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Almost coinciding families and gaps in $\mathcal{P}(\omega)$

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1. Introduction. In [DSV], Dow, Simon and Vaughan introduced the notion of almost coinciding families and showed the following Proposition 1-3.

Proposition 1 [DSV, Theorem 2.4]. If $d = \omega_1$, then there exists a nontrivial almost coinciding family indexed by $\omega_\omega$.

Proposition 2 [DSV, Theorem 3.1]. The proper forcing axiom (PFA) implies that every almost coinciding family indexed by $\omega_\omega$ is trivial.

Proposition 3 [DSV, Theorem 4.1, Lemma 4.2, 4.3]. If there exists a nontrivial almost coinciding family indexed by $\omega_\omega$, then there exists an unfilled $(b,b)$-gap in $\mathcal{P}(\omega)$. So, in Kunen's model of "ZFC + Martin's Axiom + $2^\omega = \omega_2 + \mathcal{A}$ unfilled $(\omega,\omega)$-gap", there doesn't exist a nontrivial almost coinciding family indexed by $\omega_\omega$.

In this paper, we shall show

Theorem 1. Let $P$ be the poset $\{ p : \exists x \subseteq \omega_2 (|x|<\omega \& p:x \rightarrow 2) \}$ adjoining $\omega_2$ Cohen generic reals. Then, in $V^P$, there doesn't exist a nontrivial almost coinciding family indexed by $\omega_\omega$.

Theorem 2. Let $\kappa = \kappa^{<\kappa}$ and $\omega_1 < \kappa$. Then, there is a poset $P$ with the $\omega_1$-chain condition such that, in $V^P$, $2^\omega = \kappa + \text{Martin's Axiom} + \exists$ unfilled $(\kappa,\kappa)$-gap + there doesn't exist a nontrivial
almost coinciding family indexed by $\omega_\omega$.

Since, in Theorem 1, $\Pi_2$ " $b = \omega_1 + d \cong \omega_2$ " , the assumption $d = \omega_1$ in Proposition 1 can't be replaced by $b = \omega_1$.

Question. Is " ZFC + $d > \omega_1$ + there is a nontrivial almost coinciding family indexed by $\omega_\omega$ " consistent?

2. Definitions and the proof of Theorem 1. Let $\omega$ be the set of natural numbers and $\omega_\omega$ the set of all functions on $\omega$.
$\forall x ( \cdots x \cdots )$ means that $\{ x ; \text{not} \cdots x \cdots \}$ is finite.
Define the pseudo-ordering $< \omega \omega \omega$ by

$$f < g \iff \forall n < \omega \ ( f(n) < g(n) ).$$

Let $F$ be a subset of $\omega_\omega$. $F$ is said to be bounded, if there exists a $g \in \omega_\omega$ such that $\forall f \in F \ ( f < g )$. $F$ is called a dominating family if, for any $g \in \omega_\omega$, there exists $f \in F$ such that $g < f$.

The cardinals $b$ and $d$ are defined by

$$b = \min \{ |F| ; \ F \text{ is not bounded } \},$$
$$d = \min \{ |F| ; \ F \text{ is a dominating family } \}.$$

For $f \in \omega_\omega$, $L_f$ denotes the set $\{ (n,m) \in \omega \times \omega \ ; \ m \leq f(n) \}$.

Define the quasi-ordering $\subset^*$ and the equivalence relation $\sim$ by

$$X \subset^* Y \iff X \setminus Y \text{ is finite},$$
$$X \sim Y \iff X \Delta Y \text{ is finite}.$$

