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725巻 1990年 152-162

# SUBANALYTIC GEOMETRY AND PL TOPOLOGY

by

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Let S denote the smallest family of subsets of all Euclidean sapces which satisfies the following conditions:

(i) Every analytic set in a Euclidean space is an element of S.

(ii) For elements $X_1$ and $X_2$ in $\mathbb{R}^n$ of S, $X_1 \cup X_2$ and $X_1 - X_2$ are elements of S.

(iii) Let X in $\mathbb{R}^n$ be an element of S, and let $p:\mathbb{R}^n \to \mathbb{R}^m$ be a linear map such that the restriction of p to the closure of X is proper. Then p(X) is an element of S.

Definitions. We call an element of S subanalytic. A subanalytic map between subanalytic sets is a continuous map with subanalytic graph.

Examples. Examples of subanalytic sets are a semianalytic set in a Euclidean space and a polyhedron included and closed in a Euclidean space [H]. Examples of subanalytic maps are a semianalytic map and a PL (=piecewise linear) map. Embedding closedly a polyhedron in some Euclidean space we give a unique subanalytic structure to the polyhedron.

Remark. A subset X of $\mathbb{R}^n$ is subanalytic if and only if X is a finite union of sets of the form Im $f_1$-Im $f_2$, where $f_1$ and $f_2$ are proper analytic maps from analytic manifolds to $\mathbb{R}^n$[H].

Subanalytic sets and subanalytic maps form a category. The subanalytic category has strong relations to the PL category of polyhedrons and PL maps. For the study of the PL topology, the subanalytic category is useful. Conversely we use many techniques of PL topology for the study of the subanalytic category. The following results show the relations.

(1) Subanalytic Hauptvermutung [S-Y]. Two polyhedrons which are subanalytically homeomorphic are PL homeomorphic.

(2) Subanalytic triangulation of a subanalytic function [$S_1$]. Let X be a closed subanalytic set in a Euclidean space, and let f be a subanalytic function on X. Then f admits a subanalytic triangulation, namely, there exist a polyhedron P and a subanalytic homeomorphism $\pi : P \to X$ such that $f \circ \pi$ is PL.

(3) Subanalytic triangulation of a complex analytic function. Let X be a complex analytic set in an open set of $\mathbb{C}^n$. Then a complex analytic function on X admits a subanalytic triangulation, namely, there exist a polyhedron P and a subanalytic imbedding $\pi : P \to \mathbb{C}^n$ such that $\pi(P)$ equals X and $f \circ \pi$ is PL. (Here X is not necessarily subanalytic.)

We want to restate these facts without using analyticity terminology and with only $C^\infty$ terminology. Consequently some important properties of subanalytic sets and subanalytic maps become clear. I am now preparing for a paper which contains the proofs and other facts.

## §1. Subanalytic Hauptvermutung

Let X and Y be polyhedrons and let $\pi : X \to Y$ be a homeomorphism. We explain conditions on $\pi$ in terminology of $C^\infty$ Whitney stratification. Under these conditions X and Y become PL homeomorphic, and of course a subanalytic homeomorphism satisfies these conditions. For simplicity of notations we assume X and Y to be compact.

Example. Before stating the conditons we consider a simple case of two polyhedrons which are homeomorphic but not necessarily PL homeomorphic. Let M and N be compact $C^\infty$ manifolds which are h-cobordant. Assume dim $M\geq 4$. Then it is known that $M\times\mathbb{R}$ and $N\times\mathbb{R}$ are $C^\infty$ diffeomorphic. Let X and Y denote the suspensions of M and N respectively, and let $x_1,x_2\in X$ and $y_1,y_2\in Y$ be the vertexes. Then by the above diffeomorphism we have a homeomorphism $\pi:X\to Y$ such that $\pi(x_i)=y_i$, i=1, 2, and the restriction of $\pi$ to $X-\{x_1,x_2\}$ is a $C^\infty$ diffeomorphism onto $Y-\{y_1,y_2\}$.

Let $M^p$ and $N^p$ mean the $C^\infty$ triangulations of M and N, respectively, in the sense of Cairns-Whitehead, let $X^p$ and $Y^p$ denote the suspensions of $M^p$ and $N^p$, and define $x_1^p$, $x_2^p$, $y_1^p$, $y_2^p$ and $\pi^p:X^p\to Y^p$. Then it is easy to see that $X^p$ and $Y^p$ are PL homeomorphic if and only if $M^p$ and $N^p$ are PL homeomorphic. Hence $X^p$ and $Y^p$ are homeomorphic but not necessarily PL homeomorphic. When are they PL homeomorphic? The following condition on $\pi$ is sufficient.

