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SINGULARITIES FOR HAMILTON-JACOBI EQUATION AND LEGENDRIAN UNFOLDINGS

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0.Introduction

In this talk we will consider the Cauchy problem for Hamilton-Jacobi equation :

$$\frac{\partial z}{\partial t}+H(t,x_1,\dots,x_n,\frac{\partial z}{\partial x_1},\dots,\frac{\partial z}{\partial x_n})=0 \tag{1}$$

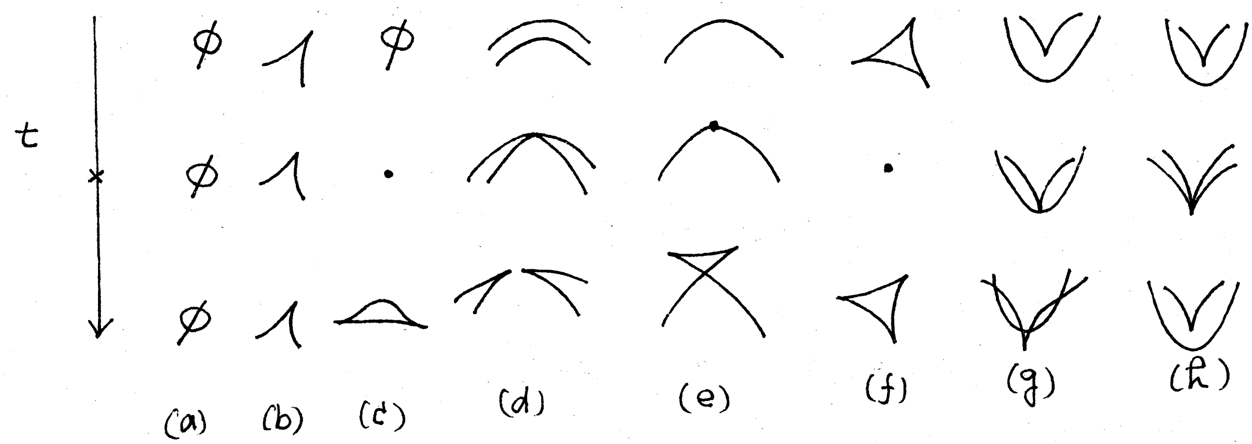
$$z(0,x_1,\dots,x_n)=\phi(x_1,\dots,x_n), \tag{2}$$

where H and ϕ are C^∞-functions.

It is well-known that, even for smooth initial data, the Cauchy problem (1) (2) does not admit a smooth solution for all t. Therefore we consider a "geometric solution" which is constructed by the method of characteristic equation and it may have the singularities.

Recently Tsuji ([4] [5]) studied the behavior of solutions near the singular point on the base space in the case where $n \leq 2$.(Nakane also treated for the general n [3].) He assumed that these singularities are folds or cusps. But, other singularities may be appeared in generic. In fact, by the consequence of our theorem, the generic singularities for geometric solutions for (1) (2) is equal to generic metamorphoses of an 1-parameter caustics which are classified by Arnol'd ([1],[2]).

The picture of these are the following :



The case (C) in the above picture is the process of the appearance of the generic singularities of the solution from the non-singular solution. Because the initial data of the Cauchy problem (1) (2) is smooth, then this process must exist for a neighbourhood of some t_0

I study this problem in the framework of the theory of Legendrian unfoldings. Then we now introduce the theory of Legendrian unfoldings.

All map germs and diffeomorphisms considered here are class C^∞, unless stated otherwise.

1. Legendrian unfoldings

Notations:

1) $J^1(\mathbb{R}^n,\mathbb{R}) \cong \mathbb{R}^{2n+1} \ni (t_1,\dots,x_n,z,p_1,\dots,p_n)$: 1-jet bundle.

2) The canonical 1-form on $J^1(\mathbb{R}^n,\mathbb{R})$: $\theta = dz - \sum_{i=1}^n p_i dx_i$.

