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Instantons and representations of an associative algebra

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In this note we show that instantons on \( S^4 \) can be identified with some representations of an associative algebra.

Let \( A \) be the free algebra over \( \mathbb{C} \) generated by two elements \( q \), \( p \). We define a new product \( \ast \) in \( A \) as follows:

\[ f_1 \ast f_2 = f_1(pq - qp)f_2, \quad f_1, f_2 \in A. \]

Then \( (A, \ast) \) is an associative algebra (with no unit), which is an extension of the Weyl algebra \( A/(pq - qp - 1) \). We consider finite dimensional representations of \( (A, \ast) \). Let \( W \) be the complex vector space of dimension \( l \), and \( h \) be a linear map from \( A \) to \( \text{End} \ W \). Then \( h \) induces a linear map

\[ \tilde{h}: A \otimes W \to A^* \otimes W \]

defined by

\[ (\tilde{h}(f_1 \otimes w), f_2) = h(f_2f_1)w, \quad f_1, f_2 \in A, \ w \in W. \]

We denote by \( H(l, k) \) the set of all algebra homomorphisms \( h: (A, \ast) \to \text{End} \ W \) such that the rank of \( \tilde{h} \) is \( k \).

Let \( P \) be the principal \( SU(l) \) bundle over \( S^4 = \mathbb{R}^4 \cup \infty \) with \( c_2 = k \), and \( \tilde{M}(SU(l), k) \) be the framed moduli space for anti-self-dual (ASD) connections on \( P \): \{ ASD connections on \( P \} / \mathcal{G}_\infty \), where \( \mathcal{G}_\infty \) stands for the group of all gauge transformations on \( P \) fixing the points in the fiber over \( \infty \). \( \tilde{M}(SU(l), k) \) is a \( 4kl \)-dimensional smooth manifold.

Our main result is the following:

**Theorem 1.** The framed moduli space \( \tilde{M}(SU(l), k) \) is diffeomorphic to \( H(l, k) \).

§1. Some remarks on a theorem of Donaldson.

Let \( X = \text{Mat}(k, k; \mathbb{C}) \times \text{Mat}(k, k; \mathbb{C}) \times \text{Mat}(l, k; \mathbb{C}) \times \text{Mat}(k, l; \mathbb{C}) \). We define the action of \( G = GL(k, \mathbb{C}) \) on \( X \) as follows:

\[ p \cdot (\alpha_1, \alpha_2, a, b) = (p\alpha_1p^{-1}, p\alpha_2p^{-1}, ap^{-1}, pb) \]
for \( p \in G \), \((\alpha_1, \alpha_2, a, b) \in X\). We call a point \( x \) in \( X \) stable when the map \( G \ni p \mapsto p \cdot x \in X \) is proper. We denote by \( X^s \) the set of all stable points in \( X \). Let

\[
\omega(\alpha_1, \alpha_2, a, b) = \text{tr}(d\alpha_1 \wedge d\alpha_2 + db \wedge da),
\]

\[
\mu = \alpha_1 \alpha_2 - \alpha_2 \alpha_1 + ba.
\]

We can show by easy computation that

\[
\omega(p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb) = \omega(\alpha_1, \alpha_2, a, b) + \text{tr}(p^{-1}dp \wedge d\mu) + \text{tr}(p^{-1}dp \wedge p^{-1}dp \cdot \mu).
\]

(This is suggested to the author by H. Nakajima from the viewpoint of hyperkähler structure.)

**Theorem (Donaldson [1]).** The framed moduli space \( \widetilde{M}(SU(l), k) \) is diffeomorphic to \( G \setminus \mu^{-1}(0) \cap X^s \).

So we deduce from geometric invariant theory [4] that \( \widetilde{M}(SU(l), k) \) is an open dense nonsingular subset of an affine algebraic variety.

Next we seek a criterion for the stability in this case. Let \( A^m \in \text{Mat}(2^m l, k; \mathbb{C}) \) be the matrix which is the column of matrices \( a\alpha_i, \cdots \alpha_i^{m} \), \( i = 0, 1 \), and \( B^m \in \text{Mat}(k, 2^m l; \mathbb{C}) \) be the matrix which is the row of matrices \( \alpha_1 \cdots \alpha_i b \), i.e.

\[
A^0 = a, \ A^1 = (a_1 a_2), \ A^2 = (a_1 a_2 a_1 a_2), \ldots , \ A^m = (A^m a_1 a_2),
\]

\[
B^0 = b, \ B^1 = (a_1 b \ a_2 b), \ B^2 = (a_1 a_1 b \ a_1 a_2 b \ a_2 a_1 b \ a_2 a_2 b), \ldots , \ B^m = (a_1 B^{m-1} a_2 B^{m-1}).
\]

We set

\[
A_m = (A^0, \ldots , A^m), \quad B_m = (B^0, \ldots , B^m).
\]

**Lemma 2.** The point \( x = (\alpha_1, \alpha_2, a, b) \in X \) is stable if and only if \( \text{rank } A_{k-1} B_{k-1} = k \).

