<table>
<thead>
<tr>
<th>Title</th>
<th>A FULL-INFORMATION BEST-CHOICE PROBLEM WITH ALLOWANCE (Studies on Decision Theory and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tamaki, Mitsushi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1990), 726: 167-180</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1990-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/101893">http://hdl.handle.net/2433/101893</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
A FULL-INFORMATION BEST-CHOICE PROBLEM WITH ALLOWANCE

Mitsushi Tamaki

0. INTRODUCTION

The basic form of the full-information best-choice problem, originally studied by Gilbert and Mosteller (1966) can be described as follows: Let $X_i, i=1, 2, \ldots, n$ be the value attached to the $i^{th}$ item and suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables from a known continuous distribution function $F$. On arrival of the $i^{th}$ item, we observe $X_i$ and decide immediately either to accept or reject this item, weighing the possibility of obtaining a better item against the risk of losing the current item. The objective is to maximize the probability of choosing the overall best, i.e., the item which has the largest value among all, assuming no solicitation of the previously rejected item. If the $n-1$ items have been rejected, the last one must be accepted. Generalizations and extensions of this problem were made by Petruccelli (1982) and Tamaki (1986).

In the real situation, though the chosen item is not the overall best, we will be satisfied with it if its value is sufficiently large compared with the overall best. This motivates our problems. In Section 1, an allowance function $\phi(.)$ will be introduced. Let $x$ be the value of the chosen item and $y$ be the largest value among all, then this selection is a success if $x \geq y - \phi(y)$. We seek an optimal strategy, which maximizes the probability of success.

1. ALLOWANCE MODEL

Here $X_1, X_2, \ldots, X_n$ are assumed to be independent and identically distributed non-negative random variables from a known continuous distribution function $F(x)$. $F(x)$ is also assumed to be increasing on the set where $0 < F(x) < 1$.

Let $Y_i = \max\{X_1, X_2, \ldots, X_i\}, 1 \leq j \leq n$, and $\phi(y)$ be a prescribed allowance function
defined on \((0, \infty)\). Then the state of the decision process after having observed
\(X_1, X_2, \ldots, X_{n-k}\) can be described as \((x, y, k), 0 \leq y \leq y_0, 0 \leq k \leq n\), if \(X_{n-k}=x\) and \(Y_{n-k}=y\) (note
that \(k\) represents the number of the remaining observations available) and choosing
\(X_{n-k}\) in this state can be regarded as a success if \(Y_{n-k} \leq Y_n\).

To make the subsequent analysis simple, we put the following two assumptions
on \(\rho(y)\).

\((A1) \ \ \rho(y)\ is\ a\ continuous\ function\ of\ y\ with\ 0 \leq \rho(y) \leq y.\)

\((A2) \ \ y - \rho(y)\ is\ non-decreasing\ in\ y.\)

\((A1)\ is\ a\ natural\ assumption.\ Now\ let
\(E=\{(x, y): y - \rho(y) \leq x \leq y\}\),
then \((A2)\ assures\ that,\ for\ each\ k,\ it\ is\ not\ optimal\ to\ accept\ X_{n-k}\ in\ state
\((x, y, k)\ for\ which\ (x, y) \notin E.\ This\ is\ easily\ seen\ because,\ under\ \(E\),\ if\ (x, y) \notin E\)
then \((x, y') \notin E\ for\ y' \leq y\ and\ because\ the\ maximum\ value\ observed\ so\ far\ does\ not\ decrease\ as\ time\ goes.\ This\ is\ why\ we\ confine\ our\ attention\ to\ state\ \((x, y, k)\ for
which\ (x, y) \in E.\ If\ \((x, y) \in E,\ x\ is\ called\ a\ candidate\ with\ respect\ to\ y\ (sometimes\ x\ is\ simply\ called\ a\ candidate).\ It\ should\ be\ noted\ that
under\ \(E\),\ if\ \(x\ is\ a\ candidate\ with\ respect\ to\ y,\ then
\(x\ is\ also\ a\ candidate\ with\ respect\ to\ y'\ when\ (x) \leq y' < y.\)