3) $J^1(\mathbb{R}^{r-n}\times\mathbb{R}^n,\mathbb{R}) \cong \mathbb{R}^{2r+1} \ni (t_1,\dots,t_{r-n},x_1,\dots,x_n,z,q_1,\dots,q_{r-n},p_1,\dots,p_n)$

4) The canonical 1-form on $J^1(\mathbb{R}^{r-n}\times\mathbb{R}^n,\mathbb{R})$: $\Theta = \theta - \sum_{j=1}^{r-n} q_j dt_j$.

5) $\pi : J^1(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}^n\times\mathbb{R}; \pi(x,z,p) = (x,z)$

Let R be an r-dimensional smooth manifold, $\mu : (R,y_0) \to (\mathbb{R}^{r-n},t_0)$ be a submersion germ and $\ell : (R,y_0) \to J^1(\mathbb{R}^n,\mathbb{R})$ be a smooth map germ. We say that the pair (ℓ,μ) is a

Legendrian family if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}^{r-n}, t_0)$ By the definition, we have the following simple lemma.

LEMMA 1.1. *Let $(\ell.\mu)$ be a Legendrian family. Then there exist unique elements $h_1, \ldots, h_{r-n} \in C^\infty_{y_0}(R)$ such that $\ell^*\theta = \sum_{i=1}^{r-n} h_i d\mu_i$, where $\mu(u) = (\mu_1(u), \ldots, \mu_{r-n}(u))$.*

Define a map germ

$$\mathcal{L} : (R, y_0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), z \circ \ell(u), h(u), p \circ \ell(u)).$$

Then it is easy to show that $\mathcal{L}$ is a Legendran immersion germ. We call $\mathcal{L}$ a *Legendrian unfolding associated with the Legendrian family* (ℓ, μ). Since $\mathcal{L}$ is a Legendrian immersion germ, it has a generating family: Let

$$F : ((\mathbb{R}^{r-n} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a smooth function germ such that $d_2 F|0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, where

$$d_2 F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \ldots, \frac{\partial F}{\partial q_k}(t, x, q)).$$

Then $C(F) = d_2 F^{-1}(0)$ is a smooth r-manifold and

$$\pi_F : (C(F), 0) \to \mathbb{R}^{r-n}$$

is a submersion germ. Here $\pi_F(t, x, q) = t$.

Define map germs

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t,x,q)=(t,x,F(t,x,q),\frac{\partial F}{\partial t}(t,x,q),\frac{\partial F}{\partial x}(t,x,q)).$$

It is easy to show that Φ_F is a Legendrian unfolding associated with $(\tilde{\Phi}_F, \pi_F)$. By the same method as the theory of Arnol'd-Zakalyukin, we can show the following proposition.

PROPOSITION 1.2. *Every Legendrian unfolding germs are constructed by the above method.*

By this proposition, we can apply the singularity theory of smooth families of function germs with distinguished parameters. In this note we will only consider the case where $r = n + 1$. This case is much easier than the other cases. We have another application of the theory of Legendrian unfoldings, in which the case of $r > n + 1$ is very important.

2. GEOMETRY OF HAMILTON-JACOBI EQUATION

In which we will treat Hamilton-Jacobi equation in the framework of the geometric theory of first order partial differential equations. *Hamilton-Jacobi equation* is defined to be a hypersurface

$$E(H)=\{(t,x,z,p,q)\in J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})|q+H(t,x,q)=0\}$$

in $J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$.

Since the equation is contact regular at every points (i.e.$\Theta|E(H)\neq 0$), a generalized Cauchy problem (GCP) has a unique solution: It is solved by the method of characteristic equations. In this case the characteristic vector field is given by

$$X_H=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i}+(\sum_{i=1}^{n}p_i\frac{\partial H}{\partial p_i}-H)\frac{\partial}{\partial z}-\frac{\partial H}{\partial t}\frac{\partial}{\partial q}+\sum_{i=1}^{n}\frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}.$$

We say that *a generalized Cauchy problem (GCP) is posed for an equation E(H)* if there is given an n-dimensional submanifold $i : L' \subset E(H)$ such that $i^*\Theta = 0$ and $X_{H,x} \notin T_x(L')$ for any $x \in L'$.