Let X and Y be included in some Euclidean space. Let $\rho^X$ and $\rho^Y$ denote the distance functions from $\{x_1,x_2\}$ on X and from $\{y_1,y_2\}$ on Y respectively. We note that $\rho^X$ and $\rho^Y$ are $C^\infty$ functions on $U-\{x_1,x_2\}$ and $V-\{y_1,y_2\}$, respectively, where U and V are some small open neighborhoods of $\{x_1,x_2\}$ in X and of $\{y_1,y_2\}$ in Y respectively. Choose U and V so that $\pi(U)\subset V$.

The condition is

$$|\mathrm{grad}\rho^X_x|\mathrm{grad}(\rho^Y\circ\pi)_x+|\mathrm{grad}(\rho^Y\circ\pi)_x|\mathrm{grad}\rho^X_x\neq 0 \quad \text{for} \quad x\in U.$$

If this is the case then M and N are $C^\infty$ diffeomorphic and hence $X^p$

and $Y^p$ are PL homeomorphic. Note that the above vector vanishes if and only if at least one of $\mathrm{grad}(\rho^Y\circ\pi)_x$ and $\rho^X_x$ vanishes or they point opposite directions. When the condition is satisfied, we say that $\rho^X$ and $\rho^Y\circ\pi$ on U satisfy the <u>condition</u> 1.

The proof is easy. It suffices to see that $\rho^{X-1}(\varepsilon)$ and $\rho^{Y-1}(\varepsilon)$ are $C^\infty$ diffeomorphic for small $\varepsilon>0$ since they are $C^\infty$ diffeomorphic to M and N respectively. Let C be an integral curve of the above defined vector field, and restrict $\rho^X$ and $\rho^Y\circ\pi$ to C. They are strictly monotone. Hence there exists a $C^\infty$ difeomorphism $\tau_C$ of C such that

$$\rho^X = \rho^Y\circ\pi\circ\tau_C \quad \text{on} \quad C\cap U',$$

where U' is a smaller neighborhood of $\{x_1,x_2\}$ in X. Define $\tau$ to be $\tau_C$ on C. Then

$$\rho^X = \rho^Y\circ\pi\circ\tau \quad \text{on} \quad U'-\{x_1,x_2\}$$

and $\tau$ is a $C^\infty$ diffeomorphism, which implies that $\pi\circ\tau$ maps $\rho^{X-1}(\varepsilon)$ to $\rho^{Y-1}(\varepsilon)$ for small $\varepsilon>0$.

<u>Remark</u>. If M, N and $\pi$ are subanalytic then $\rho^X$ and $\rho^Y\circ\pi$ satisfy the condition 1 as follows. Let Z denote the points of U where the condition 1 is not satisfied. Then it is easy to see that Z is subanalytic. Assume the closure $\overline{Z}$ of Z contains $x_1$ or $x_2$. Then by the curve selection lemma there is a subanalytic curve C in Z such that $\overline{C}$ contains $x_1$ or $x_2$. By restricting $\rho^X$ and $\rho^Y\circ\pi$ to C we reduce to the 1-dimensional case. The 1-dimensional case is clear.

<u>Conjecture</u>. Let X and Y be compact polyhedrons, let $\{X_i\}$ and $\{Y_i\}$ be $C^\infty$ Whitney stratifications of X and Y, respectively, and let $\pi:\{X_i\}\to\{Y_i\}$ be a stratified diffeomorphism, namely, $\pi(X_i)=Y_i$

and the restrictions of $\pi$ to $X_i$ are $C^\infty$ diffeomorphisms. If $\pi$ is an isomorphism in the following sense, then X and Y are PL homeomorphic.

<u>Definition of isomorphism</u>. First assume that $\{X_i\}=\{X_1, X_2\}$ and $\dim X_1<\dim X_2$. We call $\pi$ an <u>isomorphism</u> if $\rho_1^X$ and $\rho_1^Y\circ\pi$ on $U_1\cap X_2$ satisfy the condition 1, where $\rho_1^X$ and $\rho_1^Y$ are the distance functions from $X_1$ and $Y_1$, respectively, and $U_1$ is an open neighborhood of $X_1$.