Let $\mathcal{A}$, $\mathcal{B}$ be subsets of $\mathcal{P}(\omega)$. $\mathcal{A} \perp \mathcal{B}$ means that $A \cap B \sim \phi$, for any $A \in \mathcal{A}$, $B \in \mathcal{B}$. $\mathcal{A} \ll \mathcal{B}$ means that $A \subset^* B$, for any $A \in \mathcal{A}$, $B \in \mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ can be separated, if there is an $X$ such that $\mathcal{A} \ll \{X\}$ and $\mathcal{B} \perp \{X\}$. A $\kappa$-sequence $\langle X_{\alpha} \mid \alpha < \kappa \rangle$ of subsets of $\omega$ is called a $\kappa$-tower, if $X_{\alpha} \subset^* X_{\beta}$, for any $\alpha < \beta < \kappa$. A $\kappa$-sequence $\langle (X_{\alpha}, Y_{\alpha}) \mid \alpha < \kappa \rangle$ is called a $(\kappa, \kappa)$-gap, if $\langle X_{\alpha} \mid \alpha < \kappa \rangle$ and
\[ \langle Y_\alpha \mid \alpha < \kappa \rangle \text{ are towers and } \{ X_\alpha \mid \alpha < \kappa \} \sqsubseteq \{ Y_\alpha \mid \alpha < \kappa \}. \]

A \((\kappa, \kappa)\)-gap \(\langle X_\alpha, Y_\alpha \mid \alpha < \kappa \rangle\) is unfilled, if \(\{ X_\alpha \mid \alpha < \kappa \}\) and \(\{ Y_\alpha \mid \alpha < \kappa \}\) can’t be separated. Finally, an indexed set
\[ \langle \phi_f \mid f \in F \rangle \]

is called an almost coinciding family indexed by \(F\), if

(i) for any \(f \in F\), \(\phi_f : L_f \rightarrow \omega\).

(ii) for any \(f, g \in F\) \(\phi_f \upharpoonright (L_f \cap L_g) \sim \phi_g \upharpoonright (L_f \cap L_g)\).  

An almost coinciding family \(\langle \phi_f \mid f \in F \rangle\) is trivial, if there exists a \(\sigma : \omega \times \omega \rightarrow \omega\) such that \(\{ \phi_f \mid f \in F \} \ll \sigma\).

To prove Theorem 1, we need the following lemma which is a little modification of Lemma 4.3 in [DSV] and is easily verified by using Fact 2.2. which appears below.

**Lemma 2.1.** Let \(F \subset S \subset \omega_\omega\). Suppose that \(\langle \phi_f \mid f \in S \rangle\) is a nontrivial almost coinciding family indexed by \(S\) and that \(F\) is an unbounded subset of \(\omega_\omega\) which consists strictly increasing functions. Then, \(\langle \phi_f \mid f \in F \rangle\) is nontrivial.

**Fact 2.2. (well-known/clear)** Suppose that \(F\) is an unbounded subset of \(\omega_\omega\) which consists strictly increasing functions. Then, it holds that, for any infinite subset \(A\) of \(\omega\),

\[ \forall f \in \omega_\omega \ \exists g \in F \ \exists \inf n \in A \ (f(n) < g(n)). \]

Let \(Q\) be the poset \(\{ q : \exists n < \omega (q : n \rightarrow \omega) \}\) and \(P\) the poset \(\{ p : \exists x \subset \omega_2 (|x| < \omega \ \& \ p : x \rightarrow 2) \}\).

**Lemma 2.3.** Suppose that \(S\) is an unbounded subset of \(\omega_\omega\) which consists strictly increasing functions and \(\langle \phi_f \mid f \in S \rangle\) is a nontrivial almost coinciding family indexed by \(S\). Let \(g\) be the canonical generic real on \(Q\). Then, in \(V^{Q \times P}\), \(\langle \phi_f \mid f \in S \rangle\) can not be extended to an almost coinciding family indexed by \(S \cup \{g\}\).

**Proof.** To get a contradiction, suppose that
(1) \((q,p) \in Q \times P \land \phi : Q \times P\)-name,
(2) \(\models Q \times P \phi : L_g \rightarrow \omega\),
(3) \((q,p) \models Q \times P \forall f \in S \forall x \in L_f \cap L_g \ (\phi(x) = \phi_f(x))\).

