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On Reductive Dual Pairs

KOICHI TAKASE 髙瀬幸一
(宮城教育大)

§0 Introduction

The reductive dual pair is, by the definition (Howe [**H1**]), the pair (G_1, G_2) of reductive subgroup of the symplectic group $Sp(n,\mathbf{R})$ such that the centralizer of G_1 in $Sp(n,\mathbf{R})$ is G_2 and vice versa. On the other hand, there exists a non-trivial two-fold covering group $\widetilde{Sp}(n,\mathbf{R})$ of $Sp(n,\mathbf{R})$ with a projection p (the fundamental group of $Sp(n,\mathbf{R})$ is isomorphic to $\mathbf{Z}$), and a unitary representation $(\omega, L^2(\mathbf{R}^n))$ of $\widetilde{Sp}(n,\mathbf{R})$ called the Weil representation. Let $\mathcal{A}_j$ be the von-Neumann algebra generated by $\omega(\widetilde{G}_j)$ $(j=1,2)$ where $\widetilde{G}_j = p^{-1}(G_j)$ is the pull-back of G_j in $\widetilde{Sp}(n,\mathbf{R})$. It is proved (Weil [**Wi**]) that the pull-backs $\widetilde{G}_1$ and $\widetilde{G}_2$ are mutually commutative, and we have $\mathcal{A}_1 \subset \mathcal{A}_2'$ and $\mathcal{A}_2 \subset \mathcal{A}_1'$, where, as usual, $\mathcal{A}_1'$ (resp. $\mathcal{A}_2'$) denotes the commutant of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$).

Roger Howe [**H2**] proved the following theorem which plays the central role in the theory of the theta correspondence;

THEOREM. $\mathcal{A}_1 = \mathcal{A}_2'$ *or equivalently* $\mathcal{A}_2 = \mathcal{A}_1'$.

Our purpose in this note is to characterize the reductive dual pairs by the mutual commutancy of the von-Neumann algebras. The Weil representation is constructed via a natural action of the symplectic group on the Heisenberg group. But why the symplectic group, why the Heisenberg group? My original motivation of this study is to find an answer to these naive questions.

We will recall in §1 some basic facts on the Weil representation. In §2, we will give a general framework in which our characterization of the reductive dual pair is given. In §3, divided into three parts, we will give our main results (Theorem 3.2.2, 3.2.3, 3.3.3 and Corollary 3.3.4, 3.3.5).

REMARK 0.1. In this note, we will consider only over the field of real numbers. Our theory is based on Kirillov's theorem (Theorem 2.1) which holds over any local fields or over adele rings of global fields (Moore [**M**]). So the main results in this note may hold over any local fields or even over adele ring of global fields.

§1 **Review on Weil representation**

Let $(V, <,>)$ be a symplectic $\mathbf{R}$-space, that is, a finite dimensional $\mathbf{R}$-vector space V with a non-degenerate alternating bilinear form $<,>$. Let $G = Sp(V, <,>)$ be the symplectic group of $(V, <,>)$, that is the group consisting of $\sigma \in GL_{\mathbf{R}}(V)$ such that $< z\sigma, w\sigma > = < z, w >$ for all $z, w \in V$. Let $H = H(V, <,>)$ be the Heisenberg group associated with $(V, <,>)$. The group H is defined as follows; $H = V \times \mathbf{R}$ as a topological space and the group operation is defined by $(z,t)\cdot(w,u) = (z+w, t+u+ < z,w > /2)$. The center $Z(H)$ of H is identified with $\mathbf{R}$ via $(0,t) = t$. The quotient group $H/Z(H)$ is isomorphic to V, so the Heisenberg group is a two-step-nilpotent real Lie group which is connected and simply connected.

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of H. By Schur's lemma, the restriction of π to the center of H is a character χ_π of the center (the central character of π). If $\chi_\pi = 1$, then π factors through $H/Z(H)$ which is abelian, and so we have $\dim\pi = 1$. We have

THEOREM 1.1. *(Stone-von Neumann) The set* $\{\pi \in \widehat{H} | \dim\pi > 1\}$ *correspond bijectively to the set* $\{1 \neq \chi \in \widehat{\mathbf{R}}\}$ *via the mapping* $\pi \mapsto \chi_\pi$.

Let χ be a non-trivial character of $\mathbf{R}$ and $(\pi, \mathcal{H})$ the irreducible unitary representation of H corresponding to χ by Theorem 1.2. The group G acts on H as an automorphism group by $(z,t)\cdot\sigma = (z\sigma, t)$ for $\sigma \in G$ and $(z,t) \in H$. For any $\sigma \in G$, the twisted representation $(\pi^\sigma, \mathcal{H})$ of H is defined by $\pi^\sigma(h) = \pi(h\cdot\sigma)$ for all $h \in H$. Then, by Theorem 1.1, the two representations π and π^σ are unitarily equivalent. So there exists a unitary operator $W_\chi(\sigma) \in U(\mathcal{H})$ of $\mathcal{H}$ such that $\pi(h\cdot\sigma) = W_\chi(\sigma)^{-1} \circ \pi(h) \circ W_\chi(\sigma)$ for all $h \in H$. The unitary operator $W_\chi(\sigma)$ is well-defined up to the scalar multiplication.

For any $\sigma, \tau \in G$, by Schur's lemma, there exists a $\alpha_\chi(\sigma,\tau) \in T = \{z \in \mathbf{C} \mid |z| = 1\}$ such that $W_\chi(\sigma) \circ W_\chi(\tau) = \alpha_\chi(\sigma,\tau) \cdot W_\chi(\sigma \cdot \tau)$. Then $\alpha_\chi : G \times G \to T$ is a 2-cocycle, and the cohomology class $[\alpha_\chi] \in H^2(G,T)$ is well-defined. It is proved by Weil [**Wi**] that the cohomology class $[\alpha_\chi]$ has order 2 in $H^2(G,T)$. Then there exists a 2-fold covering group $p : \widetilde{G} \to G$ and a group homomorphism $\widetilde{W_\chi} : \widetilde{G} \to U(\mathcal{H})$ such that $W_\chi \circ p = \widetilde{W_\chi}$. More explicitly, there exists a mapping $\beta : G \to T$ such that $\alpha_\chi(\sigma,\tau)^2 = \beta(\tau)\beta(\sigma\tau)^{-1}\beta(\sigma)$ for all $\sigma, \tau \in G$. Then $\widetilde{G} = \{(\varepsilon,\sigma) \in T \times G \mid \varepsilon^2 = \beta(\sigma)^{-1}\}$ with the group law $(\varepsilon,\sigma)\cdot(\eta,\tau) = (\varepsilon\eta\alpha_\chi(\sigma,\tau), \sigma\tau)$, and $p(\varepsilon,\sigma) = \sigma$ (see Remark 1.6 below). The representation $\widetilde{W_\chi}$ is called the Weil representation associated with χ.