Next assume $\{X_i\}=\{X_1,X_2,X_3\}$ and $\dim X_1<\dim X_2<\dim X_3$. Then we call $\pi$ an <u>isomorphism</u> if the condition 1 is satisfied for the five pairs $\rho_1^X$ and $\rho_1^Y\circ\pi$ on $U_1\cap X_2$, $\rho_1^X$ and $\rho_1^Y\circ\pi$ on $U_1\cap X_3$, $\rho_2^X$ and $\rho_2^Y\circ\pi$ on $U_2\cap X_3$, $\rho_2^X$ and $\rho_2^Y\circ\pi$ on $U_2\cap X_3\cap\rho_1^{X-1}(\varepsilon)$, and $\rho_2^X$ and $\rho_2^Y\circ\pi$ on $U_2\cap X_3\cap (\rho_1^Y\circ\pi)^{-1}(\varepsilon)$ for every small $\varepsilon>0$. Here $U_i$ are open neighborhoods of $X_i$, and $\rho_i^X$, $\rho_i^Y$ are the distance functions from $X_i$, $Y_i$ respectively.

We define an isomorphism for general $\{X_i\}$ in the same way.

I could not prove the conjecture and I proved it under a condition. But I believe that the conjecture is true. Before explaining the condition we remark the following.

<u>Remark</u>. Let X and Y be compact subanalytic sets, and let $\pi:X\to Y$ be a subanalytic homeomorphism. Then there exist $C^\infty$ Whitney stratifications $\{X_i\}$ of X and $\{Y_i\}$ of Y such that $\pi:\{X_i\}\to\{Y_i\}$ is a stratified diffeomorphism. In this case $\pi$ is automatically an isomorphism.

<u>Remark</u>. Let X, Y, $\{X_i\}$, $\{Y_i\}$ and $\pi$ be the same as in the conjecture. Assume $\dim X_1<\dim X_2<\cdots$. Let $\infty\gg\varepsilon_1\gg\varepsilon_2\gg\cdots>0$. Then for each $\ell$

$$X(\ell)=\rho_1^{X-1}([\varepsilon_1,\infty[)\cap\cdots\cap\rho_\ell^{X-1}([\varepsilon_\ell,\infty[)\cap\rho_{\ell+1}^{X-1}([0,\varepsilon_{\ell+1}])\cap$$

$$\rho_{\ell+2}^{X-1}([0,\varepsilon_{\ell+2}])\cap\cdots$$

and $$Y(\ell)=\rho_1^{Y-1}([\varepsilon_1,\infty[)\cap\cdots\cap\rho_\ell^{Y-1}([\varepsilon_\ell,\infty[)\cap\rho_{\ell+1}^{Y-1}([0,\varepsilon_{\ell+1}])\cap$$
$$\rho_{\ell+2}^{Y-1}([0,\varepsilon_{\ell+2}])\cap\cdots$$

are $C^\infty$ diffeomorphic $C^\infty$ manifolds possibly with corners.

We prove this in the same way as the suspension case. Moreover we can construct a $C^\infty$ diffeomorphism from $X(\ell)$ to $Y(\ell)$ which maps a face of $X(\ell)$ to the corresponding face of $Y(\ell)$. If $X(\ell)$ and $Y(\ell)$ are all finite unions of PL balls then we can modify the diffeomorphisms so that they form a piecewise diffeomorphism from X to Y.

<u>Condition 2</u>. For simplicity of notations we assume that $\pi$ in the conjecture is extended to a $C^\infty$ map between the ambient spaces. (This assumption is not strong because we can replace $\pi:X\to Y$ by the projections graph $\pi\to X,Y$.) For each stratum $X_i$, $\overline{X}_i$ is isomorphic to a finite union of simplexes.

<u>Theorem 1</u>. Under the condition 2 the conjecture holds true.

<u>Remark</u>. If $\pi$ is subanalytic then the condition 2 can be satisfied. Under the condition 2 $X(\ell)$ and $Y(\ell)$ in the remark are finite unions of PL balls.

## §2. Subanalytic triangulation of a subanalytic function

We generalize the concept of subanalytic sets. Let $\mathcal{X}$ be a family of subsets of Euclidean spaces such that the following conditions are satisfied. An $\mathcal{X}$-<u>set</u> means an element of $\mathcal{X}$. We define an $\mathcal{X}$-<u>map</u> between $\mathcal{X}$-sets, an $\mathcal{X}$-<u>function</u>, an $\mathcal{X}$-<u>triangulation</u> of an $\mathcal{X}$-set, and an $\mathcal{X}$-triangulation of an $\mathcal{X}$-function in the same way as the subanalytic case.

(i) All algebraic sets in Euclidean spaces are $\mathcal{X}$-sets.

