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On the regularity of a germ of analytic mapping

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Let (X, o) be a germ of analytic space (reduced and of pure dimension n) at the origin of $\mathbb{C}^N$; let $F: (\mathbb{C}^N, o) \to (\mathbb{C}^p, o)$ a germ of analytic mapping and $f = F|X$ the restriction of F to X. We denote *sing* F the singular set of F, i.e. the germ of points $x \in \mathbb{C}^N$ such that $dF(x)$ has a rank $< r(F)$, $r(F)$ meaning the generic rank of F. Many results on F or f are true and well known when *sing* $F = \emptyset$ or when F is flat. In this paper, we give examples where these results can be extended with an hypothesis on the codimension of *sing* F.

1) If the rank of F is constant $(= r)$, F admits a factorisation $(\mathbb{C}^N, o) \xrightarrow{h} (\mathbb{C}^r, o) \xrightarrow{g} (\mathbb{C}^p, o)$, where h is a submersion and g an immersion. In the general situtaion, we associate to F a differential form Ω_F of degree r ; if the codimension of *sing* Ω_F in $\mathbb{C}^N$ is ≥ 3 and if Ω_F is decomposable, there exists a factorisation by a generic submersion and a generic immersion. If $codim_{\mathbb{C}^N}\ sing\, F \geq 2$ and if there exists a formal factorisation $F = \overline{g} \circ \overline{h}$, then there exists an analytic factorisation which approximates the formal one. These results are an easy consequence of Malgrange's Frobenius theorem.

2) If s is the generic rank of f, there does not exist in general a factorisation : $(X, o) \to (Y, o) \overset{i}{\hookrightarrow} (\mathbb{C}^p, o)$, where (Y, o) is an analytic germ, reduced and of pure dimension s at the origin of $\mathbb{C}^p$ and i is the canonical injection. Nevertheless, this is true if F is a flat morphism and if $codim_{\mathbb{C}^N} X = codim_{\mathbb{C}^p} f(X)$. We prove analogous results when (X, o) is a complete intersection, an hypothesis about the codimension of *sing* F taking the place of the flatness.

3) At last, let $y = (y_1, \dots, y_p)$ (*resp.* $x = (x_1, \dots, x_N)$) a system of coordinates at the origin of $\mathbb{C}^p$ (*resp.* $\mathbb{C}^N$) and let $\overline{N}$ a sub-modulus of $\mathbb{C}[[y]]^q$. Let us suppose that $(\overline{N} \circ F)\, \mathbb{C}[[x]]$ is generated on $\mathbb{C}[[x]]$ by elements of $\mathbb{C}\{x\}^q$ ($\mathbb{C}\{x\}$ is the ring of convergent series in x) ; then, if F is flat, $\overline{N}$ is also analytic, i.e. is generated on $\mathbb{C}[[y]]$ by convergent series. The same is true when hypothesis about the codimension of *sing* F take the place of the flatness.

1 - A factorisation theorem.

Let $r = r(F)$ be the generic rank of $F : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ and let $\Lambda^r\{x\}$ be the free modulus on $\mathbb{C}\{x\}$ composed with germs at $0 \in \mathbb{C}^N$ of holomorphic differential forms of degree r.

Lemma 1.1 : *There exists a differential form* $\Omega_F \in \Lambda^r\{x\}$, $r = r(F)$, *unic modulo multiplication by inversible elements of* $\mathbb{C}\{x\}$ *, such that* :
(1) $\mathrm{codim}_{\mathbb{C}^N} \operatorname{sing} \Omega_F \geq 2$.

(2) $\forall I = (i_1,\dots,i_r)$, $1 \leq i_1 < \dots < i_r \leq p$, *there exits* $\theta_I \in \mathbb{C}\{x\}$ *such that* $d F_I = \theta_I.\Omega_F$.
($dF_I = d F_{i_1} \wedge \dots \wedge d F_{i_r}$ *and* $\operatorname{sing} \Omega_F = \{x; \Omega_F(x) = 0\}$).

Proof : For every I such that $d F_I \neq 0$, we can write $d F_I = \theta'_I . \Omega_I$ where $\theta'_I \in \mathbb{C}\{x\}$ and Ω_I is a form such that $\mathrm{codim}_{\mathbb{C}^N} \operatorname{sing} \Omega_I \geq 2$. Let I, J be such that $d F_I \neq 0, d F_J \neq 0$; the generic rank of F being r, we have $\Omega_J = \alpha . \Omega_I$ with α mieromorphic at the origin of $\mathbb{C}^N$; but α is holomorphic in $\mathbb{C}^N \setminus \operatorname{sing} \Omega_I$, so $\alpha \in \mathbb{C}\{x\}$. Permuting I and J, we see that α is inversible and the lemma follows.

Let Θ_F be the ideal generated by all the θ_I in $\mathbb{C}\{x\}$ and let us denote $V(\Theta_F)$ the germ of zeros of Θ_F ; obviously :

$$\operatorname{sing} F = V(\Theta_F) \cup \operatorname{sing} \Omega_F .$$

If $\mathrm{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 2$ and if $F = g \circ h$, where $h : (\mathbb{C}^n,0) \to (\mathbb{C}^r,0)$ is a generic submersion and $g : (\mathbb{C}^r,0) \to (\mathbb{C}^p,0)$ is a generic immersion, then we may choose $\Omega_F = d h_1 \wedge \dots \wedge d h_r$; Θ_F is then the ideal of $\mathbb{C}\{x\}$ generated by all the determinants of order r of the matrix $(dg) \circ h$.

