y_\ell \wedge y_m$$
$$-\,\delta_{k\ell}x_i \wedge x_j \leftrightarrow y_m \wedge y_n - \delta_{km}x_i \wedge x_j \leftrightarrow y_n \wedge y_\ell - \delta_{kn}x_i \wedge x_j \leftrightarrow y_\ell \wedge y_m.$$

Hence we have $\theta_0(r) = -\overline{d}(r) = -3\delta_{i\ell}\delta_{jm}\delta_{kn}$ (here we have used the assumptions that $i < j < k$ and $\ell < m < n$). Also we have $d([\xi,\eta]) = 0$. Therefore we obtain $\lambda_f([\xi,\eta]) = -2\delta_{i\ell}\delta_{jm}\delta_{kn}$.

(V) The case where $\xi = x_i \wedge x_j \wedge y_k$, $\eta = x_\ell \wedge x_m \wedge y_n$.

In this case we have

$$r(\xi,\eta) = -\,\delta_{in}x_j \wedge y_k \leftrightarrow x_\ell \wedge x_m - \delta_{jn}y_k \wedge x_i \leftrightarrow x_\ell \wedge x_m$$
$$+\,\delta_{k\ell}x_i \wedge x_j \leftrightarrow x_m \wedge y_n + \delta_{km}x_i \wedge x_j \leftrightarrow y_n \wedge x_\ell$$

and hence we obtain $\theta_0(r) = \overline{d}(r) = 0$. Also we have $d(r) = 0$ and therefore $\lambda_f([\xi,\eta]) = 0$.

(VI) The case where $\xi = x_i \wedge x_j \wedge y_k$, $\eta = x_\ell \wedge y_m \wedge y_n$.

In this case we have

$$\begin{aligned} r(\xi,\eta) = & -\delta_{im}x_j \wedge y_k \leftrightarrow y_n \wedge x_\ell - \delta_{in}x_j \wedge y_k \leftrightarrow x_\ell \wedge y_m \\ & - \delta_{jm}y_k \wedge x_i \leftrightarrow y_n \wedge x_\ell - \delta_{jn}y_k \wedge x_i \leftrightarrow x_\ell \wedge y_m \\ & + \delta_{k\ell}x_i \wedge x_j \leftrightarrow y_m \wedge y_n \end{aligned}$$

and hence $\theta_0(r) = \delta_{k\ell}\delta_{im}\delta_{jn}$. Recalling the conditions $i < j$ and $m < n$, we compute

$$\begin{aligned} \overline{d}(r) = & \delta_{im}\delta_{jk}\delta_{n\ell} - \delta_{in}\delta_{jk}\delta_{\ell m} - \delta_{jm}\delta_{ik}\delta_{\ell n} \\ & + \delta_{jn}\delta_{ik}\delta_{\ell m} - 3\delta_{im}\delta_{jn}\delta_{k\ell} \\ d(r) = & 8(\delta_{jk}\delta_{\ell m}\delta_{in} - \delta_{jk}\delta_{n\ell}\delta_{im} - \delta_{ik}\delta_{\ell m}\delta_{jn} + \delta_{ik}\delta_{n\ell}\delta_{jm}). \end{aligned}$$

Hence we can conclude that $\lambda_f([\xi,\eta]) = 0$.

In the above list we omit four other cases like $\xi = y_i \wedge y_j \wedge y_k$ and $\eta = y_\ell \wedge y_m \wedge y_n$ because of a symmetry in the above computations with respect to x and y. Now we can easily read off from the above list that the only possible pairs of (ξ,η) with non-zero λ_f is $(x_i \wedge x_j \wedge x_k, y_i \wedge y_j \wedge y_k)$ and $(y_i \wedge y_j \wedge y_k, x_i \wedge x_j \wedge x_k)$ and in these cases the values are -2 and 2 respectively. Therefore we can conclude that

$$\lambda_f([\varphi,\psi]) = 2\delta_f(\varphi,\psi) - 2\delta_f(\psi,\varphi).$$

This completes the proof of Proposition 4.5.