Because \(Q \times P\) satisfies the \(\omega_1\)-chain condition, there exists an \(A \subset \omega_2\) such that

\[\models A \leq \omega \land p \in P \upharpoonright A \land \phi\text{ is a }Q \times P \upharpoonright A\text{-name.}\]

By using (3), for each \(f \in S\), take an \(n_f < \omega\) and \((q_f,p_f)\) in \(Q \times P \upharpoonright A\) such that

(4) \(\text{dom}(q_f) \subseteq n_f \land (q_f,p_f) \subseteq (q,p)\),
(5) \((q_f,p_f) \models Q \times P \forall x \in L_f \cap L_g \setminus (n_f \times \omega) \ (\phi(x) = \phi_f(x))\).

Since \(Q \times P \upharpoonright A \leq \omega\) and \(S\) is unbounded in \(\omega\omega\), there exist an \(n' < \omega\), \((q',p') \in Q \times P \upharpoonright A\) and a subset \(F\) of \(S\) such that

(6) \(F\) is unbounded in \(\omega\omega\),
(7) \(\forall f \in F \ (n_f = n' \land q_f = q' \land p_f = p')\).

By (6) and Lemma 2.1,

(8) \(\langle \phi_f \mid f \in F \rangle\) is nontrivial.

**Claim 1.** \(\forall x \in L_f \cap L_g \setminus (n' \times \omega) \ (\phi_f(x) = \phi_h(x))\), for any \(f, h \in F\).

**Proof of Claim 1.** Let \(f, h \in F\) and \(x = (m,k) \in L_f \cap L_h\) and \(n' \leq m\). Take \(q'' \in Q\) such that

\(q'' \leq q' \land m \in \text{dom}(q'') \land q''(m) > k\).

Then, since \((q'',p') \models "\forall x \in L_g \cap L_f \setminus (n' \times \omega)"\), it holds that

\((q'',p') \models \phi (x) = \phi_f (x)\).

Similarly, \((q'',p') \models \phi (x) = \phi_h (x)\). Hence, \(\phi_f (x) = \phi_h (x)\). QED.

By Claim 1, it holds that

\(\pi = \cup \{ \phi_f \upharpoonright (L_f \setminus (n' \times \omega)) \mid f \in F \}\) is a function.

So, \(\langle \phi_f \mid f \in \omega \rangle\) is trivial. This contradicts (8).

**Proof of Theorem 1.** To get a contradiction, suppose that

\(\models \text{"} (\phi_f \mid f \in \omega)\text{ is a nontrivial almost coinciding family indexed by } \omega \text{"} \).
Since $\neg p" b = \omega_1 "$, we can take an $A \subseteq \omega_2$ and a $P \vDash A$-name $S$ such that $\models \omega_1 \subseteq \omega_1$ and $\neg p" S$ is an unbounded subset of $\omega$ consisting increasing functions & $\models S_1 = \omega_1 "$. Since $P$ satisfies the $\omega_1$-chain condition, there exists a $B \subseteq \omega_2$ such that

$A \subseteq B$ & $\models \omega_1 \subseteq S$ & $\models \phi_f \ | f \in S$ is a $P \vDash B$-name.

Since $\neg p" S$ is unbounded and consists of increasing functions ",

by Lemma 2.1,

$\neg p" <\phi_f \ | f \in S$ is nontrivial ".

From this and the fact that the formula "$x$ is nontrivial" is $\Pi_1$, it holds that

$\neg p" P \vDash \phi_f \ | f \in S$ is nontrivial ".

Since $P \vDash (\omega_2 \setminus B)$ is isomorphic to $P$, by replacing a ground model $V$ to $V^P \vDash B$, we can assume that $S$ and $<\phi_f \ | f \in S$ are sets in $V$.