DEFINITION 1.2. *A pair of groups (G_1, G_2) is called a reductive dual pair in $G = Sp(V, <,>)$ if*

1) G_j is a reductive subgroup of G ($j = 1,2$),

2) G_2 is the centralizer of G_1 in G and vice versa.

The reductive dual pair is the direct sum of the irreducible reductive dual pairs, and the irreducible reductive dual pairs are completely classified (Howe [**H1**])

Let (G_1, G_2) be a reductive dual pair in $G = Sp(V, <,>)$, and put $\widetilde{G}_j = p^{-1}(G_j) \subset \widetilde{G}$. The following proposition is proved by Weil [**Wi**];

PROPOSITION 1.3. *$\widetilde{G}_1$ and $\widetilde{G}_2$ are mutually commutative.*

Let $\mathcal{A}_j$ be the von-Neumann algebra generated by $\widetilde{W_\chi}(\widetilde{G}_j)$, that is, $\mathcal{A}_j = \widetilde{W_\chi}(\widetilde{G}_j)''$. Here we used the usual notations; $\mathcal{S}' = \{T \in \mathcal{L}(\mathcal{H}) \mid T \circ S = S \circ T \text{ for all } S \in \mathcal{S}\}$ for all the subset $\mathcal{S}$ of the C^*-algebra $\mathcal{L}(\mathcal{H})$ of the bounded operators on $\mathcal{H}$. Then we have $\mathcal{A}_1 \subset \mathcal{A}_2'$ and $\mathcal{A}_2 \subset \mathcal{A}_1'$ by Proposition 1.3. The following theorem is proved by Howe [**H2**];

THEOREM 1.4. *$\mathcal{A}_1 = \mathcal{A}_2'$ or equivalently $\mathcal{A}_2 = \mathcal{A}_1'$.*

The meaning of the mutual commutancy of the von-Neumann algebra is this;

PROPOSITION 1.5. *Let G_j be a locally compact unimodular group of type I ($j = 1,2$),* *and $(\omega, \mathcal{H})$ a unitary representation of $G_1 \times G_2$. Let $\mathcal{A}_j$ be the von-Neumann algebra generated by $\omega(G_j)$ $(j = 1, 2)$. Suppose that $\mathcal{A}_1 = \mathcal{A}_2'$ (or equivalently $\mathcal{A}_2 = \mathcal{A}_1'$). Then*

1) $(\omega, \mathcal{H})$ is multiplicity-free,

2) for any $\pi_1 \in \widehat{G}_1$, there exists at most one $\pi_2 \in \widehat{G}_2$ such that $\pi_1 \otimes \pi_2$ is a subrepresentation of ω.

Because of Theorem 1.4 and Proposition 1.5, the Weil representation restricted to the reductive dual pair works as the graph of the theta correspondence, and this is the basis of the theory of theta correspondence. So what is important is not the mutual centralizer of groups ($Z_{Sp}(G_1) = G_2,\ Z_{Sp}(G_2) = G_1$) but the mutual commutancy of the von-Neumann algebras ($\mathcal{A}_1 = \mathcal{A}_2',\ \mathcal{A}_2 = \mathcal{A}_1'$). Proposition 1.5 is considered as the infinite dimensional version of Weyl's reciprocity law which is the basis of his famous book Weyl [**Wy**] (see Remark 1.7 below). So the theory of the theta correspondence is the infinite dimensional (or transcendental) invariant theory (Howe [**H1**]).

REMARK 1.6. Depending on the normalization of $W_\chi(\sigma)$, we have the following two explicit formula of α_χ ;

EXPLICIT FORMULA I. Let X be a Lagrangean subspace of V, that is, a subspace of V such that $< z, w >= 0$ for all $z, w \in X$ and $\dim_{\mathbf{R}} X = \frac{1}{2}\dim_{\mathbf{R}} V$. For any Lagrangean subspace X' and X'' of V, define a quadratic form $Q_{X,X',X''}$ on $X \times X' \times X''$ by $Q_{X,X',X''}(x, y, z) =< x, y > + < y, z > + < z, x >$. We will denote by $[X, X', X'']$ the element of the Witt group $W_{\mathbf{R}}$ over $\mathbf{R}$ which contains the quadratic form $Q_{X,X',X''}$. The Witt group $W_{\mathbf{R}}$ is the cyclic group of infinite order whose generator is $Q_1(x) = x^2$ $(x \in \mathbf{R})$. Let γ_χ be the group homomorphism from $W_{\mathbf{R}}$ to $\mathbf{C}^\times$ such that $\gamma_\chi(Q_1) = \exp(\pi\sqrt{-1} \cdot \mathrm{sign}(a)/4)$ where $\chi(x) = \exp(2\pi\sqrt{-1} \cdot ax)$. Then $\alpha_\chi(\sigma, \tau) = \gamma_\chi([X, X\tau, X\sigma\tau])^{-1}$ for all $\sigma, \tau \in G$. For the details, see Lion-Vergne [**LV**].

EXPLICIT FORMULA II. Let $\mathcal{H}_n$ be the Siegel upper half space of degree n on which G acts by $\sigma(W) = (aW + b)(cW + d)^{-1}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $W \in \mathcal{H}_n$. Put $\mathcal{X} = \{T \in M_n(\mathbf{C}) \mid {}^tT = T,\ \mathrm{Re}(T) > 0\}$. Then $\mathcal{X}$ is a connected simply connected complex manifold, and there exists uniquely a holomorphic function $\det^{1/2}$ on $\mathcal{X}$ such that

1) $(\det^{1/2}T)^2 = \det T$ for all $T \in \mathcal{X}$,

2) $\det^{1/2}T = (\det T)^{1/2}$ for all $T \in \mathcal{X} \cap M_n(\mathbf{R})$.