Lemma 4.6. *Let V be a handlebody of genus g and fix any diffeomorphism $\partial V \cong \Sigma_g$. Let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g relative to an embedded disc $D^2 \subset \Sigma_g$ and let $\mathcal{N}_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which can be extended to diffeomorphisms of V (under the above identification $\partial V = \Sigma_g$). Let H_y be the kernel of the homomorphism $H = H_1(\Sigma_g;\mathbf{Z}) \to H_1(V;\mathbf{Z})$, which is induced from the inclusion $\Sigma_g = \partial V \to V$, and choose any symplectic basis*

$x_1, \cdots, x_g, y_1, \cdots, y_g$ of H such that $y_i \in H_y$ for all $i = 1, \cdots, g$. Define the subgroup W_y of $\Lambda^3 H$ to be the one generated by the elements of the forms $x_i \wedge x_j \wedge y_k$, $x_i \wedge y_j \wedge y_k$ and $y_i \wedge y_j \wedge y_k$. Then we have

$$\tau_2(\mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}) = W_y.$$

We omit the proof of the above lemma because it follows from Lemma 2.5 of [14] by an easy argument.

Proof of Theorem 4.3. By the assumption that $f : \Sigma_g \to M$ is a Heegaard embedding, both of the pieces V_+ and V_- are handlebodies of genus g. Now as before choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of H such that x_i is homologous to 0 in V_- and y_i is homologous to 0 in V_+ for all i. Also fix an embedded disc $D^2 \subset \Sigma_g$ and choose any lifts $\widetilde{\varphi}, \widetilde{\psi} \in \mathcal{I}_{g,1}$ of the given elements $\varphi, \psi \in \mathcal{I}_g$. Now write

$$\tau_2(\widetilde{\varphi}) = \sum_{i<j<k} a_{ijk}\ y_i \wedge y_j \wedge y_k + \xi_x$$
$$\tau_2(\widetilde{\psi}) = \sum_{i<j<k} b_{ijk}\ x_i \wedge x_j \wedge x_k + \eta_y$$

with respect to the basis of $\Lambda^3 H$ which is associated to the symplectic basis of H chosen above. If we apply Lemma 4.6 to the pair $(V_+, f(\Sigma_g))$, then we can conclude that there exists two elements $\varphi_+, \psi_+ \in \mathcal{I}_{g,1}$ such that (i) φ_+ extends to a diffeomorphism of V_+ (here we identify $f(\Sigma_g)$ with ∂V_+) and $\tau_2(\varphi_+) = -\sum a_{ijk} y_i \wedge y_j \wedge y_k$ (ii) ψ_+ extends to a diffeomorphism of V_+ and $\tau_2(\psi_+) = -\eta_y$. The same argument applied to the pair $(V_-, f(\Sigma_g))$ implies the existence of elements $\varphi_-, \psi_- \in \mathcal{I}_{g,1}$ such that (iii) φ_- extends to a diffeomorphism of V_- and $\tau_2(\varphi_-) = -\xi_x$ (iv) ψ_- extends to a diffeomorphism of V_- and $\tau_2(\psi_-) = -\sum b_{ijk} x_i \wedge x_j \wedge x_k$. Now set $\varphi_1 = \varphi_- \widetilde{\varphi}, \psi_1 = \widetilde{\psi}\psi_+$, $\varphi_2 = \varphi_- \widetilde{\varphi}\varphi_+$ and $\psi_2 = \psi_- \widetilde{\psi}\psi_+$. Then since φ_+ and ψ_+ (resp. φ_- and ψ_-) extends to diffeomorphisms of V_+ (resp. V_-), we have $M_{\varphi_1\psi_1} = M_{\varphi\psi}$, $M_{\varphi_2} = M_\varphi$ and