Since ro($P$) is isomorphic to ro($Q \times P$), by Lemma 2.3, in $V^P$, $<\phi_f \ | f \in S$ can't be extended to an almost coinciding family indexed by $\omega_\omega$. But this contradicts the fact that, in $V^P$, $<\phi_f \ | f \in \omega_\omega$ is an almost coinciding family. \qed

3. The proof of Theorem 2.

Lemma 3.1. The following (a), (b) and (b') are equivalent.

(a) There exists a nontrivial almost coinciding family indexed by $\omega_\omega$.

(b) There exist a dominating family $F \subseteq \omega_\omega$ and an indexed set $< (A_f, B_f) \ | f \in F$ such that

(b.1) $\forall f \in F ( (A_f, B_f) is a partition of \ L_f )$,

(b.2) $\{ A_f ; f \in F \}$ and $\{ B_f ; f \in F \}$ can't be separated,

(b.3) $\forall f, g \in F ( if f < g then A_f \subseteq^* A_g \ & \ B_f \subseteq^* B_g )$
(b') For any dominating family $F \subseteq \omega_\omega$, there exists an indexed set 
\( \langle (A_f, B_f) \mid f \in F \rangle \) which satisfy (b.1)~(b.3).

Proof. It is easy to see that (b) and (b') are equivalent to the following (c) and (c'), respectively.

(c) There exists a dominating family $S \subseteq \omega_\omega$ and a nontrivial 
almost coinciding family $\langle \phi_f \mid f \in S \rangle$ such that, for every 
f \in S, \phi_f : L_f \rightarrow 2.

(c') For any dominating family $S \subseteq \omega_\omega$, there exists a nontrivial 
almost coinciding family $\langle \phi_f \mid f \in S \rangle$ such that, for every 
f \in S, \phi_f : L_f \rightarrow 2."

Also, it is easy to see that (c) and (c') are equivalent.

So, it suffices to show that (c) and (a) are equivalent. The 
implication from (c) to (a) is clear. To show from (a) to (c), let 
\( \langle \phi_f \mid f \in \omega_\omega \rangle \) be a nontrivial almost coinciding family indexed 
by $\omega_\omega$. For each finite sequence $s = \langle a_i \mid i < n \rangle : n \rightarrow \omega$, 
s* denotes the finite sequence 
\( \langle 0, 1, \cdots, 0, 1, \cdots, 0 \rangle \).

For each $g : \omega \rightarrow \omega$, $\phi : L_g \rightarrow \omega$ and $n < \omega$, let $s_{\phi, n}$ denotes 
\( \langle \phi(n, i) \mid i < g(n) \rangle \). For each $f : \omega \rightarrow \omega$, define $f^- : \omega \rightarrow \omega$ 
and $\Psi_{f^-} : L_{f^-} \rightarrow 2$ by

\[
f^-(n) = \text{the length of } (s_{\phi_f, n})^*,
\]
\[
\Psi_{f^-} = \text{the unique } \phi : L_{f^-} \rightarrow 2 \text{ such that, for any } n < \omega,
\]
\[
(s_{\phi_f, n})^* = s_\phi, n^*.
\]

Then, it is easy to see that $\{ f^- ; f \in \omega_\omega \}$ is a dominating 
subset of $\omega_\omega$ and $\langle \Psi_{f^-} \mid f \in \omega_\omega \rangle$ is a nontrivial almost 
coinciding family indexed by $\{ f^- ; f \in \omega_\omega \}$. \qed

The next lemma is due to Kunen(see [B, p.931 Theorem 4.2]).
Lemma 3.2. Let $T$ is an unfilled $(\omega_1, \omega_1)$-gap, then there is a poset $P$ with the $\omega_1$-chain condition such that, $|P| = \omega_1$ and in $V^P$, $T$ remains unfilled for any generic extension preserving $\omega_1$.

In fact, any finite product of any such posets satisfy the $\omega_1$-chain condition (see Appendix A). So, we get

Lemma 3.3. There is a poset $Q$ such that

1. $Q$ satisfies the $\omega_1$-chain condition,
2. $|Q| \leq \omega_1$,
3. for any unfilled $(\omega_1, \omega_1)$-gap $T$,

$\models \quad Q" T$ remains unfilled for any generic extension preserving $\omega_1"$.