Our result is a corollary of the singular Frobenius's theorem :

Theorem 1.2 (Malgrange, [3]) : *Let* $\omega_1,\ldots,\omega_r$ *be in* $\Lambda^1\{x\}$ *and let us put* $\Omega=\omega_1\wedge\ldots\wedge\omega_r$. *We suppose that for* $i=1,\ldots,r$, $d\omega_i\wedge\Omega=0$. *Then* :

(1) *If* $\operatorname{codim}_{\mathbb{C}^N}$ sing $\Omega\geq 3$, *the system* $\{\omega_1,\ldots,\omega_r\}$ *is integrable, i.e. there exist* $f_1,\ldots,f_r\in\mathbb{C}\{x\}$ *such that* :

$$(\omega_1,\ldots,\omega_r)\,.\,\mathbb{C}\{x\}=(df_1,\ldots,df_r).\mathbb{C}\{x\}.$$

(2) *If* $\operatorname{codim}_{\mathbb{C}^N}$ sing $\Omega\geq 2$ *and if the system* $\{\omega_1,\ldots,\omega_r\}$ *is formally integrable (i.e. there are formal series* $\overline{f}_1,\ldots,\overline{f}_r\in\mathbb{C}[[x]]$ *such that* $(\omega_1,\ldots,\omega_r).\mathbb{C}[[x]]=(d\overline{f}_1,\ldots,d\overline{f}_r).\mathbb{C}[[x]]$), *then the system is integrable.*

We use also the following result (cf [3] or [4]) :

Lemma 1.3 : *Let* $h:(\mathbb{C}^N,0)\to(\mathbb{C}^r,0)$ *be a germ of holomorphic mapping such that* $r=r(h)$ *and such that* $\operatorname{codim}_{\mathbb{C}^N}$ sing $h\geq 2$. *Then, if* $f:(\mathbb{C}^N,0)\to\mathbb{C}$ *verifies* $df\wedge dh_1\wedge\ldots\wedge dh_r=0$, *we have* $f=g\circ h$, *with* $g:(\mathbb{C}^r,0)\to\mathbb{C}$ *analytic.*

Proposition 1.4 :

(1) *Let us suppose that* $\operatorname{codim}_{\mathbb{C}^N}$ sing $\Omega_F\geq 3$ *and let us suppose that* Ω_F *is decomposable, i.e.* $\Omega_F=\omega_1\wedge\ldots\wedge\omega_r$, *with* $\omega_i\in\Lambda^1\{x\}$. *Then, there exists a factorisation* $F=g\circ h$, *where* $h:(\mathbb{C}^N,0)\to(\mathbb{C}^r,0)$ *and* $g:(\mathbb{C}^r,0)\to(\mathbb{C}^p,0)$ *are analytic.*

(2) *Conversally , if F admits such a factorisation and if* $\operatorname{codim}_{\mathbb{C}^N}$ sing $F\geq 2$, Ω_F *is decomposable.*

Proof : The system $(\omega_1,\ldots,\omega_r)$ is locally integrable in $\mathbb{C}^N\setminus V(\Theta_F)$, because $dF_I=\theta_I\,.\,\Omega_F$ and so $d\omega_i\wedge\omega_1\wedge\ldots\wedge\omega_r=0$ for $i=1,\ldots,r$. By theorem 1.2, we may suppose that $\omega_i=dh_i$, $i=1,\ldots r$, with $h_i\in\mathbb{C}\{x\}$, $h_i(0)=0$. At last, for every $g=1,\ldots,p$, $dF_j\wedge dh_1\wedge\ldots\wedge dh_r=0$ and so $F_j=g_j(h_1,\ldots,h_r)$ with g_j analytic, by lemma 1.3. The converse (2) is obvious.

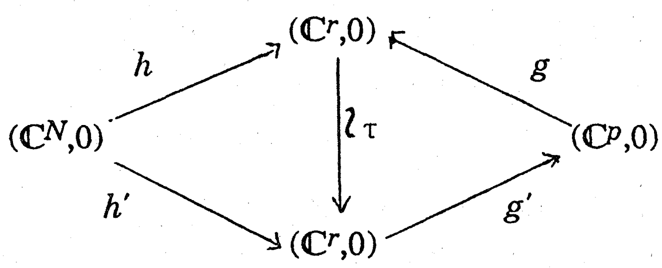
Proposition 1.5 : *Let us suppose that* $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 2$ *and let us suppose that* F *admits a formal factorisation* $F = \bar{g} \circ \bar{h}$ ($\bar{h} : (\mathbb{C}^N,0) \to (\mathbb{C}^r,0)$ *and* $\bar{g} : (\mathbb{C}^r,0) \to (\mathbb{C}^p,0)$). *Then* F *admits an analytic factorisation* $F = g \circ h$ *and we may choose* g *and* h *as closely as we wish to* $\bar{g}$ *and* $\bar{h}$.

Proof : From the hypothesis, Ω_F admits a formal decomposition : $\Omega_F = \bar{\lambda} \,.\, d\bar{h}_1 \wedge \ldots \wedge d\bar{h}_r$, with $\bar{\lambda} \in \mathbb{C}[[x]]$ and $\bar{\lambda}(0) \neq 0$. By Artin's approximation theorem [1], Ω_F is decomposable, i.e. $\Omega_F = \omega_1 \wedge \ldots \wedge \omega_r$ with $\omega_i \in \Lambda^1\{x\}$ and the system $\{\omega_1, \ldots, \omega_r\}$ is formally integrable. By the part (2) of theorem 1.2, the system is integrable and we conclude as in the proof of 1.4.

Proposition 1.6 : *Let* $\underline{F} : \mathbb{C}^N \supset U \to \mathbb{C}^p$ *be an holomorphic mapping with generic rank* r ; *we suppose that the set of singular points of* $\underline{F}$ *has codimension* ≥ 3. *Then, the set* Γ *of points* $x \in U$ *such that the germ* $\underline{F}_x : (U,x) \to (\mathbb{C}^p, \underline{F}(x))$ *is factorisable in the sense of* 1.4, *is the compliment of a closed analytic subset of* U.

Proof : The result being of local nature, we may suppose that there exists $\underline{\Omega} \in \Lambda^r(U)$ such that $\forall x \in U$, the germ $\underline{\Omega}_x$ induced by $\underline{\Omega}$ in x, is a differential form $\Omega_{\underline{F}_x}$. By 1.4, the point x belongs to Γ if and only if the equation : $\underline{\Omega}_x = \omega_1 \wedge \ldots \wedge \omega_r$ admits an holomorphic solution. The proposition results from a general theorem about the solutions of a system of analytic equations depending analytically of a parameter (cf [6]).