$M_{\psi_2} = M_\psi$. Hence

$$\begin{aligned}\lambda_f(\varphi_1\psi_1) =&\lambda_f(\varphi\psi)\\ \lambda_f(\varphi_2) =&\lambda_f(\varphi)\\ \lambda_f(\psi_2) =&\lambda_f(\psi).\end{aligned}$$

Also observe that both of the elements φ_2 and ψ_2 belong to $\mathcal{K}_{g,1}$ because $\tau_2(\varphi_2) = \tau_2(\psi_2) = 0$. Similarly we have $M_{\varphi_1\psi_1} = M_{\psi_-\varphi_1\psi_1\varphi_+}$ and hence

$$\lambda_f(\varphi_1\psi_1) = \lambda_f(\psi_-\varphi_1\psi_1\varphi_+).$$

On the other hand we have

$$\psi_-\varphi_1\psi_1\varphi_+ = \psi_-\varphi_2\psi_-^{-1}[\psi_-, \varphi_+^{-1}]\varphi_+^{-1}\psi_2\varphi_+$$

and therefore

$$\lambda_f(\psi_-\varphi_1\psi_1\varphi_+) = \lambda_f(\psi_-\varphi_2\psi_-^{-1}) + \lambda_f([\psi_-, \varphi_+^{-1}]) + \lambda_f(\varphi_+^{-1}\psi_2\varphi_+)$$

by Proposition 2.1. But we have $\lambda_f(\psi_-\varphi_2\psi_-^{-1}) = \lambda_f(\varphi_2)$ and $\lambda_f(\varphi_+^{-1}\psi_2\varphi_+) = \lambda_f(\psi_2)$ by Corollary 2.3,(i). If we combine the above equations, we see that

$$\lambda_f(\varphi\psi) = \lambda_f(\varphi) + \lambda_f(\psi) + \lambda_f([\psi_-, \varphi_+^{-1}]).$$

Here recall that $\tau_2(\psi_-) = -\sum b_{ijk}x_i \wedge x_j \wedge x_k$ and $\tau_2(\varphi_+) = -\sum a_{ijk}y_i \wedge y_j \wedge y_k$. Hence $\lambda_f([\psi_-, \varphi_+^{-1}) = \lambda_f([\varphi_+, \psi_-]) = \sum a_{ijk}b_{ijk} = 2\delta_f(\varphi, \psi)$ by Proposition 4.5 and the definition of δ_f. Therefore

$$\lambda_f(\varphi\psi) = \lambda_f(\varphi) + \lambda_f(\psi) + 2\delta_f(\varphi, \psi)$$

as required. This completes the proof.

Remark 4.7. In the general case where the embedding $f : \Sigma_g \to M$ is *not* Heegaard, the formula of Theorem 4.3 seems to be not true in general anymore. However we can still conclude that the mapping $\overline{\lambda}_f : \mathcal{I}_g \to \mathbf{Z}/2$, which is the mod 2 reduction of λ_f, is a homomorphism. This follows from the fact that we can modify the embedded surface $f(\Sigma_g)$ to a Heegaard one by adding 1-handles from both sides.