**Lemma 2'.** The point \( x = (\alpha_1, \alpha_2, a, b) \in X \) is stable if and only if \( \text{rank } A_m B_n = k \) for some \( m, n \).

**Proof:** We can test the stability of a point by the following:
**Hilbert Criterion ([1,4]).** The point $x \in X$ is stable for $G$ if and only if for all $g \in G$ and integers $(w_1, \ldots, w_k) \neq (0, \ldots, 0)$:

$$g \begin{pmatrix} t^{w_1} \\ \vdots \\ t^{w_k} \end{pmatrix} g^{-1} \cdot x \to \infty \quad \text{as} \quad t \to \infty.$$

**Claim:** If $\text{rank } A_{m+1} = \text{rank } A_m$, then $\text{rank } A_{m'} = \text{rank } A_m$ for all $m' \geq m$. Similarly, if $\text{rank } B_{m+1} = \text{rank } B_m$, then $\text{rank } B_{m'} = \text{rank } B_m$ for all $m' \geq m$.

**Proof:** Assume that $\text{rank } A_{m+1} = \text{rank } A_m$. Then the row vectors in $A^{m+1}$ can be written by the linear combinations of the row vectors in $A_m$. So the row vectors in $A^{m+2} = \begin{pmatrix} A^{m+1} & \alpha_1 \\ A^{m+1} & \alpha_2 \end{pmatrix}$ are the linear combinations of the row vectors in $A_m\alpha_1, A_m\alpha_2$, which are the row vectors in $A_{m+1}$. So $\text{rank } A_{m+2} = \text{rank } A_{m+1}$. The claim follows by induction.

Now we go back to the proof of Lemma 2, 2'. First we assume that $\text{rank } A_{k-1} = k' < k$. If $k = 1$, then $a = 0$ and

$$t^{-1} \cdot (\alpha_1, \alpha_2, a, b) = (\alpha_1, \alpha_2, 0, t^{-1}b) \to (\alpha_1, \alpha_2, 0, 0) \quad \text{as} \quad t \to \infty.$$

This implies that $(\alpha_1, \alpha_2, a, b)$ is not stable.

If $k > 1$, we deduce from the Claim that $\text{rank } A_{k-2} = k'$. So

$$A_{k-1}g = \begin{pmatrix} A_{k-2}g \\ A_{k-1}g \end{pmatrix} = \begin{pmatrix} A' & 0 \\ * & 0 \end{pmatrix},$$

for some $g \in G$, where the column vectors in $A'$ are linearly independent. Particularly, $ag = (\ast \ 0)$. Since the row vectors in $A_{k-2}\alpha_1$ are the ones in $A_{k-1}$,

$$(A' \ 0)g^{-1}\alpha_1g = (\ast \ 0).$$

This implies that $g^{-1}\alpha_1g = \begin{pmatrix} \ast & 0 \\ \ast & \ast \end{pmatrix}$. Similarly we get $g^{-1}\alpha_2g = \begin{pmatrix} \ast & 0 \\ \ast & \ast \end{pmatrix}$. So

$$\begin{pmatrix} 1_{k'} \\ t^{-1}1_{k-1} \end{pmatrix} \cdot (g^{-1}\alpha_1g, g^{-1}\alpha_2g, ag, g^{-1}b)$$

converges as $t \to \infty$. Therefore if $\text{rank } A_{k-1} < k$, then $x = (\alpha_1, \alpha_2, a, b)$ is not stable. Similarly, if $\text{rank } B_{k-1} < k$, $x$ is not stable.
Next we assume that \((\alpha_1, \alpha_2, a, b)\) is not stable. From the Hilbert Criterion we get some \(g \in G, (w_1, \ldots, w_k)\) such that
\[
\left(\begin{array}{c}
t^{w_1} \\
\vdots \\
t^{w_k}
\end{array}\right) : (g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, ag, g^{-1}b)
\]
converges as \(t \to \infty\). We may assume that \(w_1 \geq \ldots \geq w_k\). If \(w_{k'} > 0 > w_{k'+1}\), we deduce that
\[
ag = (\ast \ 0), \ g^{-1}\alpha_1 g = (\ast \ 0), \ g^{-1}\alpha_2 g = (\ast \ 0).
\]
This implies that \(A_m g = (\ast \ 0)\). Similarly, if \(w_{k'} > 0 \geq w_{k'+1}\), then \(g^{-1}B_n = \begin{pmatrix} 0 & * \\ & * \end{pmatrix}\). Therefore if \((\alpha_1, \alpha_2, a, b)\) is not stable, then rank \(A_m B_n < k\) for all \(m, n\).

§2 The proof of Theorem 1.