Remark 1.7 : Let us suppose that $F : (\mathbb{C}^N,0) \to (\mathbb{C}^p,0)$ admits a factorisation by $(\mathbb{C}^r,0)$, with $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 2$. Then this factorisation is unic, in the following sense : if $F = g \circ h$, $F = g' \circ h'$ are two factorisations, there is a unic analytic difféomorphism $\tau : (\mathbb{C}^r,0) \xrightarrow{\sim} (\mathbb{C}^r,0)$ such that the following diagram is commutative :



1.8. Special Cases

(1.8.1.) Let us suppose that $V(\Theta_F) = \emptyset$; for instance, let us suppose that $\theta_{(1,\ldots,r)}(0) = 0$. Then we may choose $\Omega_F = dF_1 \wedge \ldots \wedge dF_r$ and if $j > r$, we get $F_j = g_j\ (F_1,\ldots,F_r\ )$, with g_j analytic. So $F = g \circ h$, where g is the immersion $\mathbb{C}^r \ni (z_1,\ldots,z_r\ ) \to (z_1,\ldots,z_r\ ; g_{r+1}(z),\ldots,g_p(z))$. The converse is obvious and we get an equivalence :

$(V(\Theta_F) = \emptyset) \Leftrightarrow \mathrm{codim}_{\mathbb{C}^N}\, \mathrm{sing}\, F \geq 2$ *and there exists a factorisation* $F = g \circ h$, *where* $g : (\mathbb{C}^r,0) \to (\mathbb{C}^p,0)$ *is an immersion.*

(1.8.2) Let us suppose that sing $\Omega_F = \emptyset$; the form Ω_F is generically decomposable and non singular and so, by remark 1.9, it is decomposable, and we may apply 1.4. We get that $F = g \circ h$ where h is a submersion and the converse is obvious :

(sing $\Omega_F = \emptyset$) $\Leftrightarrow$ *There exists a factorisation* $F = g \circ h$ *where* $h : (\mathbb{C}^N,0) \to (\mathbb{C}^r, 0)$ *is a submersion.*

(1.8.3) Let us suppose that the rank of F at 0 is $r-1$.Then Ω_F is decomposable ; indeed, with a convenient choice of coordinates, we may suppose that $F_1 = x_1,\ldots,F_{r-1} = x_{r-1}$ and so $\Omega_F = dx_1 \wedge \ldots \wedge dx_{r-1} \wedge \omega$, and we may apply 1.4.

1.9 A decomposable form must verify obvious conditions. Let E be a vector space of dimension N on $\mathbb{C}$ and let $e_1,\dots,e_N$ be a basis of E. Let us consider the mapping :

$$\mathbb{C}^{Nr} \simeq E^r \ni (\omega_1,\dots,\omega_r) \to \Omega = \omega_1 \wedge\dots\wedge \omega_r \in \Lambda^r E \simeq \mathbb{C}^{\binom{N}{r}}.$$

Obviously, $\omega_1 \wedge\dots\wedge \omega_r = \omega'_1 \wedge\dots\wedge \omega'_r$ if and only if there exists a matrix $M \in G\ell(N, \mathbb{C})$ with determinant 1 such that

$$(\omega_1,\dots,\omega_r)\ M = (\omega'_1,\dots,\omega'_r).$$

Outside $\phi^{-1}(0)$ the mapping ϕ is a fibering with fiber of dimension r^2-1 and $\phi^1(0)$ is the set of matrices $(\omega_1, \dots,\omega_r)$ with rank $< r$ and so $\phi^{-1}(0)$ is an algebraic variety in E^r of codimension $N-r+1$. The image of ϕ is an algebraic variety in $\Lambda^r E$, of dimension $Nr-r^2+1$, with an isolated singularity at the origin.

If $\Sigma\, \theta_I\, e_I$ $(e_I = e_{i_1} \wedge\dots\wedge e_{i_r})$ is the generic point of $\Lambda^r E$ and if U_I is the open set $\theta_I \neq 0$, then $\mathrm{Im}\,\phi \cap U_I$ is regular and is the transverse intersection of $\binom{N}{r} - (Nr-r^2+1)$ algebraic hypersurfaces $F_{I,\alpha} = 0$, where $F_{I,\alpha}$ is homogeneous of degree r (if $r=2$, $\mathrm{Im}\,\phi = \{\Omega\,;\,\Omega \wedge \Omega = 0\}$).

So, a decomposable differential form Ω must verify $\binom{N}{r} - (Nr-r^2+1)$ independent conditions. Conversally, if these conditions are full filled and if Ω is non singular, then Ω is decomposable. If Ω is singular and decomposable $(\Omega = \omega_1 \wedge\dots\wedge \omega_r)$, every irreducible component of $\mathrm{sing}\,\Omega = \{x; \mathrm{rank}\,(\omega_1 \wedge\dots\wedge \omega_r) < r\}$ has codimension $\leq N-r+1$.

Another condition is the following one. Let $\nu(\Omega) = \inf_I \nu(\theta_I)$ be the infimum of the multiplicities at the origin of the θ_I $(\Omega = \sum_I \theta_I dx_I$ and let $\mathcal{J}(\Omega)$ be the ideal generated in $\mathbb{C}\{x\}$ by θ_I's. If $\nu(\Omega) = s < r$ and if $\Omega = \omega_1 \wedge\dots\wedge \omega_r$, then $r-s$ forms ω_i are linearly independant at the origin and by choosing suitable coordinates :

$$\Omega = (\sum_I \theta_I^* \, dx_I) \wedge dx_{N-(r-s)+1} \wedge \ldots \wedge dx_N$$

with $I = (i_1, \ldots, i_s)$, $1 \leq i_1 < \ldots < i_s \leq N-(r-s)$; so the minimal number of generators of $\mathcal{I}(\Omega)$ is $\leq \binom{N-(r-s)}{s}$.

For instance, if $N = 3$, $r = 2$, the conditions $F_{I,\alpha}(\Omega) = 0$ are vacant. A form Ω with an isolated singularity at the origin is not decomposable ; the form $\Omega = xy\, dx\, dy + y^2\, dy\, dz + z\, dx\, dz$, with the x–axis as a line of singularities, is not decomposable. Nevertheless, it is decomposable at every point outside the origin (if $x \neq 0$, $\Omega = (x\, dx - y\, dz) \wedge (y\, d\, y + \frac{z}{x}\, dz)$).