Define
\[\beta(x) = \sup \{(y, x) \in E\}.\]
\(\beta(x)\ then\ represents\ the\ maximal\ value\ of\ y,\ for\ which\ x\ remains\ a\ candidate.\ It\ follows\ from\ \(A1)\ and\ \(A2)\ that\ \beta(x)\ is\ increasing\ in\ x\ where\ F(\beta(x)) < 1,\ but\ possibly\ have\ several\ discontinuity\ points.\ To\ guarantee\ that\ \beta(x)\ is\ a\ continuously
increasing function, (A2) must be replaced by

\[(A2)' \quad y - \rho(y) \text{ is increasing in } y.\]

Two typical cases of \(\rho(y)\) which satisfy (A1) and (A2) are as follows. Corresponding \(\beta(x)\) is also given:

**Example 1 (proportional allowance)**

\[\rho(y) = ry, \quad y \geq 0\]
\[\beta(x) = x/\tau, \quad x \geq 0 \quad \text{where } \tau = 1-r \quad \text{and } 0 \leq r < 1.\]

**Example 2 (constant allowance)**

\[\rho(y) = \min(y, c), \quad y \geq 0 \quad \text{and } c > 0\]
\[\beta(x) = x + c, \quad x \geq 0.\]

Example 1 satisfies \((A2)'\) but Example 2 does not satisfy \((A2)'\).

In state \((x, y, k)\), we have two alternatives: acceptance (stopping) and rejection (continuance). Let \(s_k(x, y)\) be the probability of success when we accept the candidate and \(c_k(x, y)\) be the corresponding probability when we assume continuation in an optimal manner. It is easy to see that \(s_k(x, y)\) \((c_k(x, y)\)) depends on \((x, y)\) only through \(x\) \((y)\). So we simply write \(s_k(x)\) and \(c_k(y)\) for them. Put, for \((x, y) \in \mathbb{E}\) and \(0 \leq k \leq n,

\[v_k(x, y) = \max\{s_k(x), \ c_k(y)\} \tag{1.2}\]

Then we have the following recursive relations

\[s_k(x) = F(\beta(x))^x, \quad 0 \leq k \leq n \tag{1.3}\]
\[c_k(y) = F(y - \rho(y))\ c_{k-1}(y) + \int_y^{\infty} v_{k-1}(t, y) \ dF(t) + \int_0^y v_{k-1}(t, t) \ dF(t) \tag{1.4}\]

with the boundary condition \(c_0(y) = 0\).

Eq. (1.3) is immediate since all the remaining observations must have values not greater than \(\beta(x)\) for our selection \(X_{n-k}\) to be a success. After leaving state \((x, y, k)\), \(X_{n-k+1}\) is observed but rejected if it is not a candidate, i.e.,
\(X_{n-k+1} < y - \rho(y)\). Otherwise state makes transition into \((t, y, k-1)\) or \((t, t, k-1)\), depending on whether \(X_{n-k+1} = t < y\) or \(X_{n-k+1} = t \geq y\). This yields Eq. (1.4). Note that Eq. (1.2) for \(k = n - 1\) is defined only for \(x = y\) due to \(X_1 = Y\), and the probability of success is calculated as \(\int_0^\infty v_{n-1}(x, x) dF(x)\).

We start with the following lemma.

Lemma 1.1. Assume that (A1) and (A2) hold. Then, for each \(k\),

\[c_k(y)\text{ is continuous and non-increasing in } y.\]

Proof. Denote by \(\sigma_i\), \(i = 1, 2\), an optimal strategy followed after leaving state \((x, y_1, k)\), where \(y_1 > y_2 \geq x\) and compare the following two situations:

situation 1: We leave \((x, y_1, k)\) and use \(\sigma_i\).

situation 2: We leave \((x, y_2, k)\) and use \(\sigma_i\).

It is easy to see from (1.1) that the success in situation 1 is also a success in situation 2. Thus the probability of success in situation 2 is at least as large as \(c_k(y_1)\), and consequently \(c_k(y_2) \geq c_k(y_1)\). Continuity follows by induction on \(k\) from (1.4).