The next lemma follows from Lemma 3.3 and the standard forcing arguments.

Lemma 3.4. Let $\omega_1 < \kappa$, $\kappa^{<\kappa} = \kappa$ and $\delta < \kappa$. Suppose that $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$, $\mathcal{B} = \langle B_\xi \mid \xi < \delta \rangle$ are sequences of subsets of $\mathcal{P}(\omega)$ such that $\mathcal{A}$ and $\mathcal{B}$ are towers and $\mathcal{A} \perp \mathcal{B}$.

Then, there exist a poset $Q$ and $Q$-names $f, B$ such that

1. $Q$ satisfies the $\omega_1$-chain condition & $|Q| = \kappa$,
2. $\models " 2^\omega = \kappa + \text{Martin's Axiom } "$,
3. $\models " f \in \omega_\omega " \& \models " h \prec f "$, for any $h \in \omega_\omega$,
4. $\models " \mathcal{A} \perp (B) \& \mathcal{B} \ll (B) "$,
5. whenever $X \subseteq \omega$ and $\mathcal{A} \perp (X)$, $\models " B \not\subseteq^* X "$,
6. if $T$ is an unfilled $(\omega_1, \omega_1)$-gap, then, in $V^Q$, $T$ remains unfilled for any generic extension preserving $\omega_1$".

(Outline of a proof) Let $Q_1$ be the poset as in Lemma 3.3. Since $|Q_1| = \kappa$ and $Q_1$ satisfies the $\omega_1$-chain condition, it holds that $\models Q_1" \kappa = \kappa^{<\kappa}"$. So, in $V^{Q_1}$, take a poset $Q_2$ such that $Q_2$
satisfies the $\omega_1$-chain condition and (10)~(13) except that $\mathop{\mathcal{M}}_{Q_2}$, take a poset $Q_3$ such that $Q_3$ satisfies the $\omega_1$-chain condition and $\mathop{\mathcal{M}}_{Q_3} \kappa = \kappa < \kappa$ & Martin's Axiom". (Such a poset exists under the assumption that $\kappa = \kappa^+ \omega_1$ (see e.g., [B2, Remark after Lemma 3.5, p.16])]

Then, the poset $Q = Q_1 \times Q_2 \times Q_3$ is as required. \hfill \Box

To prove Theorem 2, assume that $\kappa = \kappa^+ \omega_1$. By replacing the ground model to a certain generic extension, we may assume that there exists a $\kappa$-tower $\mathcal{A} = \{ A_\alpha \mid \alpha < \kappa \}$ in $P(\omega)$.

By using Lemma 3.4, we can construct a $\kappa$-stage finite support iteration $P_\alpha$, $Q_\alpha$ and $P_\alpha$-names $f_\alpha$, $B_\alpha$ (for $\alpha < \kappa$) such that

\begin{enumerate}
  \item[(9')] $Q_\alpha$ satisfies the $\omega_1$-chain condition & $1Q_\alpha \downarrow = \kappa$.
  \item[(10')] $\{ B_\xi \mid \xi < \alpha \}$ is a tower & $\mathcal{A} \downarrow \{ B_\xi \mid \xi < \alpha \}$.
  \item[(11')] for any $X \subseteq \omega$, if $\mathcal{A} \downarrow \{X\}$, then $\mathop{\mathcal{M}}_{Q_\alpha} \exists^* X$.
  \item[(12')] $Q_\alpha$ forces "$2^\omega = \kappa + \text{Martin's Axiom}".
  \item[(13')] $Q_\alpha$ forces "$f_\alpha \in \omega^\omega$ & $s < f_\alpha$", for any $s \in \omega^\omega$.
  \item[(14')] if $T$ is an unfilled $(\omega_1, \omega_1)$-gap, then $Q_\alpha$ forces that
     "$T$ remains unfilled for any generic extension preserving $\omega_1$".
\end{enumerate}

Set $P = \operatorname{dir lim} \{ P_\alpha \mid \alpha < \kappa \}$. It is easy to see that $P$ satisfies the requirement in Theorem 2 except that $\mathop{\mathcal{M}}_P$ "there doesn't exist a nontrivial almost coinciding family indexed by $\omega^\omega$".