If $N = 4$, $r = 2$, there is one condition $F_{i,\alpha} = 0 : \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\,\theta_{23} = 0$.

This condition is not sufficient, but I do not know if the hypotheses that Ω is decomposable at every point outside the origin, implies that Ω is decomposable.

1.10 In this paragraph, we give some upper bounds for $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F$

(1.10.1). First, every irreducible component of $F^{-1}(0)$ has codimension $\leq$ r = r(F) (indeed, if F is the germ at 0 of $\underline{F} : U \to C^p$, the generic codimension of the fiber $\underline{F}_\xi^{-1}\ \underline{F}(\xi)$ is r and this codimension is a lower semi-continuous function of ξ) ; after, $F^1(0) \setminus \operatorname{sing} F$ is a regular variety of codimension r. Accordingly :

$$r(F) \geq \operatorname{codim}_{\mathbb{C}^N} F^{-1}(0) \geq \inf (r(F), \operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F)$$

(1.10.2) If $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 2$ and if $V(\Theta_F) \neq \emptyset$, there is an inclusion :

$$F^{-1}(0) \subset V(\Theta_F).$$

Accordingly , if $V(\Theta_F) \neq \emptyset$, we get $\operatorname{codim}_{\mathbb{C}^N} V(\Theta_F) \leq r$, and so :

$$\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \leq r \quad \text{(we suppose } r \geq 1\text{)}.$$

Indeed, if $F^{-1}(0) \not\subset V(\Theta_F)$ there exists an holomorphic curve $\mathbb{C} \ni t \to x(t) \in \mathbb{C}^N$ such that $x(0) = 0$ and $x(t) \in \underline{F}^{-1}(0) \setminus V(\Theta_F)$ if $t \neq 0$. From (1.8.1), the morphism $\underline{F}_t : (\mathbb{C}^N, (x(t)) \to (\mathbb{C}^p, 0)$ ($t \neq 0$ small enough) admits a factorisation through a germ Σ_t of analytic variety of dimension r at the origin of $\mathbb{C}^p$. All Σ_t are equal to a Σ and $F = \underline{F}_0$ admits a factorisation through Σ. From (1.8.1), $V(\Theta_F) = \phi$, c.q.f.d.

(1.10. 3) Let us suppose that $\Omega_{\underline{F}}$ is decomposable. From 1.9, if sing $\Omega_F \neq \emptyset$:

$$\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} \Omega_F \leq N-r+1.$$

Nevertheless, if sing $\Omega_F = \emptyset$, there is from (1.8.2) a factorisation $F = g \circ h$, where h is a submersion, and $V(\Theta_F)$ is the germ of zeros of the ideal $h^* \mathcal{J}$, where $\mathcal{J}$ is the ideal generated by all determinants of order r of the matrix $p \times r\,(dg_1,\ldots,dg_p)$.

So, if $V(\Theta_F) \neq \emptyset$, $\operatorname{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p-r+1$.

Accordingly, if sing $F \neq \emptyset$ and if Ω_F is decomposable :

$$\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \leq \sup\,(p,n)-r+1.$$

The codimension being lower semi-continuous :

Let us suppose that there exist points $x \in$ sing F, as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable (this is true if d_x F has rank $r-1$, *cf* (1.8.3)). *Then* :

$$\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \leq \sup\,(p,n)-r+1.$$

We have also the following remarks :

Let us suppose there exist points $x \in$ sing Ω_F , as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable ; then

$$\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} \Omega_F \leq n-r+1$$

Let us suppose there exist points $x \in V(\Theta_F)$, as closely as we wish from the origin, such that $\Omega_{\underline{F}_x}$ is decomposable ; if $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} \Omega_F \geq 3$, *then* :

$$\operatorname{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p-r+1.$$

Remarks 1.11 : The proposition 1.5 is false in general if we suppose that the dimension of the space by which we factorise is not equal to $r = r\,(F)$. For example (cf [2]), there exists an analytic morphism $F : (\mathbb{C}^2,0) \to (\mathbb{C}^4,0)$

with $F_1 = x_1$, $F_2 = x_1 \cdot x_2$, $F_3 = x_1 x_2 e^{x_2}$, $F_4 = \overline{\phi}(F_1, F_2, F_3)$, where $\overline{\phi}$ is formal and cannot be choosen analytic.Then $r(F) = 2$, $\ker F^* = 0$ and $\ker \hat{F}^*$ is generated by $y_4 - \overline{\phi}(y_1, y_2, y_3)$. So, there is a formal factorisation of $F : (\mathbb{C}^2,0) \xrightarrow{\overline{h}} (\mathbb{C}^3,0) \xrightarrow{\overline{g}} (\mathbb{C}^4,0)$, where $\mathbb{C}^3 = \{y \in \mathbb{C}^4 ; y_4 = 0\}$ and $\overline{g}$ is the graph of $\overline{\phi}$; if $F : g \circ f$ is an analytic factorisation of F near of the preceding one, g is (as $\overline{g}$) an immersion and $\ker F^* \neq 0$, but this is false. In this example, $\operatorname{codim}_{\mathbb{C}^2} \operatorname{sing} F = 1$, but we may modify it, in such a way that $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 2$.

2 - On the regularity of a germ of analytic mapping.

In this paragraph and the following one, we suppose that F is a generic submersion, i.e. $r(F) = p$.

2.1. Let us suppose that $(X,0)$ is irreductible and let us suppose that $f : F|X : (X,0) \to \mathbb{C}^p$ has generic rank s. If $\mathcal{O}_X$ is the ring of germs of holomorphic functions on $(X,0)$ and if $f^* : \mathbb{C}\{y\} \to \mathcal{O}_X$ is the homomorphism induced by f, there is inequalities :

$$s \leq s' = \dim (\hat{\mathcal{O}}_X / \ker \hat{f^*}) \leq s'' = \dim (\mathcal{O}_X / \ker f^*).$$

Let us recall the following result (Gabrielov, [2]) :

Theorem 2.2 : *If* $s = s'$, we get $s = s' = s''$, *i.e if the topological dimension of the image* $f(X)$ *is equal to its formal dimension, then it is also equal to its analytical dimension* ; *therefore* : $\ker \hat{f^*} = \widehat{\ker f^*}$.

