数理解析研究所講究録
725巻 1990年 163-190

Residual currents and tensor products of holonomic systems

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The primary purpose of this note is to provide an elementary introduction to the theory of $\mathcal{D}_X$-Modules. This note is not meant to be a survey. We try to explain some basic notions in the theory of $\mathcal{D}_X$-modules. We mainly focus our attention on concrete examples of regular holonomic $\mathcal{D}_X$-Modules which are related with residual currents. It is hoped that some of the basic features of holonomic systems emerge.

§ 0. Products of residual currents.

To begin with let us recall the notion of residual current.

Let $(X, \mathcal{O}_X)$ be a complex manifold and let $f \in \Gamma(X, \mathcal{O}_X)$ be a holomorhic function. Dolbeault [6] and Herrera-Liebermann [10] showed, by making use of a desingularization of the hyperfurface defined by $f = 0$, that for any compactly supported smooth test differential form ϕ, the limits

$$\lim_{\varepsilon\to+0}\int_{|f|>\varepsilon}\frac{\phi}{f} \quad \text{and} \quad \lim_{\varepsilon\to+0}\int_{|f|=\varepsilon}\frac{\phi}{f}$$

exist.

The linear functional $[\frac{1}{f}]$ defined by

$$[\frac{1}{f}] : \quad \phi \longrightarrow \lim_{\varepsilon \to +0} \int_{|f|>\varepsilon} \frac{\phi}{f}$$

is called the principal value of $\frac{1}{f}$. And the current defined by

$$\phi \longrightarrow \lim_{\varepsilon \to +0} \int_{|f|=\varepsilon} \frac{\phi}{f}$$

is called a residual current. Note that

$$\bar{\partial}[\frac{1}{f}](\phi) = \lim_{\varepsilon \to 0} \int_{|f|=\varepsilon} \frac{\phi}{f}.$$

Recently Passare [22] defined and intensively studied "products of residual currents". Let r be a product of residual currents of the form

$$r = \bar{\partial}[\frac{1}{f_1}] \wedge \bar{\partial}[\frac{1}{f_2}] \wedge \cdots \wedge \bar{\partial}[\frac{1}{f_k}], \qquad f_1, f_2, \ldots, f_k \in \Gamma(X, \mathcal{O}_X).$$

Passare showed in particular the following fact.

Theorem(local version). Assume that $f_1 = f_2 = \cdots = f_k = 0$ is a complete intersection at x. Then a germ of holomorphic function $h \in \mathcal{O}_x$ annuls the residual current r if and only if the holomorphic function h belongs to the ideal generated by $f_1, f_2, \ldots, f_k$ i.e. $h \in \mathcal{O}_x(f_1, f_2, \ldots, f_k)$. Here $\mathcal{O}_x$ denotes the stalk at x of the sheaf of holomorphic functions.

For incetance, in $C^2 = \{(x, y) \mid x, y \in C\}$, let us consider the following residual current.

$$r = \bar{\partial}[\frac{1}{y^2}] \wedge \bar{\partial}[\frac{1}{y-x^2}]$$

Since $y^2 r = 0$ and $(y-x^2)r = 0$, you get $x^4 r = 0$. Hence in particular the current r is supported at the origin. Therefore r is a linear combination of derivatives of Dirac's delta function. In fact, one can verify the following equality.

$$(\bar{\partial}[\frac{1}{y^2}] \wedge \bar{\partial}[\frac{1}{y-x^2}]\ dx\wedge dy)\phi \ = -(2\pi i)^2 \{ \frac{1}{6} \frac{\partial^3 \phi}{\partial x^3}(0,\ 0) + \frac{\partial^2 \phi}{\partial x \partial y}(0,\ 0) \} .$$

But it seems dificult, in general, to get such an explicit formula. We are thus interested in the following problem.

How to calculate or characterize a residual current?

Now let us explain our basic idea.
Let r be a residual current in the sence of Passare [22]. Let $\mathcal{D}_X$ be the sheaf on X of rings of holomorphic linear partial differential operators with holomorphic coefficients. We write $\mathcal{J}_r \subset \mathcal{D}_X$ the sheaf of annihilator ideals of the residual current r.

$$\mathcal{J}_{r,x} = \{ P \in \mathcal{D}_{X,x} \mid \ Pm = 0 \} ,$$

where $\mathcal{D}_{X,x}$ denotes the stalk at x of $\mathcal{D}_X$. Then the current r can be regarded as a distribution solution for the system of linear partial differential equations : $\quad Pr = 0 \qquad$ for any $P \in \mathcal{J}_r$.

If the annihilator ideal $\mathcal{J}_r$ is determined, one can employ the theory of linear partial differential equations to study the residual current r. And if in particular the dimension of the vector space of distribution solutions of this system above is equal to one, the ideal $\mathcal{J}_r$ characterizes the residual current r.

In this report we will restrict ourselves mainly to the two dimensional case and we will derive a regular holonomic system and its generator whose distribution solutions may equal to a constant multiple of the residual current r. In other wards we will derive a left $\mathcal{D}_X$-ideal which may equal to the annihilator ideal of the residual current r.

We will use the notions of

(i) $\mathcal{D}_X$-Modules and algebraic local cohomologies,

(ii) regular holonomic distributions,

(iii) tensor products of holonomic systems,

and

(iv) blow-up and blow-down of holonomic systems.