5. Determination of $H^1(\mathcal{K}_g; \mathbf{Z})^{\mathcal{M}_g}$

Recall that we denote $\mathcal{K}_g$ for the subgroup of the mapping class group $\mathcal{M}_g$ generated by all the Dehn twists on separating simple closed curves on Σ_g. The purpose of this section is to determine $H^1(\mathcal{K}_g; \mathbf{Z})^{\mathcal{M}_g}$ which is the group consisting of all homomorphisms $r : \mathcal{K}_g \to \mathbf{Z}$ such that $r(\psi) = r(\varphi\psi\varphi^{-1})$ for all $\psi \in \mathcal{K}_g$ and $\varphi \in \mathcal{M}_g$. Namely we will determine all the integer valued additive invariants for elements of $\mathcal{K}_g$ which are invariant under the natural action of $\mathcal{M}_g$. We also determine the related group $H^1(\mathcal{K}_{g,1}; \mathbf{Z})^{\mathcal{M}_{g,1}}$ which is the set of all $\mathcal{M}_{g,1}$-invariant homomorphisms $\mathcal{K}_{g,1} \to \mathbf{Z}$. We first consider the group $\mathcal{K}_{g,1}$. Recall that we have a homomorphism $d : \mathcal{K}_{g,1} \to \mathbf{Z}$ which is the "core" of the Casson invariant from the point of view of our approach (see § 2 and [14] for details). It has the property that for any BSCC map φ of genus h, we have $d(\varphi) = 4h(h-1)$. It follows that d is an element of $H^1(\mathcal{K}_{g,1}; \mathbf{Z})^{\mathcal{M}_{g,1}}$. Next we construct another element $d' \in H^1(\mathcal{K}_{g,1}; \mathbf{Z})^{\mathcal{M}_{g,1}}$ by making use of Johnson's homomorphism $\tau_3 : \mathcal{K}_{g,1} \to \overline{T}$ (see § 2 and [14]).

Proposition 5.1. *The following two types of correspondences*

(i) $\bar{d}'((a \wedge b)^{\otimes 2}) = -3(a \cdot b)^2$

(ii) $\bar{d}'(a \wedge b \leftrightarrow c \wedge d) = -4(a \cdot b)(c \cdot d) - 2(a \cdot c)(b \cdot d) + 2(a \cdot d)(b \cdot c)$

define a well-defined $\mathrm{Sp}(2g; \mathbf{Z})$*-invariant homomorphism* $\bar{d}' : T \to \mathbf{Z}$*. Moreover it is trivial on* $\mathrm{Ker}(T \to \overline{T})$ *so that it defines a homomorphism* $\bar{d}' : \overline{T} \to \mathbf{Z}$ *(we use the same letter).*

Definition 5.2. *We define a homomorphism* $d' : \mathcal{K}_{g,1} \to \mathbf{Z}$ *to be the composition*

$$\mathcal{K}_{g,1} \xrightarrow{\tau_3} \overline{T} \xrightarrow{\bar{d}'} \mathbf{Z}.$$

It is easy to see that d' *is* $\mathcal{M}_{g,1}$*-invariant so that it is contained in* $H^1(\mathcal{K}_{g,1}; \mathbf{Z})^{\mathcal{M}_{g,1}}$*.*

Proof of Proposition 5.1. First of all it is easy to check that the two formulae (i) and (ii) are consistent with the relation $a \wedge b \leftrightarrow a \wedge b = 2(a \wedge b)^{\otimes 2}$. Next observe that type (ii) correspondence is linear with respect to any variables and also it is skew symmetric with respect to a, b and c, d respectively. Moreover the value remains to be unchanged if we interchange the two pairs (a, b) and (c, d). It follows that the correspondence is well-defined. That the resulting homomorphism $\bar{d}'$ is $\mathrm{Sp}(2g; \mathbf{Z})$-invariant follows directly from the definition. Finally we check that $\bar{d}'$ is trivial on $\mathrm{Ker}(T \to \overline{T})$. Recall that $\mathrm{Ker}(T \to \overline{T})$ is generated by the elements of the form

$$a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c.$$

But a direct computation shows that the value of the homomorphism $\bar{d}'$ on this element is 0. This completes the proof.

Proposition 5.3. *Let* $\varphi \in \mathcal{K}_{g,1}$ *be a BSCC map of genus* h*. Then we have* $d'(\varphi) = h(2h+1)$*.*

Proof. Let ω be the bounding simple closed curve on $\Sigma_g \setminus D^2$ correspondint to the element φ. Choose a symplectic basis $u_1, \cdots, u_h, v_1, \cdots, v_h$ of the homology of the subsurface which ω bounds. Then by Proposition 1.1 of [14], we have

$$\tau_3(\varphi) = -(u_1 \wedge v_1 + \cdots + u_h \wedge v_h)^{\otimes 2}.$$

On the other hand, a direct computation shows that

$$\bar{d}'((u_1 \wedge v_1 + \cdots + u_h \wedge v_h)^{\otimes 2}) = -h(2h+1).$$

Hence $d'(\varphi) = h(2h+1)$, completing the proof.