The morphism f is regular (Gabrelov's definition) if $s = s' = s''$; if $(X,0)$ is reduced, the morphism f is regular if it is regular in restriction to each irreducible component of $(X,0)$. The morphism f is regular if f is finished or if $(X,0)$ and f are algebraic ; here is another condition :

Proposition 2.3 : *Let us suppose that* $(X,0)$ *is irreducible* ; *the morphism* f *is regular under every following hypothesis* :

(1) $r(f) = \text{codim}_X f^{-1}(0)$ (*the inequality* $r(f) \geq \text{codim}_X f^{-1}(0)$ *is always true*)

(2) F *is a flat morphism and* $\text{codim}_{\mathbb{C}^N} X = \text{codim}_{\mathbb{C}^p} f(X)$.

Proof : (1) Let $\Sigma \subset \mathbb{C}^N$ be a generic plane of codimension $n-s$ ($s = r(f)$) passing through the origin ; then, every irreducible component $X_{\Sigma,i}$ *of* $X \cap \Sigma$ has codimension $(n-s)+(N-n) = N-s$ in $\mathbb{C}^N$, so has dimension s and $X_{\Sigma,i} \cap f^1(0) = (0)$. If $g = f\,|X_{\Sigma,i}$, $g : X_{\Sigma,i} \to \mathbb{C}^p$ is a finite morphism, and its rank is s.

The kernel of $g^* : \mathbb{C}\{y\} \to \mathcal{O}_{X_{\Sigma,i}}$) is a prime ideal $\mathfrak{P}_{\Sigma,i}$ such that $\mathbb{C}\{y\}/\mathfrak{P}_{\Sigma,i}$ has dimension s. Generically, $\underline{X}_{\Sigma,i}$ contains points x as closely as we wish to the origin, which are regular for $\underline{X}_{\Sigma,i}$ and $\underline{X}$ with :

$$\text{rank } d_x \underline{f} = \text{rank } d_x \underline{g} = s$$

(the notations $\underline{X}$, $\underline{g}$ etc... mean sets, functions etc.., the germs of which at the origin being X, g...). If $\varphi \in \mathfrak{P}_{\Sigma,i}$, $\varphi \circ f$ is null on $X_{\Sigma,i}$ and $\underline{\varphi} \circ \underline{f}$ is null on $\underline{X}$ in the neighborhood of every x. Therefore, $\varphi \circ f = 0$ and $\mathfrak{P}_{\Sigma,i} \subset \ker f^*$. The inverse inclusion is obvious because $\mathbb{C}\{y\}/\mathfrak{P}_{\Sigma,i}$ has dimension s, and the morphism f is regular.

(2) The morphism F being flat, $\text{codim}_{\mathbb{C}^N} F^{-1}(0) = p$, so $\text{codim}_{\mathbb{C}^N} f^{-1}(0) \geq p$ and $\text{codim}_X f^{-1}(0) = \text{codim}_{\mathbb{C}^N} f^{-1}(0) - \text{codim}_{\mathbb{C}^N} X \geq p - \text{codim}_{\mathbb{C}^p} f(X) = r(f)$.

Therefore $r(f) = \text{codim}_X f^{-1}(0)$ and the result is a consequence of (1).

Example 2.4 : Let $\varphi_1(x_n), \ldots, \varphi_{n-1}(x_n) \in \mathbb{C}\{x_n\}$ be germs, algebraically independent on $\mathbb{C}$, such that $\varphi'_1(0) = \ldots = \varphi'_{n-1}(0) = 1$. Let us consider the morphism f :

$$(\mathbb{C}^n,0) \ni (x_1,\ldots,x_n) \to (x_1,\ldots,x_{n-1}, x_1\,\varphi_1(x_n),\ldots,x_{n-1}\,\varphi_{n-1}(x_n)) \in \mathbb{C}^{2n-2}.$$

Then $r(f) = n$ and sing $f = f^{-1}(0)$ is the x_n–axis ; so $\text{codim}_{\mathbb{C}^N} f^{-1}(0) = n-1$;

besides, the rank of df at 0 is $n-1$. The morphism f is not regular if $n > 2$; more precisely, $\ker \hat{f^*} = 0$. Indeed, let $g \in \mathbb{C}[[y]]$ be such that :

$$g(x_1,\ldots,x_{n-1}, x_1\,\varphi_1(x_n),\ldots,x_{n-1}\,\varphi_{n-1}(x_n)) = 0.$$

If $g = \sum_{\nu=1}^{\infty} g_\nu$ is the decomposition of g in homogeneous polynomials, and if $x_1 = t\,\xi_1$,..., $x_{n-1} = t\,\xi_{n-1}$:

$$\sum_{\nu=1}^{\infty} t^\nu g_\nu(\xi_1,\ldots,\xi_{n-1}, \xi_1\,\varphi_1(x_n),\ldots,\xi_{n-1}\,\varphi_{n-1}(x_n)) = 0$$

so $g_\nu(\xi_1,\ldots,\xi_{n-1}, \xi_1\,\varphi_1(x_n),\ldots,\xi_{n-1}\,\varphi_{n-1}(x_n)) = 0$, i.e. $g_\nu = 0$, $\forall \nu$.

This example, a variant of Osgood's example, shows that it is difficult to improve 2.3. Nevertheless, in 2.3 (2), we may replace the hypothesis of flatness on F by a condition of regularity on F.