First we give the map \(\varphi\) from \(\overline{M}(SU(l), k)\) to \(H(l, k)\). Let
\[
h(f) = \varphi(\alpha_1, \alpha_2, a, b)(f) = af(\alpha_1, \alpha_2)b
\]
for \((\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0) \cap X^s\). \(\varphi\) is \(G\)-invariant. Since \(\mu(\alpha_1, \alpha_2, a, b) = 0\),
\[
h(f_1 * f_2) = h(f_1(pq - qp)f_2)
\]
\[
= af_1(\alpha_1, \alpha_2)(\alpha_2 \alpha_1 - \alpha_1 \alpha_2)f_2(\alpha_1, \alpha_2)b
\]
\[
= af_1(\alpha_1, \alpha_2)abf_2(\alpha_1, \alpha_2)b
\]
\[
= h(f_1)h(f_2).
\]
We give \(i: C^k \to A^* \otimes C^l\), \(j: A \otimes C^1 \to C^k\) by
\[
(i(v), f) = af(\alpha_1, \alpha_2)v
\]
\[
j(f \otimes w) = f(\alpha_1, \alpha_2)bw
\]
for \(f \in A\), \(v \in V\), \(w \in W\). Then we have \(\tilde{h} = i \circ j\). Lemma 2' implies that \(i\) is injective and that \(j\) is surjective, so rank \(\tilde{h} = k\). Therefore \(h \in H(l, k)\).

On the other hand, the inverse \(\psi: H(l, k) \to \overline{M}(SU(l), k)\) is defined as follows. For \(h' \in H(l, k)\), we set \(V = \text{Coim} \overline{h'} \cong \text{Im} \overline{h'} \cong C^k\). Let
\[
\overline{h}' = i' \circ j', \quad i': V \to A^* \otimes W,
\]
\[
j': A \otimes W \to V.
\]
For \( f \in A \) we define \( \langle f \rangle \in \text{Hom}(V, W) \), \( |f\rangle \in \text{Hom}(W, V) \) by
\[
\langle f \rangle(v) = (i'(v), f), \quad v \in V,
\]
\[
|f\rangle(w) = j'(f \otimes w), \quad w \in W.
\]

We set \( a' = (1|, b' = |1) \). The multiplications by \( q, p \) in \( A \) induce linear maps \( \alpha_1', \alpha_2' \in \text{End} V \) respectively:
\[
\alpha_1'|f\rangle = |qf\rangle, \quad \alpha_2'|f\rangle = |pf\rangle
\]
for \( f \in A \). If \( |f\rangle = 0 \), then \( h(f'f) = 0 \) for all \( f' \in A \). So \( \alpha_1', \alpha_2' \in \text{End} V \) are well-defined. We get
\[
\psi(h') = (\alpha_1', \alpha_2', a', b') \in X
\]
by fixing the basis of \( V, W \). Since
\[
\bigcap_{f \in A} \ker a'f(\alpha_1', \alpha_2') = \bigcap_{f \in A} \ker \langle f \rangle = 0,
\]
\[
\sum_{f \in A} \text{Im} f(\alpha_1', \alpha_2')b' = \sum_{f \in A} \text{Im} |f\rangle = V,
\]
we deduce from Lemma 2' that \( \psi(h') \) is stable. Since \( h': (A, \ast) \to \text{End} W \) is an algebra homomorphism, we have
\[
\langle f_1|\alpha_1'\alpha_2' - \alpha_2'\alpha_1' + b'a'|f_2\rangle = h'(f_1(gp - pq)f_2) + \langle f_1|1\rangle\langle 1|f_2\rangle
\]
\[
= -h'(f_1 \ast f_2) + h'(f_1)h'(f_2)
\]
\[
= 0.
\]
Therefore \( \psi(h') \in G \backslash \mu^{-1}(0) \cap X^s \).

If \( (\alpha_1', \alpha_2', a', b') = \psi(h') \),
\[
a'f(\alpha_1', \alpha_2')b' = \langle 1|f(\alpha_1', \alpha_2')|1\rangle
\]
\[
= \langle 1|f \rangle
\]
\[
= h'(f).
\]
Hence \( \varphi \circ \psi(h') = h' \).

If \( h' = \varphi(\alpha_1, \alpha_2, a, b) \), we can take \( i' = i, j' = j \) by the stability. Then
\[
\langle f \rangle = af(\alpha_1, \alpha_2), \quad |f\rangle = f(\alpha_1, \alpha_2)b.
\]
This implies that
\[
\langle 1 \rangle = a, \quad |1\rangle = b,
\]
\[
|gf\rangle = \alpha_1f(\alpha_1, \alpha_2)b = \alpha_1|f\rangle,
\]
\[
|pf\rangle = \alpha_2f(\alpha_1, \alpha_2)b = \alpha_2|f\rangle.
\]
Hence \( \psi \circ \varphi = \text{id} \).
REFERENCES