Theorem 5.4. *The group $H^1(\mathcal{K}_{g,1}\,; \mathbf{Z})^{\mathcal{M}_{g,1}}$ is a free abelian group of rank two and over the rationals it is generated by d and d'.*

Proof. It is easy to see that two BSCC maps of the same genus are conjugate each other in $\mathcal{M}_{g,1}$. On the other hand, according to Johnson [5], $\mathcal{K}_{g,1}$ is generated by all BSCC maps of genus one and two. Hence the rank of $H^1(\mathcal{K}_{g,1}\,; \mathbf{Z})^{\mathcal{M}_{g,1}}$ is at most two. Since we have already constructed two elements d, d' of this group which are clearly linearly independent, we are done.

Remark 5.5. We can restate the above theorem as follows. Namely a function $f(h)$ of h, where h stands for the genera of BSCC maps in $\mathcal{K}_{g,1}$, extends to an $\mathcal{M}_{g,1}$-invariant homomorphism $\mathcal{K}_{g,1} \to \mathbf{Z}$ if and only if $f(h) = ph + qh^2$ for some $p, q \in \mathbf{Z}$.

Remark 5.6. Although the two homomorphisms $d, d' : \mathcal{K}_{g,1} \to \mathbf{Z}$ are seemingly similar each other, the essential meaning of them are completely different. More precisely, for each element $\varphi \in \mathcal{K}_{g,1}$ let us consider the manifold W_φ which is a Σ_g-bundle over the circle with its monodromy diffeomorphism equal to φ. By the assumption that $\varphi \in \mathcal{K}_{g,1}$, the homology group of W_φ is the same as that of $S^1 \times \Sigma_g$. But higher order intersectional properties may be different. In fact as is explained in Johnson [8], his homomorphisms τ_k can be interpreted as the invariants which measure higher order Massey products of the manifolds like W_φ. Since our invariant d' is a quotient of τ_3, we can conclude that d' can be read off from a certain higher order Massey product of W_φ. In particular it is a *local invariant* in the sense that it is computable by means of integrations of forms. On the contrary we have recently found that the invariant d can be interpreted as the Hirzebruch's signature defect (see [4]) of the 3-manifold W_φ with respect to a certain canonical framing of its tangent bundle (see [16] for details). If we recall here that the signature defect is closely related with the η-invariant of Atiyah-Patodi-Singer [1] which is by no means a local invariant, we may say that the invariant d should be an essentially global invariant. At least we can say that d is a much more deep invariant than d'.

Now we consider the case of closed surfaces.

Theorem 5.7. *The group $H^1(\mathcal{K}_g;\mathbf{Z})^{\mathcal{M}_g}$ is isomorphic to $\mathbf{Z}$ whose rational generator d_0 can be defined as follows. Let ω be a separating simple closed curve on Σ_g such that the genera of the two compact surfaces which are obtained by cutting Σ_g along ω are h and $(g-h)$. Then we have $d_0(D_\omega) = h(g-h)$ where D_ω is the Dehn twist on ω.*

