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TRANSVERSALITY OF LINEAR HOLOMORPHIC VECTOR FIELD ON $\mathbb{C}^n$

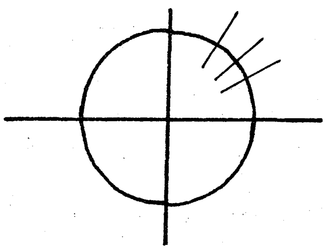
TOSHIKAZU ITO *

(Ryukoku University)

INTRODUCTION

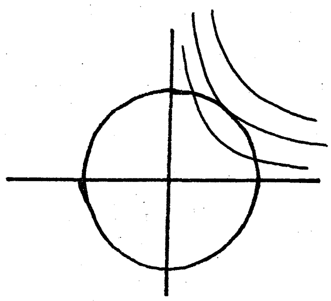
Let $\mathcal{F}$ be the holomorphic foliation on $\mathbb{C}^2 - \{0\}$ defined by a linear vector field $X = \sum_{i=1}^{2} \lambda_i z_i \partial/\partial z_i$ ($\lambda_i \in \mathbb{C}$, $\lambda_i \neq 0$) on $\mathbb{C}^2$. Let us begin by recalling a well-known fact ([1],[2]) :

FACT If λ_1/λ_2 does not belong to $\mathbb{R}_- = \{$ negative real numbers $\}$, then the 3 dimensional unit sphere S^3 in $\mathbb{C}^2$ is transverse to $\mathcal{F}$. On the other hand, if $\lambda_1/\lambda_2 \in \mathbb{R}_-$, S^3 is not transverse to $\mathcal{F}$.



$\lambda_1/\lambda_2 \notin \mathbb{R}_-$

Fig. 1



$\lambda_1/\lambda_2 \in \mathbb{R}_-$

Fig. 2

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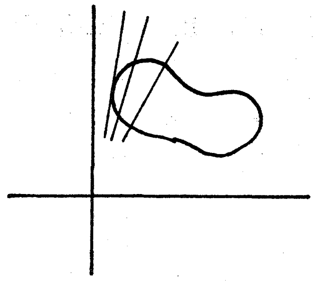
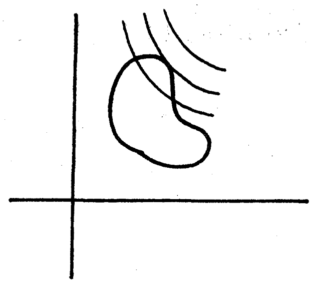


$\lambda_1/\lambda_2 \notin \mathbf{R}_-$

Fig. 3



$\lambda_1/\lambda_2 \in \mathbf{R}_-$

Fig. 4

These pictures suggest to us a question :

QUESTION Let M be a connected closed 2 or 3 dimensional smooth manifold. Is there a smooth map ϕ ; $M \longrightarrow \mathbf{C}^2 - \{0\}$ such that ϕ is transverse to $\mathfrak{F}$?

In this note, we shall give an answer to this question. This note is divided into three sections. In §1, we give a definition of transversality of maps to a holomorphic vector field on $\mathbf{C}^n$ and some examples. In §2, we investigate a non-existence of transversal maps. Finally in §3, we have a structure theorem on an existence of transverse maps on condition that λ_1/λ_2 is a real positive number.

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§ 1 DEFINITION OF TRANSVERSALITY OF MAPS TO A HOLOMORPHIC VECTOR FIELD ON $\mathbf{C}^n$ AND WELL-KNOWN EXAMPLES

Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n$ defiened by the solutions of a holomorphic vector field X on $\mathbf{C}^n$. Let M be a smooth manifold of dimension 2n-2 or 2n-1 and ϕ a smooth map from M in $\mathbf{C}^n$.

DEFINITION 1.1 We say that the map ϕ ; M $\longrightarrow$ $\mathbf{C}^n$ is transverse to $\mathcal{F}$ if the following identity satisfies for all points p ε M :

$$\phi_*(T_p(M)) + T_{\phi(p)}(\mathcal{F}) = T_{\phi(p)}(\mathbf{C}^n)$$

We shall here give the well-known examples.

EXAMPLE 1.2 ([1],[2]) Let $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ a linear holomorphic vector field on $\mathbf{C}^n$. We assume that $\lambda_i \notin \mathbf{R}\lambda_j$ for i ≠ j. The convex hull of $\Lambda = \{ \lambda_1, \ldots , \lambda_n \}$ in $\mathbf{C}$ is denoted $\mathcal{H}(\Lambda)$.

(i) If the origin 0 in $\mathbf{C}$ belongs to $\mathcal{H}(\Lambda)$, then the 2n-1 dimensional unit sphere S^{2n-1} in $\mathbf{C}^n$ is not transverse to $\mathcal{F}$.

(ii) If the origin 0 in $\mathbf{C}$ does not belong to $\mathcal{H}(\Lambda)$, then S^{2n-1} is transverse to $\mathcal{F}$.

§ 2 NON-EXISTENCE OF TRANSVERSE MAPS

First, the Fig.3 and 4 lead us to a theorem : Poincaré-

Hopf type theorem for holomorphic vector field.