Put $\det^{m/2}T = (\det^{1/2}T)^m$ for all $T \in \mathcal{X}$ and $m \in \mathbf{Z}$. We have

$$\det{}^{-1}T = \int_{\mathbf{R}^n} \exp(-\pi x \cdot T \cdot {}^t x)dx$$

for all $T \in \mathcal{X}$. Put

$$\begin{aligned}\gamma(W', W) =&\det{}^{-1/2}\left(\frac{W' - \overline{W}}{2\sqrt{-1}}\right) \cdot (\det \operatorname{Im}W')^{1/4} \cdot (\det \operatorname{Im}W)^{1/4},\\ \varepsilon(\sigma; W', W) =&\gamma(\sigma(W'), \sigma(W))/\gamma(W', W)\end{aligned}$$

for all $W, W' \in \mathcal{H}_n$ and $\sigma \in G$. Then the cohomology class $[\alpha_\chi] \in H^2(G, T)$ contains the 2-cocycle α_W for all $W \in \mathcal{H}_n$ where

$$\alpha_W(\sigma, \tau) = \varepsilon(\tau^{-1}; \sigma^{-1}(W), W)$$

for all $\sigma, \tau \in G$. We have $\alpha_W(\sigma, \tau)^2 = \beta_W(\tau) \cdot \beta_W(\sigma\tau)^{-1} \cdot \beta_W(\sigma)$ for all $\sigma, \tau \in G$ where $\beta_W(\sigma) = \det J(\sigma^{-1}, W)/|\det J(\sigma^{-1}, W)|$ with $J(\sigma, W) = cW + d$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. In this case, $\widetilde{G}_W = \{(\varepsilon, \sigma) \in T \times G \mid \varepsilon^2 = \beta_W(\sigma)^{-1}\}$ with group law $(\varepsilon, \sigma) \cdot (\eta, \tau) = (\varepsilon\eta\alpha_W(\sigma, \tau), \sigma\tau)$ is a connected Lie group and $p : \widetilde{G}_W \to G$ with $p(\varepsilon, \sigma) = \sigma$ is a 2-fold covering group as a topological group. The groups $\widetilde{G}_W$ for any $W \in \mathcal{H}_n$ are isomorphic each other. For the details, see Satake [**S1**].

REMARK 1.7. Let K be an algebraically closed field and V a K-vector space of finite dimension. Let A be a semi-simple K-subalgebra of $\operatorname{End}_K(V)$. Put

$$B = \{b \in \operatorname{End}_K(V) \mid a \circ b = b \circ a \text{ for all } a \in A\}.$$

Then V is a left $A \otimes_K B$-module by $(a \otimes b)v = a \circ b(v)$ for $a \in A$, $b \in B$ and $v \in V$. We have

1) B is a semi-simple K-algebra,

2) $A = \{a \in \mathrm{End}_K(V) \mid a \circ b = b \circ a \text{ for all } b \in B\}$,

3) $V = \oplus_{j=1}^{r} M_j \otimes_K N_j$ as a $A \otimes_K B$-module where $M_1, \cdots, M_r$ (resp. $N_1, \cdots, N_r$) is the complete system of representatives of the simple A-modules (resp. B-modules) modulo isomorphism.

This is Weyl's reciprocity law.

§2 A generalization

Let N be a connected simply connected nilpotent Lie group. Let L be a topological group acting continuously on N from right as an automorphism group. Then we have a continuous group homomorphism $\rho : L \to Aut(N)$. The differential of ρ is a representation $d\rho : L \to GL_{\mathbf{R}}(\mathcal{N})$ of L on the Lie algebra $\mathcal{N} = Lie(N)$ of N. Let $<,>$ be the natural pairing of $\mathcal{N}$ and its (real) dual space $\mathcal{N}^*$. The contragradient representation of $d\rho$ is denoted by $d^*\rho : L \to GL_{\mathbf{R}}(\mathcal{N}^*)$, that is, $< X, d^*\rho(\sigma)F >=< Xd\rho(\sigma), F >$ for $X \in \mathcal{N}, F \in \mathcal{N}^*$ and $\sigma \in L$. Let Ad^* be the co-adjoint representation of N, that is, the contragradient representation of the adjoint representation $Ad : N \to GL_{\mathbf{R}}(\mathcal{N})$. For any $F \in \mathcal{N}^*$, put $N_F = \{n \in N \mid Ad^*(n)F = F\}$. Then the Lie algebra of N_F is $\mathcal{N}_F = \{X \in \mathcal{N} \mid < [X,Y], F >= 0 \text{ for all } Y \in \mathcal{N}\}$.

The unitary equivalence classes of the irreducible unitary representations of N is described by (Kirillov [**K1**])

THEOREM 2.1. *There exists a bijection between $\widehat{N}$ and the orbit space $Ad^*(N)\backslash\mathcal{N}^*$ of the co-adjoint representation of N.*

The bijection of Theorem 2.1 is defined as follows (Kirillov [**K1**]). Let Ω be a $Ad^*(N)$-orbit in $\mathcal{N}^*$, and take an element $F \in \Omega$. The orbit Ω is a symplectic manifold and its tangent space $T_F(\Omega) = \mathcal{N}/\mathcal{N}_F$ at $F \in \Omega$ has a symplectic structure induced by the alternating form $B_F(X,Y) =< [X,Y], F >$ on $\mathcal{N}$. There exists a $\mathbf{R}$-Lie subalgebra $\mathcal{N}_F \subset \mathcal{H} \subset \mathcal{N}$ such that $B_F(X,Y) = 0$ for all $X, Y \in \mathcal{H}$ and $\dim(\mathcal{H}/\mathcal{N}_F) = \frac{1}{2}\dim T_F(\Omega)$, that is, $\mathcal{H}/\mathcal{N}_F$ is a Lagrangean subspace of $T_F(\Omega)$. Put $H = \exp\mathcal{H}$ and define a unitary character λ_F of H by $\lambda_F(\exp X) = \exp 2\pi\sqrt{-1} < X, F >$. Then the induced representation $\mathrm{Ind}_H^N \lambda_F$ is an irreducible unitary representation of N, and, up to unitary equivalence, it depends only on the orbit Ω. Then the mapping $\Omega \mapsto \mathrm{Ind}_H^N \lambda_F$ gives the bijection of Theorem 2.1.