§ 1. $\mathcal{D}_X$-Modules and algebraic local cohomologies.

Let us recall some basic notions and fix our notation.
We start with a linear partial differential equation with an unknownfunction u : $Pu = 0$, where P is a holomorphic linear partial differential operator with holomorphic coefficients. Notice that $QPu = 0$ holds for any linear partial differential operator Q. Let us consider the left $\mathcal{D}_X$-linear homomorphism from $\mathcal{D}_X$ to $\mathcal{D}_X$ defined by P :

$$\mathcal{D}_X \ni QP \xleftarrow{\quad P \quad} Q \in \mathcal{D}_X$$

Here $\mathcal{D}_X$ denotes the sheaf on X of holomorphic linear partial differential operators with holomorphic coefficients.

If we denote by $\mathcal{J}$ the sheaf of left $\mathcal{D}_X$-ideals generated by P, we have Ker(P : $\mathcal{D}_X \longrightarrow \mathcal{D}_X$) = 0 and Im(P : $\mathcal{D}_X \longrightarrow \mathcal{D}_X$) = $\mathcal{J}$.

If we set

$$\mathcal{M} = \mathrm{Coker}(P : \mathcal{D}_X \longrightarrow \mathcal{D}_X) = \mathcal{D}_X / \mathcal{J} ,$$

then $\mathcal{M}$ becomes a left $\mathcal{D}_X$-Module. And the exact sequence

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}_X \xleftarrow{P} \mathcal{D}_X \longleftarrow 0.$$

can be regarded as a finite presentation of the left $\mathcal{D}_X$-Module $\mathcal{M}$. In this sence $\mathcal{M}$ is an intrinsic object.

Notice that $\mathcal{D}_X$ is a coherent sheaf of rings and the each stalk $\mathcal{D}_{X,x}$ is a Noetherian ring.

Let $\mathcal{F}$ be a left $\mathcal{D}_X$-Module. The solutions of the equation Pu = 0 belonging to $\mathcal{F}$ can be considered as follows. From the exact sequence above we get an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{F}) \xrightarrow{P} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{F})$$

$$\begin{array}{ccc} \wr\!\| & & \wr\!\| \\ \mathcal{F} \ni g & \longrightarrow & Pg \in \mathcal{F}. \end{array}$$

It follows that

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})_x = \mathrm{Ker}\,(P : \mathcal{F} \longrightarrow \mathcal{F})_x = \{ g \in \mathcal{F}_x \mid Pg = 0 \} .$$

Therefore determining the solutions of Pu = 0 is equivalent to determining the sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$. Incidentally, we have

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}) = \text{Coker } (P : \mathcal{F} \longrightarrow \mathcal{F}),$$

in the sence of homological algebra.

More generally let

$$\sum_{j=1}^{N} P_{ij} \, u_j = 0 \qquad i = 1, 2, \ldots, N_1$$

be a system of linear partial differential equations, where P_{ij} denote linear partial differential operators and u_j denote unknown functions. We can associate to this system a coherent left $\mathcal{D}_X$-Module $\mathcal{M}$ with finite presentation :

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}_X^{N} \longleftarrow \mathcal{D}_X^{N_1}$$

Let $\mathcal{F}$ be a left $\mathcal{D}_X$-Module. We can consider the sheaf of $\mathcal{D}_X$-homomorphisms from $\mathcal{M}$ to $\mathcal{F}$ and its extensions :

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}) \quad \text{and} \quad \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}) \qquad k \geqq 1.$$

Let

$$0 \leftarrow \mathcal{M} \longleftarrow \mathcal{D}_X^{N} \xleftarrow{\;P\;} \mathcal{D}_X^{N_1} \xleftarrow{\;P_1\;} \mathcal{D}_X^{N_2} \xleftarrow{\;P_2\;}$$

be a (local) projective resolution of $\mathcal{M}$, where P_j denote matrices of linear partial differential operators. It is easy to see that $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ is the solutions sheaf and each cohomology group $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})$ gives the obstructions of the solvability of P_{k-1} with compatibility condition given by P_k.

For this reason we mean by a system of linear partial differential equations a coherent left $\mathcal{D}_X$-Module.

Example 1.

The structure sheaf $\mathcal{O}_X$ is naturally endowed with a structure of left $\mathcal{D}_X$-Module by differentiation. For instance, let $X = \{(x, y) \mid x, y \in C\} = C^2$. For any germ $f \in \mathcal{O}_{X,x}$ of holomorphic function we have

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} \in \mathcal{O}_{X,x} \quad \text{and} \quad \frac{\partial}{\partial y} f = \frac{\partial f}{\partial y} \in \mathcal{O}_{X,x}.$$

But if you regard f as an element of $\mathcal{D}_{X,x}$, f becomes a linear partial differential operator of order zero and

$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} + f \frac{\partial}{\partial x}, \qquad \text{etc.}$$

Hence we have
$$\mathcal{O}_X = \mathcal{D}_X / (\mathcal{D}_X \frac{\partial}{\partial x} + \mathcal{D}_X \frac{\partial}{\partial y}).$$

We also have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = C_X \quad \text{and} \quad \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0 \quad \text{for } k \geqq 1.$$