THEOREM 2.1. (CLASSICAL EXISTENCE THEOREM). *A GCP $i : L' \subset E(H)$ has a unique solution, that is, there is a Legendrian submanifold $L \subset E(H)$, $L' \subset L$ and any two such Legendrian submanifolds coincide in a neighbourhood of L'.*

3. GENERALIZED CAUCHY PROBLEM ASSOCIATED WITH THE TIME PARAMETER

For any $c \in (\mathbb{R}, 0)$, we set

$$E(H)_c = \left\{(c, x, z, -H(c, x, p), p) \in J^1(\mathbb{R} \times R^n, \mathbb{R}) | (x, z, p) \in J^1(\mathbb{R}^n, \mathbb{R})\right\}.$$

Then it is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$.

We define a mapping

$$\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to E(H)_c$$

by

$$\iota_c(x, z, p) = (c, x, z, -H(c, x, p), p).$$

Then it is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n, \mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ \pi \big\downarrow & & \big\downarrow \pi_c \\ \mathbb{R}^n \times \mathbb{R} & = & \mathbb{R}^n \times \mathbb{R}. \end{array}$$

We say that a *generalized Cauchy problem associated with the time parameter (GCPT) is posed for an equation* $E(H)$ if a GCP $i : L' \subset E(H)$ with $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R}, 0)$ is posed.

REMARK. *The Cauchy problem $z(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n)$ is a GCPT. The initial submanifold is given by*

$$L_{\phi,0} = \left\{(0, x, \phi(x), -H(0, x, \frac{\partial \phi}{\partial x}), \frac{\partial \phi}{\partial x}) | x \in \mathbb{R}^n\right\} \subset E(H)_0.$$

Our purpose is formulated in the following problem:

Problem. *Classify the generic bifurcations of singularities of*

$$\pi_t| : \mathcal{L} \cap E(H)_t \to \mathbb{R}^n \times R$$

and

$$\tilde{\pi}_t| : \mathcal{L} \cap E(H)_t \to \mathbb{R}^n$$

with respect to the parameter t.

Let $i : L' \subset E(H)_0 \subset E(H)$ be a GCPT and $\mathcal{L}$ be the unique solution of L'. Since $X_{H,x} \notin T_x E(H)_c$, then $\mathcal{L}$ is transverse to $E(H)_c$ in $E(H)$ for any $c \in (\mathbb{R}, 0)$. It follows that $\mathcal{L}_c = \mathcal{L} \cap E(H)_c$ is an n-dimensional submanifold of $E(H)_c$ and it satisfies $\Theta_c|\mathcal{L}_c = 0$ (i.e.$\mathcal{L}_c$ is a Legendrian submanifold of $E(H)_c$). If we consider the local parametrization of $\mathcal{L}$, we may assume that $\mathcal{L}$ is a image of an immersion germ

$$\mathcal{L} : (\mathbb{R}^n \times \mathbb{R}, 0) \to E(H)$$

such that $\mathcal{L}|(c \times \mathbb{R}^n)$ is a Legendrian immersion germ of $E(H)_c$. Hence the coordinate representation of $\mathcal{L}$ is given by

$$\mathcal{L}(t,u) = (t, x(t,u), z(t,u), -H(t, x(t,u), p(t,u)), p(t,u)).$$

We now define the projection

$$\pi' : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\pi'(t, x, z, q, p) = (x, z, p).$$

It follwos from the above arguments that $(\pi' \circ \mathcal{L}, \pi_1)$ is a Legendrian family, where

$$\pi_1 : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

is the canonical projection. Hence $\mathcal{L}$ is a Legendrian unfolding associated with $(\pi' \circ \mathcal{L}, \pi_1)$.