(ii) If $X_1$ and $X_2$ in $\mathbb{R}^m$ and Y in $\mathbb{R}^n$ are $\mathcal{X}$-sets then so are $\overline{X}_1$, $X_1-X_2$ , $X_1 \cap X_2 \subset \mathbb{R}^m$ and $X_1 \times Y \subset \mathbb{R}^m \times \mathbb{R}^n$.

(iii) If X is a nonempty $\mathcal{X}$-set then $\dim(\overline{X}-X) < \dim X$.

(iv) Each connected component of an $\mathcal{X}$-set is an $\mathcal{X}$-set.

(v) Let X in $\mathbb{R}^m$ be an $\mathcal{X}$-set, and let $p: \mathbb{R}^m \to \mathbb{R}^n$ be a linear map such that the restriction of p to the closure of X is proper. Then p(X) is an $\mathcal{X}$-set.

(vi) Let X be a subset of $\mathbb{R}^m$. Assume that X is an $\mathcal{X}$-set locally at each point of $\mathbb{R}^m$, namely, each point of $\mathbb{R}^m$ has a neighborhood U such that $U \cap X$ is an $\mathcal{X}$-set. Then X is an $\mathcal{X}$-set.

(vii) A polyhedron $\mathcal{X}$-homeomorphic to a simplex is PL homeomorphic to a simplex.

(viii) An $\mathcal{X}$-set admits a $C^\infty$ stratification with $\mathcal{X}$-set strata.

(ix) Let an $\mathcal{X}$-set X in $\mathbb{R}^m$ be a $C^\infty$ submanifold. Then the tangent space $TX \subset \mathbb{R}^m \times \mathbb{R}^m$ is an $\mathcal{X}$-set.

Examples. The family of locally semialgebraic subsets of Euclidean spaces is the smallest example of $\mathcal{X}$. The family of subanalytic subsets of Euclidean spaces is another example and may be the biggest. There are many other examples. The family of semianalytic subsets nor the family of semialgebraic subsets is not $\mathcal{X}$. The semianalytic family fails the conditon (v). The semialgebraic family fails the condition (vi).

Theorem 2. Let X be a closed $\mathcal{X}$-set in $\mathbb{R}^m$. Then an $\mathcal{X}$-function f on X admits a unique $\mathcal{X}$-triangulation.

Corollary. A subanalytic function on a closed subanalytic set in $\mathbb{R}^m$ is subanalytically triangulable. A semialgebraic function on a compact semialgebraic set is semialgebraically triangulable.

Consider the graph of f in Theorem 2. Then for the proof it is sufficient to prove the following theorem.

Theorem 2'. Let X be a closed $\mathcal{X}$-set in $\mathbb{R}^m \times \mathbb{R}$. Then there exists an $\mathcal{X}$-imbedding $\pi$ of X into $\mathbb{R}^m \times \mathbb{R}$ of the form

$$\pi(x,t) = (\pi'(x,t),t) \quad \text{for} \quad (x,t) \in \mathbb{R}^m \times \mathbb{R}$$

such that $\pi(X)$ is a polyhedron.

For the local proof of Theorem 2' we use all conditions (i),.. .,(ix) except (vii), and we use the codition (vii) to paste the local triangulations.

If admit a $C^\omega$ stratification in the condition (viii) then we do not need all the conditions. Let $\mathcal{Y}$ be a family of subsets of Euclidean spaces such that the conditions (i), (ii), (v), (vi), (viii) and (ix) are satisfied, where in (viii) we replace $C^\infty$ by $C^\omega$. We define naturally a $\mathcal{Y}$-set, $\mathcal{Y}$-function and a $\mathcal{Y}$-triangulation. Then we have

Theorem 3. Let X be a closed $\mathcal{Y}$-set in $\mathbb{R}^m$. Then a $\mathcal{Y}$-function f on X admits a $\mathcal{Y}$-triangulation.

As the uniqueness condition (vii) is not assumed, it is difficult to paste local triangulations of f. So we need well-controlled local triangulations of f. The $C^\omega$ stratification condition assures such well-controlled local triangulations.

## §3. Subanalytic triangulation of a complex analytic function

The mapping case of Theorem 2 and its corollary is not true. For example, a blowing-up $\mathbb{R}^2 \ni (x,y) \to f(x,y)=(x,xy) \in \mathbb{R}^2$ is not triangulable. Here we call f triangulable if there exist polyhedrons $P_i$, $i=1,2$, and homeomorphisms $\tau_i : P_i \to \mathbb{R}^2$ such that $\tau_2^{-1} \circ f \circ \tau_1$ is PL. If f were triangulable then the map $\mathbb{R}^2 \ni (x,y) \to \dim f^{-1}f(x,y) \in \mathbb{N}$ would be lower semicontinuous by elementary properties of PL maps. But $\dim f^{-1}f(x,y)=1$ for $x=0$ and $=0$ for $x\neq 0$.