Let us recall the notion of algebraic local cohomology.
Let $Y = \{ f = 0\}$ be a complex hypersurface, f denotes a defining holomorphic function. We define the algebraic local cohomology sheaf by the inductive limit

$$\mathcal{H}^{1}_{[Y]}(\mathcal{O}_X) = \varinjlim_{k} \mathcal{E}xt^{1}_{\mathcal{O}_X}(\mathcal{O}_X / (f)^k, \mathcal{O}_X)$$

where $(f)^k$ denotes the $\mathcal{O}_X$-Ideal generated by f^k. We also consider

$$\mathcal{H}^{0}_{[X|Y]}(\mathcal{O}_X) = \varinjlim_{k} \mathcal{H}om_{\mathcal{O}_X}((f)^k, \mathcal{O}_X),$$

hence have a exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H}^{0}_{[X|Y]}(\mathcal{O}_X) \longrightarrow \mathcal{H}^{1}_{[Y]}(\mathcal{O}_X) \longrightarrow 0.$$

If we denote by ϕ_k be a section of $\mathcal{H}om_{\mathcal{O}_X}((f)^k, \mathcal{O}_X)_x$ defined by $\phi_k(f^k) = 1$, then ϕ_k can be identified with $\frac{1}{f^k}$. Hence a germ of $\mathcal{H}^{0}_{[X|Y]}(\mathcal{O}_X)$ can be represented by

$$h_0 + \frac{h_1}{f} + \frac{h_2}{f^2} + \cdots + \frac{h_k}{f^k}, \quad \text{where } h_0, \ldots, h_k \in \mathcal{O}_{X,x}.$$

This means that $\mathcal{H}^{0}_{[X|Y]}(\mathcal{O}_X)$ is the sheaf of meromorphic functions with pole on Y.

A germ of $\mathcal{H}^{1}_{[Y]}(\mathcal{O}_X)$ can be represented by

$$\frac{h_1}{f} + \frac{h_2}{f^2} + \cdots + \frac{h_k}{f^k} \mod \mathcal{O}_{X,x}, \quad \text{where } h_0, \ldots, h_k \in \mathcal{O}_{X,x}.$$

It is thus easy to verify that these two sheaves have the structure of left $\mathcal{D}_X$-Module.

Example 2.

Let $X = C^2 = \{(x, y) \mid x, y \in C\}$, $Y = \{(x, y) \in X \mid x = 0\}$.

If we set $n = \frac{1}{x}$, then n generates the sheaf $\mathcal{H}^0_{[X|Y]}(\mathcal{O}_X)$ over $\mathcal{D}_X$.

Since $(x\frac{\partial}{\partial x} + 1)n = 0$ and $\frac{\partial}{\partial y} n = 0$, we have

$$\mathcal{H}^0_{[X|Y]}(\mathcal{O}_X) \cong \mathcal{D}_X n = \mathcal{D}_X / (\mathcal{D}_X(x\frac{\partial}{\partial x} + 1) + \mathcal{D}_X \frac{\partial}{\partial y}).$$

Hence $\mathcal{H}^0_{[X|Y]}(\mathcal{O}_X)$ is a coherent left $\mathcal{D}_X$-Module.

It is easy to verify the followings.

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}^0_{[X|Y]}(\mathcal{O}_X), \mathcal{O}_X) = C_{X-Y} = j_! C_{X-Y}$$

and

$$\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{H}^0_{[X|Y]}(\mathcal{O}_X), \mathcal{O}_X) = 0 \quad \text{for} \quad k \geqq 1,$$

where $j : X-Y \hookrightarrow X$ is the natural open inclusion map.

Example 3.

Let $X = C^2 = \{(x, y) \mid x, y \in C\}$, $Y = \{(x, y) \in X \mid x = 0\}$.

If we set $m = \frac{1}{x}$ mod $\mathcal{O}_X$, then m generates the $\mathcal{D}_X$-Module $\mathcal{H}^1_{[Y]}(\mathcal{O}_X)$.

Since $xm = 1 \in \mathcal{O}_X$ and $\frac{\partial}{\partial y} m = 0$, we have

$$\mathcal{H}^1_{[Y]}(\mathcal{O}_X) = \mathcal{D}_X\, m = \mathcal{D}_X / (\mathcal{D}_X\, x + \mathcal{D}_X \frac{\partial}{\partial y}).$$

We also have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}^1_{[Y]}(\mathcal{O}_X), \mathcal{O}_X) = 0,$$

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{H}^1_{[Y]}(\mathcal{O}_X), \mathcal{O}_X) = C_Y,$$

and

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{H}^1_{[Y]}(\mathcal{O}_X), \mathcal{O}_X) = 0 \quad \text{for } k \geqq 2.$$

Note. Set $m' = \frac{\partial}{\partial x} m$. Then $m' \in \mathcal{D}_X m$. Hence $\mathcal{D}_X m' \subseteq \mathcal{D}_X m$. Since $m = -xm'$, we have $\mathcal{D}_X m \subseteq \mathcal{D}_X m'$. Therefore we have $\mathcal{D}_X m = \mathcal{D}_X m'$ as set. But the annihilator ideal of m' is equal to

$$\mathcal{D}_X\, x^2 + \mathcal{D}_X (x\frac{\partial}{\partial x} + 2) + \mathcal{D}_X \frac{\partial}{\partial y}.$$

In what follows we say that a $\mathcal{D}_X$-Module $\mathcal{D}_X m_1$ generated by m_1 is equal to a $\mathcal{D}_X$-Module $\mathcal{D}_X m_2$ generated by m_2 if the annihilator ideal of m_1 is equal to the annihilator ideal of m_2.