Proof. Since the natural homomorphism $H^1(\mathcal{K}_g;\mathbf{Z})^{\mathcal{M}_g} \to H^1(\mathcal{K}_{g,1};\mathbf{Z})^{\mathcal{M}_{g,1}}$ is cearly injective, in view of Theorem 5.4 it is enough to prove that there is one and only one relation between the two elements $d, d' \in H^1(\mathcal{K}_{g,1};\mathbf{Z})^{\mathcal{M}_{g,1}}$ on $\mathrm{Ker}(\mathcal{K}_{g,1} \to \mathcal{K}_g)$. More precisely, since we have

$$12h(g-h) = 4(g-1)\cdot h(2h+1) - (2g+1)\cdot 4h(h-1)$$

and since $d(\varphi) = 4h(h-1)$ and $d'(\varphi) = h(2h+1)$ for any BSCC map $\varphi \in \mathcal{K}_{g,1}$ of genus h, we have only to prove that the equality

$$(2g+1)d(\varphi) = 4(g-1)d'(\varphi)$$

holds for any $\varphi \in \mathrm{Ker}(\mathcal{K}_{g,1} \to \mathcal{K}_g)$ and that d is non trivial on $\mathrm{Ker}(\mathcal{K}_{g,1} \to \mathcal{K}_g)$. Now as was recalled in § 4, $\mathrm{Ker}(\mathcal{I}_{g,1} \to \mathcal{I}_g)$ is naturally isomorphic to $\pi_1(T_1\Sigma_g)$ and also we have an exact sequence $1 \to \mathcal{K}_{g,1} \to \mathcal{I}_{g,1} \to \Lambda^3 H \to 1$. Hence $\operatorname{Ker}(\mathcal{K}_{g,1} \to \mathcal{K}_g)$ is naturally isomorphic to $\operatorname{Ker}(\tau_2 : \pi_1(T_1\Sigma_g) \to \Lambda^3 H)$. On the other hand we have a central extension

$$0 \to \mathbf{Z} \to \pi_1(T_1\Sigma_g) \to \pi_1(\Sigma_g) \to 1$$

where the center $\mathbf{Z}$ is generated by the element $\zeta \in \mathcal{K}_{g,1}$, which is the Dehn twist on a simple closed curve on $\Sigma_g \setminus \operatorname{Int} D^2$ which is parallel to the boundary (see [14]). Since $\tau_2(\zeta) = 0$, the homomorphism τ_2 restricted to $\pi_1(T_1\Sigma_g)$ is essentially equivalent to a homomorphism $\tau_2 : \pi_1(\Sigma_g) \to \Lambda^3 H$ (we use the same letter) and $\tau_2(\gamma) = [\gamma] \wedge \omega_0$ for any $\gamma \in \pi_1(\Sigma_g)$, where $[\gamma] \in H$ is the homology class of γ and $\omega_0 \in \Lambda^2 H$ is the symplectic class (see [7]). We can now conclude that $\operatorname{Ker}(\mathcal{K}_{g,1} \to \mathcal{K}_g)$ is generated by ζ and the elements of the form $[\alpha, \beta] \in \mathcal{K}_{g,1}$ with $\alpha, \beta \in \pi_1(T_1\Sigma_g) \subset \mathcal{I}_{g,1}$. Now we check that the required relation on these generators.

Since $d(\zeta) = 4g(g-1)$ and $d'(\zeta) = g(2g+1)$ (see Proposition 5.3), the reuired relation $(2g+1)d = 4(g-1)d'$ certainly holds on ζ. Next we consider the element of the form $\varphi = [\alpha, \beta] \in \mathcal{K}_{g,1}$ with $\alpha, \beta \in \pi_1(T_1\Sigma_g)$. Write the corresponding homology classes $[\alpha], [\beta] \in H$ as

$$[\alpha] = \sum_i (a_i x_i + b_i y_i)$$
$$[\beta] = \sum_i (c_i x_i + d_i y_i)$$

where $x_1, \cdots, x_g, y_1, \cdots, y_g$ is a symplectic basis of H. If we apply Proposition 5.1,(iv) to the element $\varphi = [\alpha, \beta]$ we obtain

$$\begin{aligned} d(\varphi) &= 8(g-1)^2 [\alpha] \cdot [\beta] \\ &= 8(g-1)^2 \sum (a_i d_i - b_i c_i). \end{aligned}$$

Next we compute $d'(\varphi)$. We have $\tau_2(\alpha) = \sum_i (a_i x_i + b_i y_i) \wedge \omega_0$ and $\tau_2(\beta) = \sum_i (c_i x_i +$