Remark. We can prove \(dc_k(y)/dy \leq 0\) by induction on \(k\), assuming all differentiability required. Let \(f(t) = dF(t)/dt\), then differentiating formally both sides of (1.4) yields

\[c_k'(y) = F(y - \rho(y))c_{k-1}'(y) + (\rho'(y) - 1) f(y - \rho(y)) \{v_{k-1}(y - \rho(y), y) - c_{k-1}(y)\} + \int_{-\rho(y)}^{f(y - \rho(y))} \{v_{k-1}(t, y) / dy\} dF(t).\]

\(c_{k-1}(y)\) and \(v_{k-1}(t, y)\) are non-increasing in \(y\) from the induction hypothesis and \(1 - \rho'(y) \geq 0\) from (A2). Hence, each term in the right side of the above equation is non-positive and \(c_k'(y) \leq 0\).
Let, for $0 \leq k < n$,

$$G_k = \{(x, y) \in E : s_k(x) \geq c_k(y)\}.$$  

Then it is optimal to accept the candidate in state $(x, y, k)$ for which $(x, y) \in G_k$.

Since, under (A1) and (A2), $s_k(x)$ is increasing in $x$, where $s_k(x) < 1$ (because $\beta(x)$ is increasing), the following theorem is an immediate consequence from Lemma 1.1.

**Theorem 1.2.** Assume that (A1) and (A2) hold. Then, for $k \geq 1$, there exist two critical numbers

$$a_k = \inf \{y : s_k(y) \geq c_k(y)\},$$  \hspace{1cm} (1.5)

$$b_k = \inf \{y : s_k(y - \rho(y)) \geq c_k(y)\},$$  \hspace{1cm} (1.6)

and a non-increasing continuous function

$$f_k(y) = \inf \{x : s_k(x) \geq c_k(y)\}, \text{ for } x \leq y \leq b_k,$$  \hspace{1cm} (1.7)

such that

$$G_k = \{(x, y) : f_k(y) \leq x \leq b_k, \text{ for } x \leq y \leq b_k\} \cup \{(x, y) : y - \rho(y) \leq x \leq b_k \},$$

When $k = 0$, $a_0 = b_0 = 0$ and $G_0 = E$.

**Remarks.** (1) If $X_t$ is bounded, i.e., there exists $A$ such that $F(t) = 1$ for $t \leq A$, then $s_k(t - \rho_A(x))$ since $s_k(x) \equiv 1$ for $x \leq A - \rho_A(x)$.

(2) Assume that (A1) and (A2)' hold. Then $\beta(x)$ is continuously increasing and so is $s_k(x)$, where $s_k(x) < 1$. Hence, in this case, (1.5) - (1.7) can be reduced to the following forms:

$a_k$ is the unique root $y$ of the equation

$$s_k(y) = c_k(y),$$  \hspace{1cm} (1.5)' 

$b_k$ is the unique root $y$ of the equation

$$s_k(y - \rho(y)) = c_k(y),$$  \hspace{1cm} (1.6)'
and $\mathcal{F}_k(y)$ is the unique root $x$ of the equation

$$s_k(x) = c_k(y), \quad a_k \leq y \leq b_k.$$  \hspace{1cm} (1.7)

**Lemma 1.3.** Assume that (A1) and (A2)' hold. Then

$$G_k \in G_k, \quad 0 \leq k \leq n-1.$$  

**Proof.** Let $k^* > 0$ be the smallest $k$ such that $G_k \in \mathcal{G}$. Then, by the continuity property of $s_k(x)$ and $c_k(y)$, there exists a non-empty set (subset of $G_k$) defined by

$$\mathcal{G}_k = \{(x, y) \in G_k: x = \mathcal{F}_k(y)\}, \quad k \geq k^*.$$  

To prove the lemma, it is sufficient to show that, for $(x, y) \in \mathcal{G}_k$,

$$c_{k+1}(y) \geq s_{k+1}(x).$$  \hspace{1cm} (1.8)

From (1.4) and the assumption that $(x, y) \in \mathcal{G}_k$,

$$c_{k+1}(y) = F(y - \rho(y)) \int \left\{ s_{k+1}(x) \right\} v_k(t, y) \, dF(t)$$

$$+ \int \left\{ s_{k-1}(x) \right\} v_k(t, y) \, dF(t) + \int \left\{ s_k(t) \right\} v_k(t, y) \, dF(t)$$

$$= F(y - \rho(y)) \int \left\{ c_{k+1}(y) \right\} v_k(t, y) \, dF(t) + \int \left\{ s_k(t) \right\} v_k(t, y) \, dF(t)$$

$$= F(x) \int \left\{ s_k(t) \right\} v_k(t, y) \, dF(t)$$

$$= F(x) s_k(x) + \int \left\{ s_k(t) \right\} v_k(t, y) \, dF(t),$$

where the last equality follows since $c_k(y) = s_k(x)$ on $(x, y) \in \mathcal{G}_k$. Thus, from the monotonicity property of $s_k(t)$ with respect to $t$,