Notation.

We write $\mathcal{M}_1 \cong \mathcal{M}_2$ if the $\mathcal{D}_X$-Module $\mathcal{M}_1$ is isomorphic to the $\mathcal{D}_X$-Module $\mathcal{M}_2$. We write $\mathcal{D}_X m_1 = \mathcal{D}_X m_2$ if the $\mathcal{D}_X$-Module $\mathcal{D}_X m_1$ is equal to $\mathcal{D}_X m_2$.

Let $\mathcal{I}$ be a coherent $\mathcal{O}_X$-Ideal and Y the support of $\mathcal{O}_X/\mathcal{I}$. We define the q-th algebraic local cohomology sheaf by the inductive limit

$$\mathcal{H}^q_{[Y]}(\mathcal{O}_X) = \varinjlim_k \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X).$$

We also set

$$\mathcal{H}^q_{[X|Y]}(\mathcal{O}_X) = \varinjlim_k \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{I}^k, \mathcal{O}_X).$$

We thus have a long exact sequence

$$0 \longrightarrow \mathcal{H}^0_{[Y]}(\mathcal{O}_X) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H}^0_{[X|Y]}(\mathcal{O}_X) \rightarrow \mathcal{H}^1_{[Y]}(\mathcal{O}_X) \longrightarrow 0.$$

and isomorphisms

$$\mathcal{H}^q_{[X|Y]}(\mathcal{O}_X) = \mathcal{H}^{q+1}_{[Y]}(\mathcal{O}_X) \qquad q \geqq 1.$$

It is easy to see that these are left $\mathcal{D}_X$-Modules.

Recall that a left coherent $\mathcal{D}_X$-Module whose characteristic variety is Lagrangian is called a holonomic system.

Following result is known.

Theorem (Kashiwara [13], Mebkhout [20]).

(i) $\mathcal{H}^q_{[Y]}(\mathcal{O}_X)$ is coherent.

(ii) $\mathcal{H}^q_{[Y]}(\mathcal{O}_X)$ is a regular holonomic system

For the notion of regular singularity we refer to [16].

Theorem (Mebkhout[20]).

$R\mathcal{H}om_{\mathcal{D}_X}(R\Gamma_{[Y]}(\mathcal{O}_X), \mathcal{O}_X)) = C_Y,$

where $R\Gamma_{[Y]}$ denote the right derived functor of $\Gamma_{[Y]}$ and $R\mathcal{H}om_{\mathcal{D}_X}(\ \ ,\mathcal{O}_X)$ denotes the right derived functor fo $\mathcal{H}om_{\mathcal{D}_X}(\ \ ,\mathcal{O}_X)$.

To end this section let us recall that the sheaf $\mathcal{D}_X$ has the following description :

$$\mathcal{D}_X = \mathcal{H}^{\dim X}_{[X]}(\mathcal{O}^{(0,\ \dim X)}_{X\times X})$$

where $\mathcal{O}^{(0,\ \dim X)}_{X\times X}$ is the sheaf of (dim X)-holomorphic forms in the second variables with holomorphic functions as coefficients and X is identified with the diagonal of X×X. Note also that the sheaf $\mathcal{D}^{\infty}_X$ of rings of linear partial differential operators of infinite order can be described by

$$\mathcal{D}^{\infty}_X = \mathcal{H}^{\dim X}_{X}(\mathcal{O}^{(0,\ \dim X)}_{X\times X}).$$

Let X and Y be complex manifolds and ϕ a holomorphic map from Y to X. Kashiwara introduced the sheaf $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ by

$$\mathcal{D}_{Y\to X} = \mathcal{H}^{\dim X}_{[Y]}(\mathcal{O}_{Y\times X} \underset{\mathcal{O}_X}{\otimes} \Omega^{\dim X}_X),$$

and

$$\mathcal{D}_{X\leftarrow Y} = \mathcal{H}^{\dim X}_{[Y]}(\Omega^{\dim Y}_{Y} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y\times X}),$$

respectively, where $\Omega^{\dim X}_{X}$ (resp. $\Omega^{\dim Y}_{Y}$) signifies the sheaf of the holomorphic (dim X)-forms (resp. (dim Y)-forms).

§ 2. Regular holonomic distributions and residual currents.

In this section we try to illustrate the Reconstruction theorem for holonomic systems [16]. We shall content ourselves with discussing the most simplest case which is related to residual currents.

Let us first examine the one dimensional case. Let $X = \{z \mid z \in \mathbb{C}\}$, $Y = \{0\}$. We set

$$\mathcal{M}' = \mathcal{D}_X / \mathcal{D}_X \frac{d}{dz} \cong \mathcal{O}_X,$$

$$\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X (z \frac{d}{dz} + 1) \cong \mathcal{H}^{0}_{[X|Y]}(\mathcal{O}_X),$$

and

$$\mathcal{M}'' = \mathcal{D}_X / \mathcal{D}_X z \cong \mathcal{H}^{1}_{[Y]}(\mathcal{O}_X).$$

We thus have a short exact sequence of $\mathcal{D}_X$-Modules.

$$(*) \qquad 0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

By applying the derived functor $R\mathcal{H}om_{\mathcal{D}_X}(\ *\ ,\ \mathcal{O}_X)$ to the exact sequence above we get an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}'', \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}', \mathcal{O}_X)$$

$$\to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}'', \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}', \mathcal{O}_X) \to 0.$$