Fix a $Ad^*(N)$-orbit Ω in $\mathcal{N}^*$ with the corresponding irreducible unitary representation $(\pi, \mathcal{H})$ of N. For any $\sigma \in L$, define the twisted representation $(\pi^\sigma, \mathcal{H})$ of N by $\pi^\sigma(n) = \pi(n \cdot \sigma)$. Then the irreducible unitary representation $(\pi^\sigma, \mathcal{H})$ of N corresponds to the $Ad^*(N)$-orbit $d^*\rho(\sigma)\Omega$ in $\mathcal{N}^*$. Put $L_\Omega = \{\sigma \in L \mid d^*\rho(\sigma)\Omega = \Omega\}$ which is a closed subgroup of L. Then, for any $\sigma \in L_\Omega$, the twisted representation $(\pi^\sigma, \mathcal{H})$ is unitarily equivalent to $(\pi, \mathcal{H})$, and there exists a unitary operator $W_\Omega(\sigma) \in U(\mathcal{H})$ on $\mathcal{H}$ such that $\pi(n \cdot \sigma) = W_\Omega(\sigma)^{-1} \circ \pi(n) \circ W_\Omega(\sigma)$. The unitary operator $W_\Omega(\sigma)$ is well-defined up to scalar multiplication. By the Schur's lemma, the unitary operators $W_\Omega(\sigma)$ define a 2-cocycle $\alpha_\Omega : L_\Omega \times L_\Omega \to T = \{z \in \mathbf{C} \mid |z| = 1\}$ such that $W_\Omega(\sigma) \circ W_\Omega(\tau) = \alpha_\Omega(\sigma, \tau) \cdot W_\Omega(\sigma\tau)$ for all $\sigma, \tau \in L_\Omega$. Then the cohomology class $[\alpha_\Omega] \in H^2(L_\Omega, T)$ is well-defined. By the results of Lion [**L**], the 2-cocycle α_Ω can be expressed by the eighth root of unity, and we have $[\alpha_\Omega]^8 = 1$ in $H^2(L_\Omega, T)$. Our first problem is

PROBLEM 2.2. *Determine the order of* $[\alpha_\Omega] \in H^2(L_\Omega, T)$.

Take an integer ℓ such that $[\alpha_\Omega]^\ell = 1$ in $H^2(L_\Omega, T)$. Then we have a ℓ-fold covering group $p : \widetilde{L}_\Omega \to L_\Omega$, may be trivial, and a group homomorphism $\widetilde{W}_\Omega : \widetilde{L}_\Omega \to U(\mathcal{H})$ such that $W_\Omega \circ p = \widetilde{W}_\Omega$. They are defined as follows. Let $\overline{L}_\Omega$ be the group extension associated with the 2-cocycle α_Ω, that is, $\overline{L}_\Omega = T \times L_\Omega$ with the group operation $(\varepsilon, \sigma) \cdot (\eta, \tau) = (\varepsilon\eta\alpha_\Omega(\sigma, \tau), \sigma\tau)$. There exists a mapping $\beta : L_\Omega \to T$ such that $\alpha_\Omega(\sigma, \tau)^\ell = \beta(\tau) \cdot \beta(\sigma\tau)^{-1} \cdot \beta(\sigma)$ for all $\sigma, \tau \in L_\Omega$. Then $\widetilde{L}_\Omega = \{(\varepsilon, \sigma) \in \overline{L}_\Omega \mid \varepsilon^2 = \beta(\sigma)^{-1}\}$ which is a normal subgroup of $\overline{L}_\Omega$, and $p : \widetilde{L}_\Omega \to L_\Omega$ is the projection. The group homomorphism $\widetilde{W}_\Omega$ is defined by $\widetilde{W}_\Omega(\varepsilon, \sigma) = \varepsilon \cdot W_\Omega(\sigma)$.

Let G_1 and G_2 be subgroups of L_Ω, and put $\widetilde{G}_j = p^{-1}(G_j)$. Our second problem to be consider is

PROBLEM 2.3. *Define canonically the subgroups* G_1 *and* G_2 *of* L_Ω *such that* $\widetilde{G}_1$ *and*

$\widetilde{G}_2$ *are mutually commutative.*

Let $\mathcal{A}_j = \widetilde{W}_\Omega(\widetilde{G}_j)''$ be the von-Neumann algebra generated by $\widetilde{W}_\Omega(\widetilde{G}_j)$. If Problem 2.3 is solved, then we have $\mathcal{A}_1 \subset \mathcal{A}_2'$ and $\mathcal{A}_2 \subset \mathcal{A}_1'$. Our last problem is

PROBLEM 2.4. *Characterize the case where the equality* $\mathcal{A}_1 = \mathcal{A}_2'$ *(or equivalently* $\mathcal{A}_2 = \mathcal{A}_1'$*) holds.*

This is our general program to characterize the reductive dual pairs by the mutual commutancy of the von-Neumann algebras. The first step is to find a natural system of a nilpotent Lie group N and a topological group L operating on N. Such a natural system is constructed as follows. Let G be a semi-simple real Lie group and P a parabolic subgroup of G. The parabolic subgroup P has the Levi decomposition $P = L \cdot N$ where N is a nilpotent group and L is a reductive group. Because N is a normal subgroup of P, the group L acts on N by conjugation.

In the rest of this note, we will consider in detail the case where G is the classical group of adjoint type.

§3.1 General setting

Let A be a semi-simple $\mathbf{R}$-algebra ($\dim A < \infty$) with an involution i (i.e. anti-$\mathbf{R}$-algebra isomorphism of order two), and put

$$G = \{\sigma \in Aut_{\mathbf{R}}(A) \mid \sigma \circ i = i \circ \sigma,\ \sigma|_{Z(A)} = id\}$$

where $Z(A)$ is the center of A. The $\mathbf{R}$-algebra A is a direct sum of its simple components, and the involution i induces a permutation on the simple components. Then, because i is of order two, it is enough to consider the following two types of $\mathbf{R}$-algebras;

I) A is a simple $\mathbf{R}$-algebra,

II) $A = A_1 \oplus A_2$ is a direct sum of isomorphic simple $\mathbf{R}$-algebras A_j $(j = 1, 2)$ such that $i(A_1) = A_2$.

Then the group G exhausts all the classical simple real Lie groups of adjoint type. More precisely, if A is of type I, then G is the group of the similitude for some sesqui-linear form modulo the center. If A is of type II, then G is isomorphic to $A_1^{\times}$, the multiplicative group of A_1, modulo the center.

Let $\mathcal{G}$ be the Lie algebra of G. Fix a Cartan involution θ of $\mathcal{G}$ and the corresponding Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{V}$ ($\mathcal{K}$ is the maximal compact subalgebra of $\mathcal{G}$). Let $\mathcal{T}$ be the maximal abelian subalgebra of $\mathcal{V}$, and $(\mathcal{T}^*, \Sigma)$ the restricted root system of $\mathcal{G}$ with respect to $\mathcal{T}$. Fix a fundamental root system Ψ of $(\mathcal{T}^*, \Sigma)$.