$$c_{k+1}(y) - s_{k+1}(x)$$

$$= F(x) s_k(x) + \int \left\{ s_k(t) \right\} v_k(t, y) \, dF(t) - s_{k+1}(x)$$

$$\geq F(x) s_k(x) + \int \left\{ s_k(x) \right\} v_k(t, y) \, dF(x) - s_{k+1}(x)$$

$$= s_k(x) - s_{k+1}(x)$$

$$= \{1 - \text{E}(\rho(x))\} \left\{ F(\rho(x)) \right\}^n$$

$$\geq 0,$$

which proves (1.8).
Remark. Example 2 (constant allowance) does not satisfy (A2)', but it is easy to show by induction that \(a_n, b_n\), and \(\mathcal{J}_n(y)\) can be determined by (1.5)' - (1.7)' and Lemma 1.3 still holds if \(F(2c) \ll 1\). We can achieve success with certainty if \(F(2c) = 1\). In this case, there exists a finite number \(A\) such that \(A = \inf \{x: F(t) = 1, t \geq x \}\). Hence, we employ a strategy which accepts an item whose value exceeds \(A - c\) and, if no such item appears in the first \(n-1\) observations, accepts the last item.

What is left is to determine the sequences of the decision numbers \([a_n]\) and \([b_n]\), and the sequence of the decision function \(\mathcal{J}_n(y)\) for \(a_n \leq y \leq b_n\). Hereafter we assume (A1) and (A2)', unless otherwise specified. Letting \(k = 1\) in (1.3) and (1.4) yields

\[
S_n(x) = F(\rho(x)),
\]

\[
c_n(y) = 1 - F(y - \rho(y)).
\]

Thus, from (1.5)' - (1.7)', \(a_n\) is the unique root \(y\) of the equation

\[
F(\rho(y)) + F(y - \rho(y)) = 1,
\]

(1.9)

\(b_n\) is the unique root \(y\) of the equation

\[
F(y) + F(y - \rho(y)) = 1,
\]

(1.10)

and

\[
\mathcal{J}_n(y) = F^{-1}(c_n(y)) - \rho(F^{-1}(c_n(y))).
\]

(1.11)

For \(k \geq 2\), corresponding quantities are difficult to be obtained by solving recursively (1.2) - (1.4). Repeated use of (1.4) yields, for \(y > 0\),

\[
c_n(y) = \Sigma_{j=1}^{\infty} [F(y - \rho(y))]^{j-1} \left[ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \right] \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt.
\]

In particular, for \(y \geq b_{k-1}\),

\[
c_n(y) = \Sigma_{j=1}^{\infty} [F(y - \rho(y))]^{j-1} \left[ \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt \right] F(\rho(t))^{k-1} \, dt.
\]

(1.12)

which follows because the optimal strategy, after leaving state \((x, y, k)\), immediately accepts a candidate that appears due to the monotonicity property of
Given in Lemma 1.3. This makes it easy to calculate $b_k$.

**Lemma 1.4.** The decision number $b_k$, $1 \leq k < n$, is the unique root $y \in (2b_{k-1})$ of the equation

$$
[F(y)]^k = \sum_{i=1}^k \left( \frac{F(y) - \rho(y)}{y - \rho(y)} \right)^{k-i} \frac{1}{y} \left( \frac{F(y)}{F(t)} \right)^{k-i} dF(t).
$$

(1.13)

**Proof.** From (1.9)', $b_k$ is the value of $y$ which equates $s_k(y - \rho(y))$ with $c_k(y)$. Thus the result is immediate from (1.12), since $b_k \geq b_{k-1}$ from Lemma 1.3 and $s_k(y - \rho(y)) = (F(\beta(y - \rho(y))))^k = (F(y))^k$ from the definition of $\rho(y)$.