Since $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_Y = \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_Y = 0$, we thus have

$$(**) \qquad 0 \longrightarrow C_{X-Y} \longrightarrow C_X \longrightarrow C_Y \longrightarrow 0$$

Let us recall the following quasi-isomorphisms.

$$R\mathcal{H}om_C(C_{X-Y}, \mathcal{O}_X) = [\, 0 \to \mathcal{B}_{X-Y} \xrightarrow{\bar\partial} \mathcal{B}_{X-Y} \to 0 \,],$$

$$R\mathcal{H}om_C(C_X, \mathcal{O}_X) = [\, 0 \to \mathcal{B}_X \xrightarrow{\bar\partial} \mathcal{B}_X \to 0 \,],$$

and

$$R\mathcal{H}om_C(C_Y, \mathcal{O}_X) = [\, 0 \to \Gamma_Y\mathcal{B}_X \xrightarrow{\bar\partial} \Gamma_Y\mathcal{B}_X \to 0 \,].$$

where $\mathcal{B}_X$ denotes the sheaf of hyperfunctions and $\Gamma_Y\mathcal{B}_X$ denotes the sheaf of hyperfunctions with supports in Y. It follows that

$$R\mathcal{H}om_C(C_{X-Y}, \mathcal{O}_X) = j_*j^{-1}\mathcal{O}_X,$$

$$R\mathcal{H}om_C(C_X, \mathcal{O}_X) = \mathcal{O}_X,$$

and

$$R\mathcal{H}om_{C}(C_{Y},\ \mathcal{O}_{X}) = \mathcal{H}^{1}_{Y}(\mathcal{O}_{X})[-1]$$

where j denotes the natural open inclusion map $j : X - Y \hookrightarrow X$. We thus verified the following results.

$$R\mathcal{H}om_{C}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}',\ \mathcal{O}_{X})) = \mathcal{M}'$$

$$R\mathcal{H}om_{C}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{O}_{X})) = \mathcal{D}^{\infty}_{X} \otimes_{\mathcal{D}_X} \mathcal{M}$$

and

$$R\mathcal{H}om_{C}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}'',\ \mathcal{O}_{X})) = \mathcal{D}^{\infty}_{X} \otimes_{\mathcal{D}_X} \mathcal{M}'',$$

where $\mathcal{D}^{\infty}_{X}$ denotes the sheaf of rings of linear partial differential operators of infinite orders.

Let us reexamine the result above by using the following double complex associated with the exact sequence (**) :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{Y}\mathcal{B}_{X} & \longrightarrow & \mathcal{B}_{X} & \longrightarrow & \mathcal{B}_{X-Y} & \longrightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \longrightarrow & \Gamma_{Y}\mathcal{B}_{X} & \longrightarrow & \mathcal{B}_{X} & \longrightarrow & \mathcal{B}_{X-Y} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let start with the meromorphic function $\frac{1}{z} \in \Gamma(X-Y, \mathcal{B}_{X-Y})$. Since the sheaf $\mathcal{B}_X$ is flaby, the sheaf morphism $\mathcal{B}_X \longrightarrow \mathcal{B}_{X-Y}$ is surjective. In fact the principal value hyperfunction $[\frac{1}{z}] \in \Gamma(X, \mathcal{B}_X)$ satisfies $[\frac{1}{z}] = \frac{1}{z}$ on $X - Y$ thus in particular that $\bar\partial[\frac{1}{z}]$ belongs to $\Gamma_Y\mathcal{B}_X$. Hence the coboundary morphism

$$\mathrm{Ker}(\bar\partial : \mathcal{B}_{X-Y} \longrightarrow \mathcal{B}_{X-Y}) \longrightarrow \mathrm{Coker}(\bar\partial : \Gamma_Y\mathcal{B}_X \longrightarrow \Gamma_Y\mathcal{B}_X),$$

maps $\frac{1}{z}$ to $\bar\partial[\frac{1}{z}]$.

We thus have the following exact sequence of $\mathcal{D}^\infty_X$-Modules :

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}^\infty_X(\frac{1}{z}) \longrightarrow \mathcal{D}^\infty_X(\bar\partial[\frac{1}{z}]) \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}^\infty_X \otimes \mathcal{H}^0_{[X|Y]}(\mathcal{O}_X) \longrightarrow \mathcal{D}^\infty_X \otimes \mathcal{H}^1_{[Y]}(\mathcal{O}_X) \longrightarrow 0.$$

We also have the following exact sequence of coherent left $\mathcal{D}_X$-Modules :

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}_X(\frac{1}{z}) \longrightarrow \mathcal{D}_X(\bar\partial[\frac{1}{z}]) \longrightarrow 0.$$

It is easy to verify that the exact sequence above is isomorphic to the following one :

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H}^0_{[X|Y]}(\mathcal{O}_X) \longrightarrow \mathcal{H}^1_{[Y]}(\mathcal{O}_X) \longrightarrow 0.$$

It is hoped that the explanation above illustrate some aspects of the subjects. We refer to [16] and [14] for the reconstruction theorem for holonomic systems and for the Riemann-Hilbert correspondence for regular holonomic systems.

Nextly we briefly examine the two dimensional case.