Let $\mathcal{P}$ be the standard parabolic subalgebra of $\mathcal{G}$ corresponding to a subset S of Ψ. The parabolic subalgebra $\mathcal{P}$ has the Levi decomposition $\mathcal{P} = \mathcal{L} \oplus \mathcal{N}$ with the nilpotent part $\mathcal{N}$ and the reductive part $\mathcal{L}$. Put $N = \exp\mathcal{N}$ and $L = \{\sigma \in G \mid Ad(\sigma)H = H \text{ for all } H \in \mathcal{T}_S\}$ where $\mathcal{T}_S = \{H \in \mathcal{T} \mid \alpha(H) = 0 \text{ for all } \alpha \in S\}$. The Lie algebra of L (resp. N) is $\mathcal{L}$ (resp. $\mathcal{N}$). The reductive group L normalizes the nilpotent group N. Let $Ad_{\mathcal{N}}$ be the adjoint representation of the parabolic subgroup $P = L \cdot N$ on $\mathcal{N}$. The dual space $\mathcal{N}^*$ of $\mathcal{N}$ is identified with $\mathcal{N}$ via a non-degenerate bilinear form $< X, Y >= -B(X, \theta Y)$ where B is the Killing form of $\mathcal{G}$. Let $Ad_{\mathcal{N}}^*$ be the contragradient representation of $Ad_{\mathcal{N}}$. The group L acts from right on N via the continuous group homomorphism $\rho : L \longrightarrow Aut(N)$ such that $n \cdot \rho(\sigma) = \sigma^{-1} n \sigma$, and we will use the notations of §2. Then we have $d^*\rho(\sigma) = Ad_{\mathcal{N}}^*(\sigma)$ for all $\sigma \in L$.

Except for the cases of $\mathcal{G} = so(p, p+q, \mathbf{R})$ or $so(2p+q, \mathbf{C})$, $(\mathcal{T}_S^*, \Sigma_S)$ is a root system where $\Sigma_S = \{0 \neq \lambda|_{\mathcal{T}_S} \mid \lambda \in \Sigma\}$. Put $\Sigma_S' = \{\lambda \in \Sigma_S \mid 2\lambda \notin \Sigma_S\}$. Then the reduced root system $(\mathcal{T}_S^*, \Sigma_S')$ is of type C_m (resp. A_m) if the $\mathbf{R}$-algebra A is of type I (resp. type II) where m is the rank of the parabolic subalgebra $\mathcal{P}$. Even in the exceptional case of $\mathcal{G} = so(p, p+q, \mathbf{R})$ or $so(2p+q, \mathbf{C})$, which corresponds to a type I simple $\mathbf{R}$-algebra, $(\mathcal{T}_S^*, \Sigma_S)$ is a root system and $(\mathcal{T}_S^*, \Sigma_S')$ is of type C_m with the rank m of $\mathcal{P}$, outside some boundary cases (see Remark 3.1.3 below).

Let Λ_S^0 be the long roots in Σ_S' which are invariant under the automorphism of the Dynkin diagram of $(\mathcal{T}_S^*, \Sigma_S')$. Put $\mathcal{C} = \sum_\lambda \mathcal{G}^\lambda$ where $\sum_\lambda$ is the summation over the positive roots $\lambda \in \Sigma$ with respect to Ψ such that $\lambda|_{\mathcal{T}_S} \in \Lambda_S^0$, and $\mathcal{G}^\lambda$ is the root space of λ. Then we have

PROPOSITION 3.1.1. *$\mathcal{C}$ is an abelian subalgebra of $\mathcal{N}$ such that*

1) $Z(\mathcal{N}) \subset \mathcal{C} \subset \mathcal{N}_F$ for all $F \in \mathcal{C}$ ($Z(\mathcal{N})$ is the center of $\mathcal{N}$),

2) $N_F = \{h \in N | Ad^*_{\mathcal{N}}(h)F \in \mathcal{C}\}$ for all $F \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$,

3) $Ad^*_{\mathcal{N}}(g)\mathcal{C} = \mathcal{C}$ for all $g \in L$.

Suppose that the $Ad^*_{\mathcal{N}}(N)$-orbit Ω contains a $F \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$. Then $L_\Omega = \{g \in L | Ad^*_{\mathcal{N}}(g)F = F\}$ by 2) and 3) of Proposition 3.1.1. The group L_Ω acts on Ω fixing F, and L_Ω acts also on the tangent space $T_F(\Omega) = \mathcal{N}/\mathcal{N}_F$ of Ω at F. The operation is via $Ad_{\mathcal{N}}$. The orbit Ω is a symplectic manifold and $T_F(\Omega) = \mathcal{N}/\mathcal{N}_F$ has a symplectic structure induced by $B_F(X,Y)$ (Kirillov [**K2**,§15]). Then, for any $\sigma \in L_\Omega$, $Ad_{\mathcal{N}}(\sigma)$ induces an element of the symplectic group $Sp(T_F(\Omega), B_F)$. Using this fact, we have

PROPOSITION 3.1.2. *If Ω contains a $F \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$, then $[\alpha_\Omega]^2 = 1$ in $H^2(L_\Omega, T)$.*

By Proposition 3.1.2, there exists a two-fold covering group, may be trivial, $p : \widetilde{L}_\Omega \to L_\Omega$ of L_Ω and a group homomorphism $\widetilde{W}_\Omega : \widetilde{L}_\Omega \to U(\mathcal{H})$ such that $W_\Omega \circ p = \widetilde{W}_\Omega$.

REMARK 3.1.3. In the exceptional cases of $\mathcal{G} = so(p, p+q, \mathbf{R})$ or $so(2p+q, \mathbf{C})$, $(\mathcal{T}_S{}^*, \Sigma_S)$ may or may not be a root system. If $(\mathcal{T}^*_S, \Sigma_S)$ is a root system, the reduced root system $(\mathcal{T}^*_S, \Sigma'_S)$ is of type B_m or C_m if $q > 0$ and of type B_m, C_m or D_m if $q = 0$. Here m is the rank of the parabolic subalgebra $\mathcal{P}$.