**Remark.** Let $\rho(y) = 0$ and denote by $t_k$ the corresponding decision number $b_k$. Then, from (1.13), $t_k$ satisfies

$$
[F(y)]^k = \sum_{i=1}^k \left( \frac{F(y)}{F(t)} \right)^{k-i} \frac{1}{y} dF(t)
$$

(1.13')

or equivalently

$$
[F(y)]^k = \sum_{i=1}^k \left[ \frac{1}{y} \left( \frac{F(y)}{F(t)} \right)^{k-i} dF(t) - \frac{1}{y} (F(y))^k \right] / (k-1).
$$

This is the well known result in the full-information best-choice problem (see Gilbert and Mosteller 1966 or Sakaguchi 1973)

The following lemma provides an algorithm for calculating $s_k$ and $J_k(y)$ for $a_k \leq y \leq b_k$, when $a_k$, $b_k$, and $J_k(y)$ for $a_k \leq y < b_k$ and $1 \leq k < n$ are given. We use notation $I_j = (b_{j-1}, b_j)$, $j \geq 1$.

**Lemma 1.5.** $a_k$, $b_k$, and $J_k(y)$ for $a_k \leq y \leq b_k$ are calculated from (1.9) - (1.11). Assume that $a_k$, $b_k$, and $J_k(y)$ for $a_k \leq y < b_k$ are known, where $1 \leq k \leq n$. First solve $b_k$ from (1.13) and let $i(1 \leq i < k)$ be the integer such that $a_{k-i} \in I_i$. Then $c_k(y)$ for $a_{k-1} \leq y \leq b_k$, $a_k$, and $J_k(y)$ for $a_k \leq y \leq b_k$ can be calculated as follows:
(i) Let
\[ \lambda_q(y) = \begin{cases} \prod_{k=1}^{q} F(\gamma_k(y)), & q < k \\ 1, & q = k \end{cases} \]
and define, for \( i \leq j \leq k, \)
\[ I_{ij} = \begin{cases} [a_{i-1}, b_i], & j = i \\ I_i, & i < j \leq k. \end{cases} \]

Then, for \( y \in I_{ij}, \)
\[ c_k(y) = \lambda_k(y) \sum_{i=0}^{k-1} \{ F(y - \rho(y)) \} i \sum_{x=1}^{i} \{ F(\beta(x)) \} i \sum_{x=1}^{i} F(x) \]
\[ + \sum_{x=1}^{k-1} \lambda_{k-1}(x) \sum_{x=1}^{k} \{ F(\beta(x)) \} i \sum_{x=1}^{k} F(x), \] (1.14)
where the vacuous sum is assumed to be 0.

(ii) \( a_k \) is the unique root \( y \) in \( [a_{k-1}, b_k] \) of the equation
\[ s_k(y) = c_k(y). \]

(iii)
\[ f_k(y) = F^{-1}(\sqrt[c]{c_k(y)}) = F^{-1}(\sqrt[c]{c_k(y)}), \quad a_k \leq y \leq b_k. \] (1.15)

Proof. Fix \( y \in I_{ij} \) for given \( i \) and define
\[ d_k(y) = \begin{cases} \phi_k(y), & i \leq j \leq k-1 \\ y - \rho(y), & 0 \leq j < i \end{cases}. \]

Then it is easily seen from Lemma 1.3 that, after leaving state \( (x, y, k) \), the optimal strategy immediately accepts \( X_{n-x}, 0 \leq k < n \), if \( X_{n-x} \neq d_k(y) \). Thus
\[ c_k(y) = \sum_{i=1}^{k-1} \{ \mu_{i}^{x}(x) \} \sum_{x=1}^{i} \{ F(d_k(y)) \} i \sum_{x=1}^{i} \mu_{i}^{x}(x) F(x), \]
which, combined with (1.3), yields (1.14). (ii) and (iii) are from (1.5)' and
(1.7)'.

Remark. It is easy to see from remark of Lemma 1.3 that Lemmas 1.4 and 1.5 hold
for Example 2 (constant allowance) with \( F(2c) < 1 \).
In principle, repeated use of Lemma 1.5 successively determines $a_1$, $b_1$ and $\mathcal{F}_1(y)$ for $y \in [a_1, b_1]$

$a_2$, $b_2$ and $\mathcal{F}_2(y)$ for $y \in [a_2, b_2]$

When $F$ is a uniform distribution on $[0, 1]$, some simplification can be done in calculating the decision numbers and the decision functions for Examples 1 and 2.