Let f be a holomorphic function defined in a domain X in C^2. We set $F = \{f = 0\}$. Let us denote by $\delta(f)$ the section of $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ defined by $\frac{1}{f}$ mod $\mathcal{O}_X$. If the function f is analytically irreducible, the section $\delta(f)$ generates the $\mathcal{D}_X$-Module $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$, for $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ is simple as $\mathcal{D}_X$-Module ([8]).

We have the following result (cf. [3],[7] and [19]).

Theorem. 4

Assume that the holomorphic function f is analytically irreducible. Then $\mathcal{D}_X(\delta(f)) = \mathcal{D}_X(\bar{\partial}[\frac{1}{f}])$ holds. More precisely the annihilator ideal of $\delta(f)$ and that of the residual current $\bar{\partial}[\frac{1}{f}]$ coincides.

In particular the residual current $\bar{\partial}[\frac{1}{f}]$ is a regular holonomic distribution ([15]).

§ 3. Tensor products of holonomic systems.

In this section we examine tensor products of holonomic systems supported on plane curves.

Let us recall the notion of tensor product.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two $\mathcal{D}_X$-modules. Let X_1 and X_2 be two copies of the complex manifold X. Let p_1 (resp. p_2) be the projection from $X_1 \times X_2$ to X_1 (resp. X_2). We set

$$\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2 = \mathcal{D}_{X_1 \times X_2} \otimes_{p_1^{-1}\mathcal{D}_{X_1} \otimes p_2^{-1}\mathcal{D}_{X_2}} (p_1^{-1}\mathcal{M}_1 \otimes p_2^{-1}\mathcal{M}_2).$$

Let us denote by $\overset{L}{\otimes}$ the left derived functor of $\otimes$.

We have the following result.

Proposition (Kashiwara [13]).

For two $\mathcal{D}_X$-Modules $\mathcal{M}_1$ and $\mathcal{M}_2$, we have

$$\mathcal{M}_1 \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{M}_2 = \mathcal{D}_{X \to X_1 \times X_2} \overset{L}{\otimes}_{\mathcal{D}_{X_1 \times X_2}} (\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2).$$

Now let us examine the tensor products of two algebraic local cohomologies with support in plane curves.

Let X be a domain in $C^2 = \{(x, y) \mid x, y \in C\}$ containing the origin $P = (0, 0)$. Let F and G be two analytic plane curves (defined in X) passing through P. Let f (resp. g) be a defining holomorphic function of the curve F (resp. G). We set :

$$\mathcal{L} = \mathcal{H}^{1}_{[F]}(\mathcal{O}_{X_1}) \hat{\otimes} \mathcal{H}^{1}_{[G]}(\mathcal{O}_{X_2}).$$

We identify X with the diagonal of $X_1 \times X_2$. We choose the natural coordinates (x_1, y_1, x_2, y_2) on $X_1 \times X_2$ such that $X = \{(x_1, y_1, x_2, y_2) \mid x_1 = x_2, y_1 = y_2\}$. Since

$$\mathcal{D}_{X \to X_1 \times X_2} = \mathcal{D}_{X_1 \times X_2} / (x_1 - x_2)\mathcal{D}_{X_1 \times X_2} + (y_1 - y_2)\mathcal{D}_{X_1 \times X_2},$$

$\mathcal{D}_{X \to X_1 \times X_2} \overset{L}{\underset{\mathcal{D}_{X_1 \times X_2}}{\otimes}} \mathcal{L}$ is quasi-isomorphic to the complex

$$0 \longleftarrow \mathcal{L} \overset{\chi_1}{\longleftarrow} \begin{matrix}\mathcal{L}\\ \oplus \\ \mathcal{L}\end{matrix} \overset{\chi_2}{\longleftarrow} \mathcal{L} \longleftarrow 0.$$

where $\chi_1 = (x_1 - x_2, y_1 - y_2)$ and $\chi_2 = \begin{pmatrix} y_2 - y_1 \\ x_1 - x_2 \end{pmatrix}$.

If we denote by $1_{X \to X_1 \times X_2}$ the canonical section of $\mathcal{D}_{X \to X_1 \times X_2}$ [23], we have the following :

$$x\, 1_{X \to X_1 \times X_2} = 1_{X \to X_1 \times X_2} (\frac{1}{2}(x_1 + x_2)),$$

$$\frac{\partial}{\partial x} 1_{X \to X_1 \times X_2} = 1_{X \to X_1 \times X_2} (\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}), \quad \text{etc.}$$

It is easy to verify the following result.

Proposition 5.

Let F and G be two analytically irreducible plane curves containing the origin P. Assume that F and G meet properly at P i.e. $F \cap G = P$. Then we have the following resuts.

(i) $\mathcal{T}or_k^{\mathcal{O}_X}(\mathcal{H}^1_{[F]}(\mathcal{O}_X), \mathcal{H}^1_{[G]}(\mathcal{O}_X)) = \mathcal{T}or_k^{\mathcal{D}_{X_1 \times X_2}}(\mathcal{D}_{X \to X_1 \times X_2}, \mathcal{L}) = 0$

for any $k \geqq 1$.

(ii) The tensor product $\mathcal{H}^1_{[F]}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{H}^1_{[G]}(\mathcal{O}_X) = \mathcal{D}_{X \to X_1 \times X_2} \otimes_{\mathcal{D}_{X_1 \times X_2}} \mathcal{L}$

is isomorphic to the simple regular holonomic $\mathcal{D}_X$-Module $\mathcal{H}^2_{[P]}(\mathcal{O}_X)$

supported by the origin P.