§3.2 Parabolic subalgebra of type RDP

Definition 3.2.1. *The parabolic subalgebra $\mathcal{P}$ is called to be of type RDP if $Z(\mathcal{N}) = \mathcal{C}$ and $\mathcal{N}$ is not abelian.*

Then we have our first main results;

Theorem 3.2.2. *Suppose that the parabolic subalgebra $\mathcal{P}$ is of type RDP and that the $Ad^*_{\mathcal{N}}(N)$-orbit Ω contains a $F \in Z(\mathcal{N})$ such that $\mathcal{N}_F = Z(\mathcal{N})$. Put*

$$G_1 = \{\sigma \in L_\Omega \mid Ad^*_{\mathcal{N}}(\sigma)T = T \text{ for all } T \in Z(\mathcal{N})\}$$
$$G_2 = \{\sigma \in L_\Omega \mid [\sigma, G_1] = 1\}.$$

Then

1) *the mapping $\sigma \mapsto Ad_{\mathcal{N}}(\sigma)$ is an injective group homomorphism from G_j into $Sp(T_F(\Omega), B_F)$ $(j = 1, 2)$,*
2) *(G_1, G_2) is an irreducible reductive dual pair in $Sp(T_F(\Omega), B_F)$.*

Theorem 3.2.3. *All the irreducible reductive dual pairs are obtained by the way described in Theorem 3.2.2.*

These two theorems are proved by the classification of the simple real Lie algebras (Satake [**S2**]) and the irreducible reductive dual pairs (Howe [**H1**]), and by the case-by-case calculation.

Remark 3.2.4. If $\mathcal{P}$ is of type RDP, the nilpotent group N is a two-step-nilpotent group which may be called the Heisenberg group of higher degree. In this case, for each $Ad^*_{\mathcal{N}}(N)$-orbit Ω in $\mathcal{N}$ containing $F \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$, there exists a canonical surjective group homomorphism from N to the Heisenberg group H associated with $(T_F(\Omega), B_F)$ such that the representation $(\pi, \mathcal{H}) \in \widehat{N}$ corresponding to Ω factors through H. Then $\widetilde{W_\Omega}|_{\widetilde{G}_j}$ is, in fact, the Weil representation restricted to the reductive dual pair (G_1, G_2).

REMARK 3.2.5. The irreducible reductive dual pairs are divided into two types; type I and type II (Howe [**H1**]). The irreducible reductive dual pair obtained in Theorem 3.2.2 is of type I (resp. type II) if the **R**-algebra A is of type I (resp. type II).

§3.3 A characterization of the reductive dual pairs

DEFINITION 3.3.1. *The parabolic subalgebra $\mathcal{P}$ is called admissible if there exists a standard parabolic subalgebra $\mathcal{P}' = \mathcal{L}' \oplus \mathcal{N}'$ of type RDP such that $\mathcal{P} \subset \mathcal{P}'$ and $\mathcal{C} \subset Z(\mathcal{N}')$.*

Suppose that the parabolic subalgebra $\mathcal{P}$ is admissible and let $\mathcal{P}'$ be the standard parabolic subalgebra of type RDP as in Definition 3.3.1. Such $\mathcal{P}'$ is unique and we have $L \subset L'$ and $N' \subset N$. Define subgroups G_j of L_Ω $(j = 1, 2)$ by

$$G_1 = \{\sigma \in L_\Omega \mid Ad^*_{\mathcal{N}}(\sigma)T = T \text{ for all } T \in Z(\mathcal{N}')\}$$

$$G_2 = \{\sigma \in L_\Omega \mid [\sigma, G_1] = 1\}.$$

Put $\widetilde{G}_j = p^{-1}(G_j) \subset \widetilde{L}_\Omega$. Then we have

PROPOSITION 3.3.2. $[\widetilde{G}_1, \widetilde{G}_2] = 1$.

The proposition is proved by using the explicit formula of the cocycle $\alpha_\Omega(\sigma, \tau)$ expressed by the Maslov (or Kashiwara) index (Lion [**L**]) and then reduced to the case of the reductive dual pairs in which case the proposition is proved by Weil [**Wi**].

Let Ω be a $Ad^*_\Omega(N)$-orbit in $\mathcal{N}$ containing a $F \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$. Let $\mathcal{A}_j$ be the von-Neumann algebra generated by $\widetilde{W_\Omega}(\widetilde{G}_j)$. We have $\mathcal{A}_1 \subset \mathcal{A}'_2$ and $\mathcal{A}_2 \subset \mathcal{A}'_1$ by Proposition 3.3.2. Our main result is

THEOREM 3.3.3. *Suppose that the parabolic subalgebra $\mathcal{P}$ is admissible. Then $\mathcal{A}_1 = \mathcal{A}'_2$ (or equivalently $\mathcal{A}_2 = \mathcal{A}'_1$) if and only if $\mathcal{P}$ is of type RDP.*

The proof of the if-part of the theorem is given by Howe [**H2**]. The only-if-part of the theorem is proved by using the explicit construction of $(\pi, \mathcal{H}) \in \widehat{N}$ corresponding to Ω and the explicit formula of $W_\Omega(\sigma)$ obtained from the results of Lion [**L**].

We can prove that the mapping $\sigma \mapsto Ad(\sigma)$ is an injective group homomorphism from G_j into $Sp(T_F(\Omega), B_F)$, and we will identify G_j with its image in $Sp(T_F(\Omega), B_F)$. Then Theorem 3.3.3 is restated as follows;

COROLLARY 3.3.4. *Suppose that the parabolic subalgebra $\mathcal{P}$ is admissible. Then $\mathcal{A}_1 = \mathcal{A}_2'$ (or equivalently $\mathcal{A}_2 = \mathcal{A}_1'$) if and only if (G_1, G_2) is a reductive dual pair in $Sp(T_F(\Omega), B_F)$.*

Recalling Remark 3.2.4, we will restate Theorem 3.3.3 again

COROLLARY 3.3.5. *Suppose that the parabolic subalgebra $\mathcal{P}$ is admissible. Then $\mathcal{A}_1 = \mathcal{A}_2'$ (or equivalently $\mathcal{A}_2 = \mathcal{A}_1'$) if and only if the nilpotent group N is two-step-nilpotent or the Heisenberg group of higher degree (see Remark 3.2.4).*

These results may be an answer to the questions arised in §0.

§4 Examples

In this section, we will consider the case of G =quaternionic orthogonal group.