Remark 2.5 : If $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq p$, the morphism F is flat. Indeed, by (1.10.1), $p = r(F) = \operatorname{codim}_{\mathbb{C}^N} F^{-1}(0)$ and this means exactly that F is flat. Here is an example where $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F = p-1$ and F is not flat ; $F : \mathbb{C}^{2p-2} \to \mathbb{C}^p$ is defined by

$$F_1(x) = x_1 \,;\, \ldots \,;\, F_{p-1}(x) = x_{p-1} \,;\, F_p(x) = x_1x_p + x_2x_{p+1} + \ldots + x_{p-1}x_{2p-2}.$$

Proposition 2.6 : Let us suppose that $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 3$, where $F : (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$ is a generic submersion. If $(X,0)$ is a germ of hypersurface at the origin of $\mathbb{C}^N$ such that $\operatorname{codim}_{\mathbb{C}^p} f(X) = 1$ (= $\operatorname{codim}_{\mathbb{C}^N} X$), then $f = F\,|X$ is regular ($X = F^{-1}(Y)$, where Y is a germ of hypersurface at the origin of $\mathbb{C}^p$).

Proof : Let $\varphi = 0$ be a reduced equation of X ; the condition on the generic rank of f means that at each regular point of X, $d_x\,\varphi$ is a linear conbination of $d_x F_i$, i.e :

$$(*) \qquad d\varphi = \varphi \,.\, \omega + \sum_{i=1}^{p} \varphi_i \, dF_i$$

with $\omega \in \Lambda^1\{x\}$ and $\varphi_i \in \mathbb{C}\{x\}$.

So
$$d\omega = - \sum_{i=1}^{p} d\left(\frac{\varphi_i}{\varphi}\right) \wedge dF_i \, ;$$

and $d\omega \wedge dF_1 \wedge \ldots \wedge dF_p = 0$. By 2.9, the hypothesis $\operatorname{codim}_{\mathbb{C}^N}$ sing $F \geq 3$ implies that $\omega = d\Psi$ mod(dF), where(dF) is the submodulus of $\Lambda^1\{x\}$ generated by $dF_1, \ldots, dF_p$. From $(*)$:

$$d\,(\varphi\, e^{-\Psi}) \in (dF)$$

and from lemma 1.3 : $\varphi = e^{\Psi} \,.\, (\theta \circ F)$, with $\theta \in \mathbb{C}\{y\}$.

If Y is the hypersurface with reduced equation $\theta = 0$, then $X = f^{-1}(Y)$.

Corollary 2.7 : *If* $\operatorname{codim}_{\mathbb{C}^N}$ sing $F \geq 3$ *and if* Y *is a germ of irreducible hypersurface at the origin of* $\mathbb{C}^p$, $X = F^{-1}(Y)$ *is also irreducible (indeed, if* $X = X' \cup X''$ *is a proper decomposition of* X*, we may apply to* X' *and* X'' *the previous reasoning, and* $X' = F^{-1}(Y')$, $X'' = F^{-1}(Y'')$; *so* $Y = Y' \cup Y''$ *is a proper decomposition of* Y *which is not irreducible).*

Corollary 2.8 : *Let us suppose that* $\operatorname{codim}_{\mathbb{C}^N}$ sing $F \geq 3$ *and that* Ω_F *is decomposable (we do not suppose that* F *is a generic submersion). If* $\operatorname{codim}_{\mathbb{C}^N} V(\Theta_F) \geq r = r(F)$, F *and the restriction of* F *to every hypersurface* $(X,0)$ *of* $(\mathbb{C}^N, 0)$, *are regular morphisms.*

Proof : By 1.4, there is a factorisation, $F = g \circ h$, where h is a generic submersion and g is a generic finite immersion. So F is regular ; if X is a germ of irreducible hypersurface at the origin of $\mathbb{C}^N$, either the rank of $h|X$ is equal to r and $f = F|X$ is regular ; or this rank is $r-1$, but then we may apply 2.6 and again f is regular.

2.9. In the proof of 2.6, we used a very particular case of the following result (cf [3] or [4]). Let us suppose that $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F > q$

($F : (\mathbb{C}^\nu,0) \to (\mathbb{C}^p,0)$ is a generic submersion) and let $1 \leq s \leq r \leq q$ be integers. We put :

$$\Lambda_F^{r,s}\{x\} = \{\omega \in \Lambda^r\{x\} ;\ \omega \wedge dF_{i_1} \wedge \ldots \wedge dF_{i_{p-s+1}} = 0$$

for every $1 \leq i_1 < i_2 < \ldots < i_{p-s+1} \leq p\}$ =

$$\{\omega \in \Lambda^r\{x\} ;\ \omega = \sum_{j_1 < \ldots < j_s} \theta_{j_1 \ldots j_s} \wedge dF_{j_1} \wedge \ldots \wedge dF_{j_s}$$

with $\theta_{j_1 \ldots j_s} \in \Lambda^{r-s}\{x\}\}$.

This last equality is an easy consequence of the division lemma (Saïto, [8]) stated below. If we write $\Lambda^{r,s}(F) = \Lambda^r\{x\}/\Lambda_F^{r,s}\{x\}$ then d induces a morphism :

$$\Lambda^{r,s}(F) \to \Lambda^{r+1,s}(F) ;$$

there is an exact sequence :

$$\Lambda^{s-1}\{x\} \xrightarrow{d} \Lambda^{s,s}(F) \xrightarrow{d} \Lambda^{s+1,s}(F) \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^{q,s}\{F\}$$

and the of kernel the first d is the submodulus of $\Lambda^{s-1}\{x\}$ generated by the images of $F^* : \Lambda^{s-1}\{y\} \to \Lambda^{s-1}\{x\}$ and $d : \Lambda^{s-2}\{x\} \to \Lambda^{s-1}\{x\}$. In particular, if $q = 2$, there is an exact sequence $0 \to \mathbb{C}\{y\} \xrightarrow{F^*} \mathbb{C}\{x\} \xrightarrow{d} \Lambda^{1,1}(F) \xrightarrow{d} \Lambda^{2,1}(F)$; this sequence is used in the proof of 2.6.

The division lemma says that, if $\operatorname{codim}_{\mathbb{C}^N} \operatorname{sing} F > q$ and if $\omega \in \Lambda^q\{x\}$ is such that $\omega \wedge dF_1 \wedge \ldots \wedge dF_p = 0$, then $\omega = \sum_{i=1}^{p} \theta_i \wedge dF_i$, with $\theta_i \in \Lambda^{q-1}\{x\}$.