Let us denote by $\delta(f)$ (resp. by $\delta(g)$) the section of the sheaf $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ (resp. $\mathcal{H}^1_{[G]}(\mathcal{O}_X)$) defined by $\frac{1}{f}$ mod $\mathcal{O}_X$ (resp. $\frac{1}{g}$ mod $\mathcal{O}_X$).

We set :

$$m = 1_{X \to X_1 \times X_2} \otimes (\delta(f) \hat{\otimes} \delta(g)).$$

The following result is an immediate consequence of Proposition 5.

Proposition 6.

Under the assumption of Proposition 5, we have the following equality.

$$\mathcal{D}_X m \cong \mathcal{D}_{X \to X_1 \times X_2} \otimes_{\mathcal{D}_{X_1 \times X_2}} \mathcal{L}.$$

We can use the result above to calculate the annihilator ideal of the generator m.

Example 7 ([24]).

Set $X = C^2 = \{(x, y) \mid x, y \in C\}$, $F = \{(x, y) \mid y = 0\}$ and $G = \{(x, y) \mid y - x^2 = 0\}$. Let us denote by $\delta(y)$ (resp. $\delta(y - x^2)$) the canonical generator of $\mathcal{H}^1_{[F]}(\mathcal{O}_X)$ (resp. $\mathcal{H}^1_{[G]}(\mathcal{O}_X)$). We set :

$$m = 1_{X \to X_1 \times X_2} \otimes (\delta(y_1) \hat{\otimes} \delta(y_2 - x_2^2)).$$

Then we have

$$\mathcal{D}_X m \cong \mathcal{D}_{X \to X_1 \times X_2} \underset{\mathcal{D}_{X_1 \times X_2}}{\otimes} (\mathcal{H}^1_{[F]}(\mathcal{O}_{X_1}) \hat{\otimes} \mathcal{H}^1_{[G]}(\mathcal{O}_{X_2})),$$

and

$$\mathcal{D}_X m = \mathcal{D}_X / (\mathcal{D}_X x^2 + \mathcal{D}_X (x \frac{\partial}{\partial x} + 2) + \mathcal{D}_X y)$$

$$= \mathcal{D}_X (\frac{\partial}{\partial x} \delta(x, y)),$$

where $\delta(x, y)$ denotes Dirac's delta-function at the origin.

For the detailed account, we refer the reader to [25].

Example 8.

Put $F = \{(x, y) \in C^2 \mid y = 0\}$ and $G = \{(x, y) \in C^2 \mid y^2 - x^3 = 0\}$. If we set

$$m = 1_{X \to X_1 \times X_2} \otimes (\delta(y_1) \hat{\otimes} \delta(y_2{}^2 - x_2{}^3)),$$

then we have

$$\mathcal{D}_X m \cong \mathcal{H}^1_{[F]}(\mathcal{O}_X) \underset{\mathcal{O}_X}{\otimes} \mathcal{H}^1_{[G]}(\mathcal{O}_X).$$

By direct calculation we have

$$\mathcal{D}_X m = \mathcal{D}_X / (\mathcal{D}_X x^3 + \mathcal{D}_X(x\frac{\partial}{\partial x} + 3) + \mathcal{D}_X y)$$

$$= \mathcal{D}_X (\frac{\partial^2}{\partial x^2}\delta(x, y)).$$

Let us state a conjecture.

Conjecture.

Let F and G be analytically irreducible plane curves intersecting properly at the origin. Let f and g be holomorphic defining function of F and G respectively. If we set

$$m = 1_{X \to X_1 \times X_2} \otimes (\delta(f) \hat{\otimes} \delta(g))$$

then we have $\mathcal{D}_X m = \mathcal{D}_X r$, where r denotes the residual current

$$r = \bar{\partial}[\frac{1}{f}] \wedge \bar{\partial}[\frac{1}{g}].$$

This means that the annihilator ideal of m and that of r coincides.

§ 4. Blow-up and blow-down of holonomic systems.

In order to calculate tensor products of holonomic $\mathcal{D}_X$-Modules we use the notion of blow-up and blow-down of $\mathcal{D}_X$-Modules.

Let X and Z be two complex manifolds. Let ϕ be a proper holomorphic map from Z to X. For any coherent $\mathcal{D}_X$-Module $\mathcal{M}$, we set :

$$L\phi^* \mathcal{M} = \mathcal{D}_{Z \to X} \overset{L}{\underset{\phi^{-1}\mathcal{D}_X}{\otimes}} \phi^{-1}\mathcal{M}.$$

For any coherent $\mathcal{D}_Z$-Module $\mathcal{N}$ we set :

$$\int_\phi \mathcal{N} = R\phi_* (\mathcal{D}_{X \leftarrow Z} \overset{L}{\underset{\mathcal{D}_Z}{\otimes}} \mathcal{N}).$$

Here $R\phi_*$ is the right derived functor of of ϕ_*.

Example 9.

Let $X = C^2$ and let $\pi : \tilde{X} \longrightarrow X$ be the blowing-up of X at the origin P= (0, 0). Then we have

$$\int_\pi \mathcal{O}_{\tilde{X}} = \mathcal{O}_X \oplus \mathcal{H}^2_{[P]}(\mathcal{O}_X).$$

We have the following projection formula which we will use later.