4.1 Let $\mathbf{H}$ be the Hamilton's quaternions which is given by a matrix algebra $\mathbf{H} = \{\begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \in M_2(\mathbf{C})\}$. Let $z = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \mapsto \overline{z} = \begin{pmatrix} \overline{x} & -y \\ \overline{y} & x \end{pmatrix}$ be the canonical involution on $\mathbf{H}$ over $\mathbf{R}$. Put $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbf{H}$ and put $\tilde{z} = j \cdot \overline{z} \cdot j^{-1}$ and $z^\dagger = \tilde{\overline{z}} = j \cdot z \cdot j^{-1}$, that is, $\tilde{z} = \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix}$ and $z^\dagger = \begin{pmatrix} \overline{x} & \overline{y} \\ -y & x \end{pmatrix}$ for $z = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \in \mathbf{H}$. For any matrix $X = (x_{ij}) \in M_{m,n}(\mathbf{H})$, put ${}^tX = (x_{ji}) \in M_{n,m}(\mathbf{H})$ the transposed matrix of X and $\overline{X} = (\overline{x}_{ij})$, $\tilde{X} = (\tilde{x}_{ij})$, $X^\dagger = (x_{ij}^\dagger)$.

Quaternionic orthogonal group (GO, O) and quaternionic unitary group (U) are defined by

$$GO(E, \mathbf{H}) = \{g \in GL(n, \mathbf{H}) |\ {}^t\tilde{g}Eg = \nu(g)F,\ \nu(g) \in \mathbf{R}^\times\}$$

$$O(E, \mathbf{H}) = \{g \in GO(E, \mathbf{H}) |\ \nu(g) = 1\}$$

$$U(F, \mathbf{H}) = \{g \in GL(n, \mathbf{H}) |\ {}^t\overline{g}Fg = F\}$$

where $E, F \in M_n(\mathbf{H})$ such that ${}^t\tilde{E} = E$ and ${}^t\overline{F} = F$.

4.2 Put $J = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ with the unit matrix I_p of size p. Let $GO(2p, \mathbf{H}) = GO(J, \mathbf{H})$ be the quaternionic orthogonal group associated with J. The center of $GO(2p, \mathbf{H})$ is $\mathbf{R}^\times \cdot I_{2p}$, and put $G = GO(2p, \mathbf{H})/\mathbf{R}^\times \cdot I_{2p}$. The Lie algebra $\mathcal{G} = Lie(G)$ of G is

$$\mathcal{G} = so(2p, \mathbf{H}) = \{X \in M_{2p}(\mathbf{H}) |\ {}^t\widetilde{X} J + JX = 0\}.$$

According to the block decomposition of J, any element $g \in G$ (resp. $X \in \mathcal{G}$) is denoted by 2×2 blocks $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (resp. $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$). Then

$$\mathcal{G} = \{\begin{pmatrix} A & B \\ C & -{}^t\widetilde{A} \end{pmatrix} \in M_{2p}(\mathbf{H}) |\ B +^t \widetilde{B} = 0,\ C +^t \widetilde{C} = 0\}.$$

Let θ be a Cartan involution on $\mathcal{G}$ defined by $\theta(X) = -{}^t\overline{X}$. Corresponding Cartan decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{V}$ is

$$\mathcal{K} = \{\begin{pmatrix} A & B \\ B^\dagger & A^\dagger \end{pmatrix} \in M_{2p}(\mathbf{H}) | A +^t \overline{A} = 0,\ B +^t \overline{B} = 0\},$$

$$\mathcal{V} = \{\begin{pmatrix} A & B \\ -B^\dagger & -A^\dagger \end{pmatrix} \in M_{2p}(\mathbf{H}) |{}^t\overline{A} = A,\ {}^t\overline{B} = B\},$$

and

$$\mathcal{T} = \{\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \in M_{2p}(\mathbf{H}) | A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_p \end{pmatrix},\ a_j \in \mathbf{R}\}$$

is the maximal abelian subalgebra of $\mathcal{V}$. Define $\lambda_j \in \mathcal{T}^*$ by $\lambda_j \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} = a_j$. Then the restricted root system $(\mathcal{T}^*, \Sigma)$ is

$$\Sigma = \{\pm\lambda_i \pm \lambda_j \neq 0 |\ 1 \leq i \leq j \leq p\}.$$

The fundamental root system Ψ of $(\mathcal{T}^*, \Sigma)$ is

$$\Psi = \{\alpha_j = \lambda_j - \lambda_{j+1},\ \alpha_p = 2\lambda_p | 1 \leq j < p\}.$$

Take a proper subset S of Ψ and put

$$\{1 \leq j < p |\ \alpha_j \notin S\} = \{ r_1 <, \cdots, < r_m \} \qquad (r_0 = 0,\ r_{m+1} = p).$$

The A-part of any element of $\mathcal{G}$ is decomposed into $(m+1)\times(m+1)$ blocks A_{ij} so that the k-th diagonal block $A_{kk}\in M_{r_k-r_{k-1}}(\mathbf{H})$.

Let $\mathcal{P}=\mathcal{N}\oplus\mathcal{L}$ be the standard parabolic subalgebra of $\mathcal{G}$ corresponding to S. We will consider two cases separately;

Case I; $\alpha_p\notin S$. In this case, we have

$$\mathcal{N}=\left\{\begin{pmatrix} A & B \\ 0 & -{}^t\widetilde{A}\end{pmatrix}\in\mathcal{G}\mid A=\begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1m+1} \\ & 0 & A_{23} & \dots & A_{2m+1} \\ & & 0 & \dots & A_{3m+1} \\ & & & \ddots & \vdots \\ & & & & 0\end{pmatrix}\right\}$$

and

$$\mathcal{L}=\left\{\begin{pmatrix} A & 0 \\ 0 & -{}^t\widetilde{A}\end{pmatrix}\in\mathcal{G}\mid A=\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{m+1}\end{pmatrix}\right\}.$$

The center of $\mathcal{N}$ is

$$Z(\mathcal{N})=\left\{\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix}\in\mathcal{G}|B=\begin{pmatrix} B_1 & 0 \\ 0 & 0\end{pmatrix},\ B_1\in M_{r_1}(\mathbf{H})\right\}.$$

The special abelian subalgebra $\mathcal{C}$ defined in §3.1 is

$$\mathcal{C}=\left\{\begin{pmatrix} 0 & Q \\ 0 & 0\end{pmatrix}\in\mathcal{G}|Q=\begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{m+1}\end{pmatrix},\ Q_k\in M_{r_k-r_{k-1}}(\mathbf{H})\right\}.$$