$d_iy_i) \wedge \omega_0$ where $\omega_0 = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$. Therefore

$$\begin{aligned}\tau_3(\varphi) &= \chi(\tau_2(\alpha) \wedge \tau_2(\beta)) \\ &= \sum_{i,j,k,\ell} \{a_ic_k\chi(\xi_{ij} \wedge \xi_{k\ell}) + a_id_k\chi(\xi_{ij} \wedge \eta_{k\ell}) \\ &\qquad + b_ic_k\chi(\eta_{ij} \wedge \xi_{k\ell}) + b_id_k\chi(\eta_{ij} \wedge \eta_{k\ell})\}\end{aligned}$$

where $\xi_{ij} = x_i \wedge x_j \wedge y_j$, $\xi_{k\ell} = x_k \wedge x_\ell \wedge y_\ell$, $\eta_{ij} = y_i \wedge x_j \wedge y_j$ and $\eta_{k\ell} = y_k \wedge x_\ell \wedge y_\ell$. By Theorem 3.1 we have

$$\begin{aligned}\chi(\xi_{ij} \wedge \xi_{k\ell}) = &- \delta_{i\ell}x_j \wedge y_j \leftrightarrow x_k \wedge x_\ell - \delta_{j\ell}y_j \wedge x_i \leftrightarrow x_k \wedge x_\ell \\ &+ \delta_{jk}x_i \wedge x_j \leftrightarrow x_\ell \wedge y_\ell\end{aligned}$$

and hence $d'(\chi(\xi_{ij} \wedge \xi_{k\ell})) = 0$. Similarly we have

$$\begin{aligned}\chi(\xi_{ij} \wedge \eta_{k\ell}) = &- \delta_{ik}x_j \wedge y_j \leftrightarrow x_\ell \wedge y_\ell - \delta_{i\ell}x_j \wedge y_j \leftrightarrow y_k \wedge x_\ell \\ &- \delta_{jk}y_j \wedge x_i \leftrightarrow x_\ell \wedge y_\ell - \delta_{j\ell}y_j \wedge x_i \leftrightarrow y_k \wedge x_\ell \\ &+ \delta_{j\ell}x_i \wedge x_j \leftrightarrow y_\ell \wedge y_k\end{aligned}$$

and therefore

$$\begin{aligned}d'(\chi(\xi_{ij} \wedge \eta_{k\ell})) = &- \delta_{ik}(-4 - 2\delta_{j\ell}) - \delta_{i\ell}(4\delta_{k\ell} + 2\delta_{jk}\delta_{j\ell}) \\ &- \delta_{jk}(4\delta_{ij} + 2\delta_{j\ell}\delta_{i\ell}) - \delta_{j\ell}(-4\delta_{ji}\delta_{k\ell} - 2\delta_{j\ell}\delta_{ik}) \\ &+ \delta_{j\ell}(-2\delta_{i\ell}\delta_{jk} + 2\delta_{ik}\delta_{j\ell}).\end{aligned}$$

A direct computation shows that

$$\sum_{k,\ell} d'(\chi(\xi_{ij} \wedge \eta_{k\ell})) = \delta_{ik}2(g-1)(2g+1).$$

By an obvious anti-symmetry of the above computations with respect to x and y, we obtain

$$\begin{aligned}\sum_{k,\ell} d'(\chi(\eta_{ij}\wedge\xi_{k\ell})) &= -\,\delta_{ik}2(g-1)(2g+1) \text{ and}\\ d'(\chi(\eta_{ij}\wedge\eta_{k\ell})) &= 0.\end{aligned}$$

We can now conclude that

$$d'(\varphi) = 2(g-1)(2g+1)\sum(a_i d_i - b_i c_i).$$

Hence $(2g+1)d(\varphi) = 4(g-1)d'(\varphi)$ as required. This completes the proof.

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