2.10. It would be interesting to extend 2.6 to complete intersections. If a complete intersection $(X,0)$ of codimension k at the origin of $\mathbb{C}^N$ is defined by a reduced system of equations $\varphi_1 = \ldots = \varphi_k = 0$ and if $\operatorname{codim}_{\mathbb{C}^p} F(X) = \operatorname{codim}_{\mathbb{C}^N} X$,

then

$$d\varphi = \Omega \,.\, \varphi + \sum_{i=1}^{p} \theta_i \,.\, dF_i$$

where $\varphi = \begin{pmatrix} \varphi_1 \\ \\ \varphi_k \end{pmatrix}$, Ω is a $k \times k$ matrix with coefficients in $\Lambda^1\{x\}$, θ_i is a column vector with coefficients in $\mathbb{C}\{x\}$. So :

$$(d\Omega - \Omega \wedge \Omega)\, \varphi = 0 \quad \text{mod } (\mathrm{dF_i})\,.$$

If we may choose Ω such that $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (\mathrm{dF_i})$ and if $\mathrm{codim}_{\mathbb{C}^N} \operatorname{sing} F \geq 3$, then by the arguments of the proof of 2.9 :

$$\Omega = dM.M^{-1} \qquad \text{mod } (dF_i)$$

where M is an inversible $k \times k$ matrix with coefficients in $\mathbb{C}\{x\}$, so

$$d(M^{-1}\varphi) = 0 \qquad \text{mod } (dF_i)$$

By 2.9, $M^{-1}\varphi = \theta \circ F$, where $\theta = \begin{pmatrix} \theta_1 \\ \\ \theta_k \end{pmatrix}$, $\theta_i \in \mathbb{C}\{y\}$, $\theta_i(0) = 0$, and $X = F^{-1}(Y)$ where Y is a complete intersection.

Therefore, the main problem is finding conditions on X and F such that the integrability condition $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (dF_i)$ is verified by a suitable Ω.

3 - A criteria of analyticity for modulus.

If A is a (commutative and unitary) ring without divisors of zero, we denote by $[A]$ the quotient field of A. A modulus $\mathcal{M}$ on A, of finite type, is <u>without torsion</u> if $a \in A\backslash\{0\}$, $m \in \mathcal{M}\backslash\{0\}$ implies $a.m \neq 0$. This means also that $\mathcal{M}$ is isomorphic to a submodulus of A^r, where $r = \dim_{[A]} \mathcal{M} \otimes_A [A]$ is the generic rank of $\mathcal{M}$.

If $(X,0)$ is a germ of analytic space, we denote by $\hat{\mathcal{O}}_X$ the completion of the ring $\mathcal{O}_X$ of analytic germs on X. We shall use the following result (Tougeron, [7]) :

Theorem 3.1 : Let $f : (X,0) \to (Y,0)$ be a generic analytic submersion between two irreducible germs of analytic spaces (so $\hat{f}^* : \hat{\mathcal{O}}_Y \to \hat{\mathcal{O}}_X$ is injective). Then :

$$\{\varphi \in [\hat{\mathcal{O}}_Y] \text{ and } \varphi \circ f \in [\mathcal{O}_X]\} \Rightarrow \varphi \in [\mathcal{O}_Y].$$

A submodule $\mathcal{N}$ of $\hat{\mathcal{O}}_Y^q$ is <u>analytic</u> if it is generated on $\hat{\mathcal{O}}_Y$ by elements of $\mathcal{O}_Y^q$.

Corollary 3.2 : Under the hypothesis of 3.1, if $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$ is without torsion and if the vector space generated by $\mathcal{N} \circ f$ in $[\hat{\mathcal{O}}_X]^q$ is <u>analytic</u> (i.e. is generated by vectors with coefficients in $\mathcal{O}_X$), then $\mathcal{N}$ is analytic.

Proof : Let $\varphi_1, \dots, \varphi_s \in \mathcal{N}$ be such that $\varphi_1 \wedge \dots \wedge \varphi_s \neq 0$ and r = generic rank of $\mathcal{M} = q-s$. Then $\varphi \in \mathcal{N} \Leftrightarrow \varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_s = 0$ (because $\mathcal{M}$ is without torsion). Let us put $\varphi_1 \wedge \dots \wedge \varphi_s = \sum_I \theta_I e_I$ where $e_I = e_{i_1} \wedge \dots \wedge e_{i_s}$ is the canonical basis of $\hat{\mathcal{O}}_Y^q$.

Let us suppose that $\theta_{I_0} \neq 0$; the modulus generated by $\mathcal{N} \circ f$ being analytic, each $(\theta_I / \theta_{I_0}) \circ F$ is analytic, so by 3.1 θ_I / θ_{I_0} is analytic for every I. Therefore $\mathcal{N}$ is analytic, c.q.f.d.

This corollary admits the following extension :

Proposition 3.3 : *Let $f : (X,0) \to (Y,0)$ be a morphism between two irreducible germs of analytic spaces and let us suppose that the germ of points $x \in X$ such that f_x is not flat has codimension v in X. Let $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$ be a modulus such that the two following conditions are full filled :*

- *every prime ideal associated to $\mathcal{M}$ has height $< v$,*
- *the submodule $f^*\mathcal{N}$ generated by $\mathcal{N} \circ f$ in $\hat{\mathcal{O}}_X^q$ is analytic. Then $\mathcal{N}$ is analytic.*

Let us recall that if $\mathcal{M}$ is a modulus on a ring $A = \hat{\mathcal{O}}_Y$ and if $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}$ is associated to $\mathcal{M}$ if there is an injective map : $A/\mathfrak{p} \hookrightarrow \mathcal{M}$. The modulus $\mathcal{M}$ is <u>coprimary</u> if $a \in A \setminus \mathfrak{p} \Rightarrow (\mathcal{M} \ni m \to a.m \in \mathcal{M})$ is injective and $a \in \mathfrak{p}$ $\Rightarrow (\mathcal{M} \ni m \to a.m \in \mathcal{M})$ is nilpotent. If $\mathfrak{p}_1, ..., \mathfrak{p}_k$ are the prime ideals associated to $\mathcal{M}$ then there exist submodules $\mathcal{N}_i$ of $\mathcal{M}$, with $\mathcal{M}/\mathcal{N}_i$ $\mathfrak{p}_i$-coprimary, such that $\bigcap_{i=1}^{k} \mathcal{N}_i = 0$ (cf [5]).