For any $F=\begin{pmatrix} 0 & Q \\ 0 & 0\end{pmatrix}\in\mathcal{C}$ with $Q=\begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_{m+1}\end{pmatrix}$, we have

$$\mathcal{N}_F=\mathcal{C}\iff Q_k\in GL(r_k-r_{k-1},\mathbf{H})\quad\text{for } k=1,\cdots,m.$$

Case II; $\alpha_p\in S$. In this case, we have

$$\mathcal{N}=\left\{\begin{pmatrix} A & B \\ 0 & -{}^t\widetilde{A}\end{pmatrix}\in\mathcal{G}|\ (*)\right\}$$

where the condition (*) is

$$A=\begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1m+1} \\ & 0 & A_{23} & \dots & A_{2m+1} \\ & & 0 & \dots & A_{3m+1} \\ & & & \ddots & \vdots \\ & & & & 0\end{pmatrix},\qquad B=\begin{pmatrix} B_1 & B_2 \\ -{}^t\widetilde{B}_2 & 0\end{pmatrix},\quad B_1\in M_{r_m}(\mathbf{H}).$$

The reductive part $\mathcal{L}$ is

$$\mathcal{L} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t\widetilde{A} \end{pmatrix} \in \mathcal{G} |\ (**) \right\}$$

where the condition (**) is

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_{m+1} \end{pmatrix}, \quad A_k \in M_{r_k - r_{k-1}}(\mathbf{H}), \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B_{m+1} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_{m+1} \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} A_{m+1} & B_{m+1} \\ C_{m+1} & -{}^t\widetilde{A}_{m+1} \end{pmatrix} \in so(2(p - r_m), \mathbf{H}).$$

The center of $\mathcal{N}$ is

$$Z(\mathcal{N}) = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{G} |\ B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}, \ B_1 \in M_{r_1}(\mathbf{H}) \right\}.$$

The special abelian subalgebra $\mathcal{C}$ defined in §3.1 is

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \in \mathcal{G} |\ Q = \begin{pmatrix} Q_1 & & & \\ & \ddots & & \\ & & Q_m & \\ & & & 0 \end{pmatrix}, \quad Q_k \in M_{r_k - r_{k-1}}(\mathbf{H}) \right\}.$$

For any $F = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \in \mathcal{C}$ with $Q = \begin{pmatrix} Q_1 & & & \\ & \ddots & & \\ & & Q_m & \\ & & & 0 \end{pmatrix}$, we have

$$\mathcal{N}_F = \mathcal{C} \iff Q_k \in GL(r_k - r_{k-1}, \mathbf{H}) \quad \text{for } k = 1, \cdots, m.$$

4.3 The standard parabolic subalgebra $\mathcal{P}$ is of type RDP if and only if $\mathcal{P}$ is maximal and $\alpha_p \in S$. The standard parabolic subalgebra $\mathcal{P}$ is admissible if and only if $\alpha_p \in S$. Fix a $F = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \in \mathcal{C}$ such that $\mathcal{N}_F = \mathcal{C}$. We have only to consider Case II. Put

$$Q = \begin{pmatrix} Q_1 & & & \\ & \ddots & & \\ & & Q_m & \\ & & & 0 \end{pmatrix} \text{ with } Q_k \in GL(r_k - r_{k-1}, \mathbf{H}). \text{ Then}$$

$$G_1 = \left\{ \begin{pmatrix} I_{r_m} & & 0 & \\ & a & & b \\ 0 & & I_{r_m} & \\ & c & & d \end{pmatrix} \in GL(2p, \mathbf{H}) |\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2(p - r_m), \mathbf{H}) \right\}$$

$$G_2 = \left\{ \begin{pmatrix} x & & & \\ & I & & \\ & & {}^t\widetilde{x}^{-1} & \\ & & & I \end{pmatrix} \in GL(2p, \mathbf{H}) |\ x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{pmatrix}, \ x_k \in U(j \cdot Q_k, \mathbf{H}) \right\}$$

(I is the unit matrix of size $p - r_m$). Here ${}^t(\overline{j \cdot Q_k}) = -{}^t\overline{Q}_k \cdot j = -j \cdot^t \widetilde{Q}_k = j \cdot Q_k$ and the group $U(j \cdot Q_k, \mathbf{H})$ is well-defined. Any way, G_1 (resp. G_2) is isomorphic to $O(2(p-r_m), \mathbf{H})$ (resp. $\prod_{k=1}^m U(j \cdot Q_k, \mathbf{H})$). If (and only if) $\mathcal{P}$ is of type RDP (i.e. if and only if $m = 1$), the pair of groups (G_1, G_2) is a reductive dual pair in $Sp(T_F(\Omega), B_F)$.

4.4 Put $p = n + 2$ $(n > 0)$ and $S = \{\breve{\alpha}_1, \alpha_2, \cdots, \alpha_n, \breve{\alpha}_{n+1}, \alpha_{n+2}\}$ where $\breve{\alpha}_j$ denotes that α_j is dropped. Then $m = 2$, $r_1 = 1$, $r_2 = n + 1$. Put $F = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}$ with $Q = \begin{pmatrix} -j & & \\ & -j \cdot I_n & \\ & & 0 \end{pmatrix}$. Then we have $G_1 \simeq O(2, \mathbf{H})$ and $G_2 \simeq U(1, \mathbf{H}) \times U(n, \mathbf{H}) = Sp(1) \times Sp(n)$ ($Sp(m)$ is the compact real form of $Sp(m, \mathbf{R})$). The pair of groups (G_1, G_2) is *NOT* a reductive dual pair in $Sp(T_F(\Omega), B_F)$, but this example is particularly interesting. By the sporadic isomorphism of classical Lie algebras, we have $so(2, \mathbf{H}) \simeq sl(2, \mathbf{R}) \times sp(1)$. Then, up to a compact factor in $so(2, \mathbf{H})$, we are considering the pair $(sl(2, \mathbf{R}), sp(1) \times sp(n))$. On the other hand, Ibukiyama-Ihara [II] shows that there exists a nice correspondence between automorphic forms on $SL(2, \mathbf{R})$ and on $Sp(1) \times Sp(n)$ via Weil representation. This example suggests that the pair of groups (G_1, G_2) may play an important role in the theory of theta correspondence of automorphic forms even if they are *NOT* a reductive dual pair in $Sp(T_F(\Omega), B_F)$.

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Department of Mathematics, Miyagi University of Education, Aoba-ku, Sendai 980, Japan.