So subanalytic triangulability of a complex analytic function comes from some special properties of complex analyticity.

Let U be an open set of $\mathbb{C}^n$, let X be a closed subset of U, and let $f:X\to\mathbb{C}$ be a continuous map. Now we introduce conditions on X and f under which f is $\mathcal{X}$-triangulable. The conditions are satisfied if X and f are complex analytic and if we modify the coordinate system of $\mathbb{C}^n$.

Let $\{X_i\}$ be a finite family of compact $\mathcal{X}$-sets in $\mathbb{C}^n$. We define $\{X_i\}$ to be $\mathcal{X}$-solvable by projections by induction on n. If $n=0$ we require nothing. Assume that we have defined the $\mathcal{X}$-solvability for $n-1$. Let p denote the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ which forgets the first factor. We call $\{X_i\}$ -sovable by projections if the following conditions are satisfied. If $p^{-1}(x)\cap X_i$ is of dimension 2 for some point x of $p(X_i)$ then $X_i$ is of the form $C_1\times\cdots\times C_n$, where $C_j$ are compact polyhedrons in $\mathbb{C}$. Let I denote the index set of other $X_i$, and set

$$X' = \bigcup_{i\in I} X_i, \quad Z=p(X').$$

Then there exists a finite family of compact $\mathcal{X}$-sets $\{Y_j\}$ in $\mathbb{C}^{n-1}$ $\mathcal{X}$-solvable by projections such that $p(X_i)$ for each $i\in I$ is the union of some $Y_j$'s, for the canonical $C^\infty$ stratification $\{Z_k\}$ of Z compatible with $\{Y_j\}$,

$$p|_{X\cap p^{-1}(Z_k)}: X\cap p^{-1}(Z_k) \to Z_k$$

are covering maps, and the family of all connected components of $X\cap p^{-1}(Z_k)$ is compatible with $\{X_i\}_{i\in I}$.

Theorem 4. Let U be an open set of $\mathbb{C}^n$, let X be a closed subset of U which is an $\mathcal{X}$-set locally at each point of U, and let $f: X \to \mathbb{C}$ be an $\mathcal{X}$-function locally at each point of U. Assume that the graph of f is the union of compact $\mathcal{X}$-sets $X_i$, $i=1,2,\ldots$, and that every finte subfamily of $\{X_i\}$ is $\mathcal{X}$-solvable by projections. Then f admits an $\mathcal{X}$-triangulation.

Corollary. Let U be the same as above, and let X be a complex analytic set of U. Then a complex analytic function f on X admits a subanalytic triangulation.

The essential parts of the proof of the theorem are local triangulations and their pasting. The $\mathcal{X}$-solvability admits well-controlled local triangulations. The local triangulation follows from

Lemma. Let Y be a compact $\mathcal{X}$-set in $\mathbb{R}^2\times\mathbb{R}^n$, and $q: \mathbb{R}^2\times\mathbb{R}^n \to \mathbb{R}^n$ the projection. Assume that q(Y) is the underlying polyhedron of a simplicial complex K and for each $\sigma$ of K, $q|_{q^{-1}(\mathrm{Int}\,\sigma)}: q^{-1}(\mathrm{Int}\,\sigma) \to \mathrm{Int}\,\sigma$ is a covering map. Then there exists an $\mathcal{X}$-homeomorphism $\pi$ of $\mathbb{R}^2\times\mathbb{R}^n$ of the form

$$\pi(x,y) = (\pi'(x,y),y) \quad \text{for } (x,y)\in\mathbb{R}^2\times\mathbb{R}^n$$

such that $\pi(Y)$ is a polyhedron.

If we replace $\mathbb{R}^2$ here by $\mathbb{R}$, then this lemma is very easy to see. But for $\mathbb{R}^2$ we have too large freedom of choice of $\pi$. We need to choose a canonical method, which is not easy.

## References

[H] H. Hironaka, Subanalytic sets, Number theory, algebraic geometry and commutative algebra, in honour of Y. Akizuki, 1973, 453-493, Kinokuniya, Tokyo.

[$S_1$] M. Shiota, Piecewise linearization of real-valued subanalytic functions, Trans. Amer. Math. Soc., 312(1989), 663-679.

[$S_2$] _________, Piecewise linearization of subanalytic functions II, to appear.

[S-Y] M. Shiota-M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc., 286(1984), 727-750.