Taking account of

$$\rho(t) = \begin{cases} 
\frac{1}{t} & \text{(proportional allowance)} \\
1 + c & \text{(constant allowance)}
\end{cases},$$

we have, for $x < 1 - \rho(1)$,

$$\int (F(\rho(t)))^m dF(t) = \int \frac{1}{t} \rho(t)^m dt + \int [1 - \rho(t)]^- dt$$

$$= \begin{cases}
\int \frac{r^m}{(x/r)^{m+1}/(m+1)} dt & \text{(proportional allowance)} \\
\int (c + [1 - (x+c)]^{-1}/(m+1)) dt & \text{(constant allowance)}
\end{cases}.$$ 

Hence, the following corollary is immediate from Lemmas 1.4 and 1.5.

**Corollary 1.6.** Assume that $F(.)$ is a uniform distribution on $[0, 1]$. Let, for fixed $k$,

$$\mathcal{A}_k(y) = \begin{cases} 
\prod_{j=1}^{k-1} \mathcal{F}_j(y), & y < k \\
1, & y = k
\end{cases},$$

then (1.13) - (1.15) can be written as follows:

(i) Example 1 (proportional allowance)

$b_1$ is the unique root $y$ in $(b_1, 1]$ of the equation

$$y^k = \sum_{j=1}^k (\mathcal{F}_j y)^{k-j} \left\{ r/\mathcal{F}(1-y) / j \right\}.$$
For \( y \in I_i \) (i \( \notin \) \( S \)),

\[
c_k (y) = \Lambda_n (y) \sum_{i=0}^{r} (\bar{\tau} y)^{\frac{\beta}{1-\epsilon}} \left( r + \bar{\tau} (1 - \bar{y}^{k+1}) / (\ell + 1) \right)^{\beta / k + 1} / (\ell + 1)\]

and

\[
\mathcal{F}_k (y) = \bar{\tau} \left[ c_k (y) \right]^{1/\alpha}.
\]

(ii) Example 2 (constant allowance with \( 0 < c < 1 / 2 \))

\( b_k \) is the unique root \( y \) in \( (b_{k-1}, 1] \) of the equation

\[
y^{\beta} - \sum_{i=1}^{\beta} (y - c)^{\frac{\beta}{1-\epsilon}} \left[ c + (1 - y) / i \right] = 0.
\]

For \( y \in I_i \) (i \( \notin \) \( S \)),

\[
c_k (y) = \Lambda_n (y) \sum_{i=0}^{r} (y - c)^{\frac{\beta}{1-\epsilon}} \left( c + (1 - y^{k+1}) / (\ell + 1) \right)^{\beta / k + 1} / (\ell + 1)\]

and

\[
\mathcal{F}_k (y) = \left[ c_k (y) \right]^{1/\alpha} = c.
\]

Note that the smallest possible value of \( x \) to be accepted in state \((x, y, k)\) is \( b_k - \rho(b_k) \). Hence, we may well conjecture that \( t_k \geq b_k - \rho(b_k) \), \( k \geq 1 \), for any allowance function satisfying (A1) and (A2)', where \( t_k \) as defined in (1, 13)' is the decision number of the corresponding non-allowance problem. However, this conjecture is not true. The following corollary gives an example for which \( t_k < b_k - \rho(b_k) \) holds for some \( k \).

Corollary 1.7. Let \( \rho(y) \) be

\[
\begin{align*}
\rho(y) = 0, & \quad x \leq d \\
\rho(y) > 0, & \quad x > d
\end{align*}
\]

for fixed \( d \) such that \( t_k \leq d \) for some \( k \geq 2 \). Then \( t_k < b_k - \rho(b_k) \).

Proof. Define, for \( x > 0 \),

--- 11 ---
\[ H_a(x) = \left( F(\beta(x)) \right)^{x-1} \int_{\beta(x)}^t \left( F(t) \right)^{-1} dF(t). \]  

Then, when \( x \geq b_{k-1} - \rho(b_{k-1}) \), \( H_a(x) \) can be expressed as

\[ H_a(x) = s_k(x) - c_k(\beta(x)), \]

because substituting \( y = \beta(x) \) into \((1.12)\) yields

\[ c_k(\beta(x)) = \sum_{i=0}^{k-1} \left( F(x) \right)^{x-1} \int_{\beta(x)}^t \left( F(t) \right)^{x-1} dF(t), \quad x \geq b_{k-1} - \rho(b_{k-1}). \]