Projection Formula (cf. [11], [25]).

Let ϕ be a proper holomorphic map from Z to X. For any coherent $\mathcal{D}_Z$-Module $\mathcal{N}$, and for any coherent $\mathcal{D}_X$-Module $\mathcal{M}$, we have

$$\int_{\phi}(\mathcal{N}\overset{L}{\underset{\mathcal{D}_Z}{\otimes}} L\phi^{*}\mathcal{M}) = (\int_{\phi}\mathcal{N})\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}.$$

Note. If we set $\mathcal{N} = \mathcal{O}_Z$ then we have

$$\int_{\phi} L\phi^{*}\mathcal{M} = (\int_{\phi}\mathcal{O}_Z)\overset{L}{\underset{\mathcal{D}_X}{\otimes}}\mathcal{M}.$$

Now let us return to the two dimensional case. Let X be a domain in C^2 and let F be a plane curve on X with a holomorphic defining function f. Let $\pi : \tilde{X} \longrightarrow X$ be the blow-up of X at the origin. Let us denote by $\tilde{F}$ the total transform of F, i.e. $\tilde{F} = \{ f\circ\pi = 0\}$.

It is easy to verify the following result.

Proposition 10.

(i) $\mathcal{T}or_k^{\mathcal{D}_X}(\mathcal{D}_{\tilde{X}\to X}, \mathcal{H}^1_{[F]}(\mathcal{O}_X)) = 0$ for $k \geqq 1$.

(ii) $\mathcal{D}_{\tilde{X}\to X} \otimes \mathcal{H}^1_{[F]}(\mathcal{O}_X) = \mathcal{H}^1_{[\tilde{F}]}(\mathcal{O}_{\tilde{X}})$.

It means that the total transform of the sheaf of algebraic local cohomology supported in F is equal to the sheaf of algebraic local cohomology supported in the total transform of F.

Note that if we set $\tilde{f} = f\circ\pi$ then $\delta(\tilde{f}) = 1_{\tilde{X}\to X} \otimes \delta(f)$ generates the $\mathcal{D}_{\tilde{X}}$-Module $\mathcal{H}^1_{[\tilde{F}]}(\mathcal{O}_{\tilde{X}})$.

Example 11.

We calculate the annihilator ideal of $1_{X\to X_1\times X_2}\otimes(\delta(y_1)\hat{\otimes}\delta(y_2^2 - x_2^3))$

by making use of the projection formula.

Set $F = \{(x, y) \in C^2 \mid f(x, y) = y = 0\}$ and $G = \{(x, y) \in C^2 \mid g(x, y) = y^2 - x^3 = 0\}$. Let $\mathcal{M}$ be the sheaf of algebraic local cohomology with support in the cusp G :

$$\mathcal{M} = \mathcal{D}_X \delta(g) \cong \mathcal{H}^1_{[G]}(\mathcal{O}_X)$$

Let (u, v) be local coordinates on X which satisfy $x = v$ and $y = uv$. Since $\tilde{g} = g \circ \pi = v^2(u^2 - v)$, we have

$$\pi^* \mathcal{M} = \mathcal{D}_{\tilde{X}} \delta(\tilde{g})$$

$$= \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}}(v^2(u^2 - v)) + \mathcal{D}_{\tilde{X}}(u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + 6) + \mathcal{D}_{\tilde{X}}(u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial u} + 2u)).$$

If we set $\mathcal{N} = \mathcal{D}_{\tilde{X}} \delta(u) = \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}} u + \mathcal{D}_{\tilde{X}} \frac{\partial}{\partial v})$, then we have

$$\int_\pi \mathcal{N} = \mathcal{D}_X / (\mathcal{D}_X y + \mathcal{D}_X \frac{\partial}{\partial x}).$$

It is easy to verify that

$$\mathcal{N} \otimes_{\mathcal{O}_{\tilde{X}}} \pi^* \mathcal{M} = \mathcal{D}_{\tilde{X}} / (\mathcal{D}_{\tilde{X}} u + \mathcal{D}_{\tilde{X}} v^3 + \mathcal{D}_{\tilde{X}}(v\frac{\partial}{\partial v} + 3)),$$

$$= \mathcal{D}_{\tilde{X}} (\frac{\partial^2}{\partial v^2} \delta(u, v))$$

and

$$\int_{\pi} \mathcal{D}_{\tilde{X}}\left(\frac{\partial^2}{\partial v^2}\,\delta(u,\ v)\right) \;=\; \mathcal{D}_X \,/\, \left(\mathcal{D}_X\, x^3 + \mathcal{D}_X\left(x\frac{\partial}{\partial x} + 3\right) + \mathcal{D}_X y\right).$$

Therefore we have

$$\mathcal{D}_X\, m = \mathcal{D}_X \,/\, \left(\mathcal{D}_X\, x^3 + \mathcal{D}_X\left(x\frac{\partial}{\partial x} + 3\right) + \mathcal{D}_X\, y\right)$$

$$= \mathcal{D}_X\left(\frac{\partial^2}{\partial x^2}\,\delta(x,\ y)\right),$$

where $m = 1_{X \to X_1 \times X_2} \otimes (\delta(y_1)\widehat{\otimes}\partial(y_2{}^2 - x_2{}^3))$.

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