THEOREM 2.1 ([3]) Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n$ defined by the solutions of a holomorphic vector field X on $\mathbf{C}^n$, $n \geqq 2$. If a smooth imbedding ϕ from the $2n-1$ dimensional sphere S^{2n-1} in $\mathbf{C}^n$ is transverse to $\mathcal{F}$, then X has only one singular point in the inside of $\phi(S^{2n-1})$.

Secondly, in the special case of a linear vector field $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ on $\mathbf{C}^n$, $n \geqq 2$, we have some results.

COROLLARY 2.2 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1, \ldots, n$). A smooth imbedding ϕ ; $S^{2n-1} \longrightarrow \mathbf{C}^n - \{0\}$ which is homotope to zero in $\Pi_{2n-1}(\mathbf{C}^n - \{0\})$ is not transverse to $\mathcal{F}$.

COROLLARY 2.3 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2 - \{0\}$ defined by $X = \sum_{i=1}^{2} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1,2$). Let T^3 be the torus of dimension 3. A smooth imbedding ϕ ; $T^3 \longrightarrow \mathbf{C}^2 - \{0\}$ which satisfies the following property (#) is not transverse to $\mathcal{F}$.

Property (#) : There exists a smooth imbedding Φ ; $S^1 \times S^1 \times D^2 \longrightarrow \mathbf{C}^2$ such that Φ satisfies $\Phi|_{\partial(S^1 \times S^1 \times D^2)} = \phi$ and the image of Φ contains zero in $\mathbf{C}^2$.

THEOREM 2.4 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1, \ldots, n$).

Assume that at least one of λ_i/λ_j ($i \neq j$) is a negative real number. Let M be a closed connected smooth manifold of dimension 2n-2 or 2n-1. Then there is not a smooth map ϕ , $M \longrightarrow \mathbf{C}^n - \{0\}$ which is transverse to $\mathcal{F}$.

THEOREM 2.5 ([4]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1, \dots, n$). Assume that $\lambda_i \notin R\lambda_j$ ($i \neq j$) and $0 \in \mathcal{H}(\Lambda)$. Let M be a closed connected smooth manifold of dimension 2n-1. Then there is not a smooth map ϕ ; $M \longrightarrow C^n - \{0\}$ which is transverse to $\mathcal{F}$.

§ 3 EXISTENCE OF TRANSVERSE MAP AND STRUCTURE THEOREMS

Let $\mathcal{F}$ be the holomorphic foliation on $\mathbf{C}^n - \{0\}$ defined by a linear vector field $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1, \dots, n$). Assume that all λ_i/λ_j ($i \neq j$) are positive rational numbers. Then the 2n-1 dimensional unit sphere S^{2n-1} in $\mathbf{C}^n$ is transverse to $\mathcal{F}$ and the foliation on S^{2n-1} defined by $\mathcal{F}$ is a generalized Seifert structure. Now, we have a structure theorem.

THEOREM 3.1 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, $i = 1, \dots, n$). Assume that all λ_i/λ_j ($i \neq j$) are positive real numbers. Let M be a closed connected smooth manifold of dimension 2n-1. If a smooth map ϕ ; $M \longrightarrow \mathbf{C}^n - \{0\}$ is transverse to $\mathcal{F}$, then

M is diffeomorphic to the sphere S^{2n-1} of dimension 2n-1.

Because of the existence of 2-field on a manifold M which is transverse to $\mathcal{F}$ defined by a holomorphic vector field X on $\mathbf{C}^{2n}$, $n \geqq 1$, we have other structure theorem.

THEOREM 3.2 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2$ defined by a holomorphic vector field X on $\mathbf{C}^2$. Let M be a closed connected smooth manifold of dimension 2. If a smooth map ϕ ; $M \longrightarrow \mathbf{C}^2$ is transverse to $\mathcal{F}$, then M is diffeomorphic to the torus T^2. Moreover, in the case of a linear vector field $X = \sum_{i=1}^{2} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, i = 1,2 and $\lambda_1/\lambda_2 \notin \mathbf{R}$), we can construct a smooth map ϕ ; $T^2 \longrightarrow \mathbf{C}^2$ such that ϕ is transverse to $\mathcal{F}$.

Finally, we shall give some examples of transverse maps.

EXAMPLE 3.3 ([5]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^2 - \{0\}$ defined by $X = \sum_{i=1}^{2} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, i=1,2 and $\lambda_1/\lambda_2 \notin \mathbf{R}$) Let M be T^3 or $S^2 \times S^1$. Then we can construct a smooth map ϕ ; $M \longrightarrow \mathbf{C}^2 - \{0\}$ such that ϕ is transverse to $\mathcal{F}$. (cf. §2 Corollary 2.3)

EXAMPLE 3.4 ([4]) Let $\mathcal{F}$ be the foliation on $\mathbf{C}^n - \{0\}$ defined by $X = \sum_{i=1}^{n} \lambda_i z_i \, \partial/\partial z_i$ ($\lambda_i \neq 0$, i = 1, ... , n). Assume that $\lambda_i \notin R\lambda_j$ ($i \neq j$) and $0 \notin \mathcal{H}(\Lambda)$. then there exists a smooth imbedding ϕ ; $S^1 \times S^{2n-3} \times S^1 \longrightarrow \mathbf{C}^n - \{0\}$ which is transverse to $\mathcal{F}$.

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