If $\mathcal{M} = A^q/\mathcal{N}$ is $\mathfrak{p}$-coprimary, we define a sequence $\mathcal{N}_0 = \mathcal{N} \subset \mathcal{N}_1 \subset ... \subset \mathcal{N}_s \subset \mathcal{N}_{s+1} = A^q$ of submodules of A^q such that for every $i = 0, ..., s : \mathcal{N}_{i+1} \{\xi \in A^q ;$ $\mathfrak{p} . \xi \subset \mathcal{N}_i\}$. Then, for every $i : \mathcal{N}_{i+1}/\mathcal{N}_i$ is a modulus on $A/\mathfrak{p}$ without torsion. We prove first :

Lemma 3.4 : *With the hypothesis of* 3.3, *let us suppose that* $\mathcal{M}$ *is* $\mathfrak{p}$*–coprimary* ; *then* 3.3 *is true.*

Proof : First , we observe that $\mathfrak{p}$ is analytic. In fact, as a consequence of the flatness, the prime ideals of height $< v$ associated to $\hat{\mathcal{O}}_X^q / f^*\mathcal{N}$ are exactly the prime ideals of height $< v$ associated to $\hat{\mathcal{O}}_X / f^*\mathfrak{p}$; $f^*\mathcal{N}$ being analytic, these prime ideals are analytic, and there exist a prime ideal $\mathfrak{p}'$ of $\mathcal{O}_X$ such that $\hat{\mathfrak{p}}'$ is a minimal prime ideal containing $f^*\mathfrak{p}$. If X' is the germ of analytic set $(\subset X)$ defined by $\mathfrak{p}'$, then $\mathfrak{p}$ is the kernel of the morphism $\hat{f}^* : \hat{\mathcal{O}}_Y \to \hat{\mathcal{O}}_X / \hat{\mathfrak{p}}'$ and by Gabrielov's theorem, $\mathfrak{p}$ is analytic.

After, we prove by induction on $i = s, s-1, ..., 0$ that $\mathcal{N}_i$ is analytic. If $\mathcal{N}_{i+1}$ is analytic, let $g_1, ..., g_h$ be an analytic system of generators of $\mathcal{N}_{i+1}$ and let us write $\mathcal{N}_{i+1} = \hat{\mathcal{O}}_Y^h / \mathcal{R}_{i+1}$ where $\mathcal{R}_{i+1}$ is the modulus of relations between $g_1, ..., g_h$. Then $\mathcal{N}_{i+1} / \mathcal{N}_i = \hat{\mathcal{O}}_Y^h / \mathcal{R}'_{i+1}$, $\mathcal{R}'_{i+1} \supset \mathcal{R}_{i+1}$; by the flatness of f, $f^* \mathcal{R}'_{i+1}$ is analytic at the generic point of X' ; by 3.2, $\mathcal{R}'_{i+1}$ is analytic and so $\mathcal{N}_i$ is analytic.

Remark : In the previous proof we don't use the complete assertion that $f^*\mathcal{N}$ is analytic ; we only use that $f^*\mathcal{N}$ is analytic at the generic point of X'.

Proof of 3.3 : Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_k$ be the minimal prime ideals associated to $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$. As in the proof of 3.4, let $\mathfrak{p}'_i$ be a minimal prime ideal of $\hat{\mathcal{O}}_X$ containing $f^*\mathfrak{p}_i$; then $\mathfrak{p}'_i$ is analytic. If X'_i is the germ of analytic set defined by $\mathfrak{p}'_i$, then $\mathfrak{p}_i$ is the kernel of the morphism $\hat{f}^* : \hat{\mathcal{O}}_Y \to \hat{\mathcal{O}}_X / \mathfrak{p}'_i$ and so, by Gabrielov's theorem, $\mathfrak{p}_i$ is analytic. Let $\mathcal{N}_i$ be a submodule of $\hat{\mathcal{O}}_Y^q$ such that $\bigcap_{i=1}^{k} \mathcal{N}_i = \mathcal{N}$ and $\hat{\mathcal{O}}_Y^q / \mathcal{N}_i$ is $\mathfrak{p}_i$-coprimary.

Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_{k'}$ be the minimal prime ideals in the family $\{\mathfrak{p}_1,\ldots,\mathfrak{p}_k\}$; then, by flatness, $f^*\mathcal{N}_i$ is analytic at the generic point of X'_i, for i = 1,..., k'. By 3.4 and the remark, $\mathcal{N}_i$ is analytic if $i \le k'$.

There is an injection :

$$(*) \qquad (\bigcap_{i\le k'} \mathcal{N}_i)/(\cap \mathcal{N}_i) \to \hat{\mathcal{O}}_Y^q / \bigcap_{i>k'} \mathcal{N}_i .$$

Let $g_1,\ldots,g_h$ be a system of analytic generators of $\bigcap_{i\le k'} \mathcal{N}_i$ and let us write $\hat{\mathcal{O}}_Y^h / \mathcal{R} \simeq \bigcap_{i\le k'} \mathcal{N}_i$ where $\mathcal{R}$ is the modulus of relations between the g_i. Then $(\bigcap_{i\le k'} \mathcal{N}_i)/(\cap \mathcal{N}_i) \sim \hat{\mathcal{O}}_Y^h / \mathcal{R}'$ where $\mathcal{R}' \supset \mathcal{R}$. The prime ideal associated to $\hat{\mathcal{O}}_Y^h / \mathcal{R}'$ are among $\mathfrak{p}_{k'+1},\ldots,\mathfrak{p}_k$, because of the injection (*) and by flatness $f^*\mathcal{R}'$ is analytic outside an analytic set of codimension v .

Therefore, we may prove the result by induction on the number k of prime ideals $\mathfrak{p}_i$. By the induction hypothesis, $\mathcal{R}'$ is analytic and so $\mathcal{N}$ is analytic.

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