Considering that, from Lemma 1.1, \( H_a(x) \) is increasing in \( x \) when \( x \geq b_{k-1} - \rho(b_{k-1}) \) and that \( b_k - \rho(b_k) \) is, from \((1.6)\)', the unique root \( x \) of the equation \( s_k(x) = c_k(\beta(x)) \), we find

\[ H_a(x) \geq 0, \quad x \geq b_k - \rho(b_k). \]

Thus, to prove \( t_k < b_k - \rho(b_k) \), it suffices to show \( H_a(t_k) < 0 \). Since \( t_k \) satisfies, from \((1.13)\)',

\[ \left( F(t_k) \right)^{x-1} \int_{t_k}^t dF(t) = \left( F(t_k) \right)^{x-1} \int_{t_k}^t \left( F(t) \right)^{x-1} dF(t), \]  

we have, from \((1.16)\) and \((1.17)\),

\[ H_a(t_k) = \left( F(\beta(t_k)) \right)^{x-1} \int_{\beta(t_k)}^t \left( F(t) \right)^{x-1} dF(t) \]

\[ = \left( F(\beta(t_k)) \right)^{x-1} \int_{t_k}^t \left( F(t) \right)^{x-1} dF(t) \]

\[ = \sum_{i=0}^{k-1} \left( F(t_k) \right)^{x-1} \int_{t_k}^t \left( F(t) \right)^{x-1} dF(t) \]

\[ < 0 \text{ (when } k \geq 2), \]

where the last equality follows from \( \beta(t) = t \) for \( t \leq d \) and the inequality follows from \( \beta(t) > t \) for \( t > d \).

It is of interest but difficult to investigate how the probability of success depends on the underlying distribution \( F \) and the allowance function employed.

Before concluding this section, we ma[ke, for Examples 1 and 2, simple comparisons between a uniform distribution on \([0, 1]\) and a triangular distribution on \([0, 1]\) when \( n = 2 \). Denote by \( P(\text{Success}|F) \) the probability of success under an optimal policy.
when the underlying distribution is F. We accept the first item if $X_i < a_i$, but continue and observe the second item if $X_i > a_i$. Hence,

$$P(\text{Success} | F) = \int_0^\infty \left[ 1 - F(t) - \rho(t) \right] dF(t) + \int_0^\infty F(\rho(t)) dF(t).$$

Let

$$F_\omega(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

and

$$F_\tau(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}.$$

We have, from straightforward calculation,

(i) **Example 1 (proportional allowance)**

$$P(\text{Success} | F_\omega) = 1 - \tau / 2 + \tau^2 / 2 \sqrt{1 + \tau^2},$$

$$P(\text{Success} | F_\tau) = 1 - \tau^2 / 2 + \tau^2 / 2 \sqrt{1 + \tau^2},$$

$$P(\text{Success} | F_\tau) - P(\text{Success} | F_\omega) = \tau^2 \left[ (1 - \tau^2) + (1 - \tau) \sqrt{1 + \tau^2} \right] / 2 \left( 1 + \tau^2 \right) \left( 1 + \tau^2 \right)$$

$$\geq 0,$$

and

(ii) **Example 2 (constant allowance with $c < 1/2$)**

$$P(\text{Success} | F_\omega) = 3 / 4 + c - c^2,$$

$$P(\text{Success} | F_\tau) = 3 / 4 + 4c / 3 - 2c^2 + 4c^4 / 3,$$

$$P(\text{Success} | F_\tau) - P(\text{Success} | F_\omega) = c \left( 1 + c \right) \left( 1 - 2c \right) ^2 / 3$$

$$\geq 0.$$

Does the inequality $P(\text{Success} | F_\tau) \geq P(\text{Success} | F_\omega)$ correspond to the stochastic order relation $F_\tau(x) \geq F_\omega(x)$? The answer is negative. Let

$$F_\lambda(x) = 1 - \exp(-\lambda x), \quad x \geq 0 \text{ and } \lambda > 0.$$  

Then, for the constant allowance case,

$$P(\text{Success} | F_\lambda) = 1 - \exp(-2\lambda c) / 2 \left\{ \exp(\lambda c) + \exp(-\lambda c) \right\},$$

which is clearly increasing in \lambda. Thus $P(\text{Success} | F_\lambda) \geq P(\text{Success} | F_{\lambda'})$ goes together with $F_{\lambda'}(x) \geq F_\lambda(x)$ for $\lambda > \lambda'$. 

—— 13 ——
References