The main theorem is as follows.

THEOREM 3.1. *(1) The local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$q + H(t, x, p) = 0$$

is a Legendrian unfolding

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}).$$

(2) Let $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be a Legendrian unfolding associated with (ℓ, π_1). Then there exists a C^∞-function germ $H(t, x_1, \ldots, x_n, p_1, \ldots, p_n)$ such that $\mathcal{L}$ is a local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation

$$q + H(t, x, p) = 0,$$

where the initial condition is given by $\ell(0, u)$.

PROOF: The statement (1) is already proved by the avobe arguments. We now prove the statement (2). Taking a coordinate representation of ℓ, we have

$$\ell(t, u) = (x(t, u), z(t, u), p(t, u)).$$

Since (ℓ, π_1) is a Legendrian family, we have

$$\langle dt \rangle_{\mathcal{E}_{n+1}} \supset \langle \ell^* \theta \rangle_{\mathcal{E}_{n+1}}.$$

Hence, there exists a C^∞-function germ $H(t, u)$ such that

$$dz(t, u) - \sum_{i=1}^{n} p_i(t, u) dx_i(t, u) = h(t, u) dt.$$

By the defintion of the Legendrian unfoldings, we have

$$\mathcal{L}(t, u) = (t, x(t, u), z(t, u), h(t, u), p(t, u)).$$

We now define a C^∞-map germ

$$\tilde{\ell} : (\mathbb{R} \times \mathbb{R}^n, 0) \to T^*\mathbb{R}^n$$

by

$$\tilde{\ell}(t, u) = (x(t, u), p(t, u)).$$

Since (ℓ, π_1) is a Legendrian family, $\tilde{\ell}_t$ is a Lagrangian immersion germ with respect to the canonical symplectic structure on $T^*\mathbb{R}^n$ for any $t \in (\mathbb{R}, 0)$.

We also define a C^∞-map germ

$$\ell' : (\mathbb{R} \times \mathbb{R}^n, 0) \to \mathbb{R} \times T^*\mathbb{R}^n$$

by

$$\ell'(t, u) = (t, x(t, u), p(t, u)).$$

By the above argument, ℓ' is an immersion germ. Then

$$\ell'^* : C^\infty_{\ell'(0)}(\mathbb{R} \times \mathbb{R}^n) \to C^\infty_0(\mathbb{R} \times \mathbb{R}^n)$$

is an epimorphism. Since $h \in C^\infty_0(\mathbb{R} \times \mathbb{R}^n)$, there exists $H \in C^\infty_{\ell'(0)}(\mathbb{R} \times \mathbb{R}^n)$ such that $\ell'^*(H) = -h$. That is,

$$H(t, x(t, u), p(t, u)) = h(t, u).$$

Thus the Lgendrian unfolding

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

is a geometric solution of the Hamilton-Jacobi equation

$$q + H(t, x, p) = 0.$$

It is also a local solution of GCPT of the Hamilton-Jacobi equation whose initial condition is $\ell(0, u)$.

By this theorem, we can apply the classification theory of bifurcations of singularities of 1-parameter Legendrian unfoldings. It is corresponding to Arnol'd's theory of 1-parameter bifurcations of caustics and wave front sets.

References

1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, "Singularities of Differentiable Maps," Birkhauser, 1986.
2. V. I. Arnol'd, *Evolution of singularities of potential fows in collision-free media and the metamorphosis of caustics in three-dimensional space*, Trudy Seminara imeni I.G. Petrovskogo **No. 8** (1982), p. 21-57.
3. S. Nakane, Formation of singularities for Hamilton-Jacobi equation in several space dimensions.
4. M. Tsuji, *Solution globale et propagation des singularites pour l'equation de Hamilton-Jacobi*, C. R. Acad. Sc. Paris **289** (1979), p. 397–400.
5. M. Tsuji, *Formation of singularities for Hamilton-Jacobi equation II*, J. Math. Kyoto Univ. **26** (1986), 299–308.