To show this by a contradiction, assume that $p_0 \in P$ forces the existence of a nontrivial almost coinciding family indexed by $\omega^\omega$. Then, by Lemma 3.1, there exist $P$-names $\langle (X_\alpha, Y_\alpha) \mid \alpha < \kappa \rangle$ such that

\begin{enumerate}
  \item[(15)] $\mathop{\mathcal{M}}_P$ "$(X_\alpha, Y_\alpha)$ is a partition of $L_f \alpha$",
  \item[(16)] $\mathop{\mathcal{M}}_P$ "$X_\alpha \subseteq X_\beta$ & $Y_\alpha \subseteq Y_\beta$", if $\alpha < \beta < \kappa$.
\end{enumerate}
(17) \( p_0 \vdash " \{ X_\alpha ; \alpha < \kappa \}, \{ Y_\alpha ; \alpha < \kappa \} \) can't be separated ".

Set \( S = \{ \delta < \kappa ; \lim \delta \& \text{ cf} \delta = \omega_1 \& X_\alpha, Y_\alpha \) are Q_\delta \)-names, for any \( \alpha < \delta \). Since \( P \) satisfies the \( \omega_1 \)-chain condition, \( S \) is unbounded in \( \kappa \) and \( \omega_1 \)-closed. By (14'),

\[ p_0 \vdash \delta \" (X_\alpha, Y_{\alpha})_\delta \alpha < \delta > \) is filled \", for any \( \delta \in S \).

By this and the fact that \( P \) satisfies the \( \omega_1 \)-chain condition, it holds that, for any \( \delta \in S \), there is a \( \beta < \delta \) such that

\[ \exists P_\beta \)-name \( C \) \( ( p_0 \vdash \delta \" (X_\alpha; \alpha < \delta) \ll (C) \& (Y_\alpha; \alpha < \delta) \perp (C) \" ) \).

So, we can define the function \( \pi \) from \( S \) to \( \kappa \) by

\[ \pi(\delta) = \text{ the least } \beta < \delta \text{ such that } (x) \) holds.

For each \( \delta \in S \), take a \( P_\pi(\delta) \)-name \( C_\delta \) such that

\[ p_0 \vdash \delta \" (X_\alpha; \alpha < \delta) \ll \{ C_\delta \} \& (Y_\alpha; \alpha < \delta) \perp \{ C_\delta \} \" .\]

Since \( \pi : S \rightarrow \kappa \) is regressive, there exist a stationary set \( S' \subset S \) and \( \beta < \kappa \) such that

\[ p_0 \in P_\beta \& \pi(\delta) = \beta, \text{ for any } \delta \in S'. \]

Claim. Let \( \delta, \eta \in S' \) and \( \beta < \delta < \eta \). Then, it holds that

\[ p_0 \vdash \beta \" C_\delta \setminus (n \times \omega) = C_\eta \setminus (n \times \omega), \text{ for some } n < \omega \" .\]

Proof of Claim. To get a contradiction, let \( \delta, \eta \in S' \) and \( p_1 \leq p_0 \) such that

\[ \beta < \delta < \eta \& p_1 \vdash \beta \forall n < \omega (C_\delta \setminus (n \times \omega) \neq C_\eta \setminus (n \times \omega)). \]

Take a \( P_\beta \)-name \( g \) such that

\[ \vdash \" g : \omega \rightarrow \omega \" \& p_1 \vdash \" L_g \cap (C_\delta \triangle C_\eta) \) is infinite \".

Since \( \vdash \beta + 1 \" g < f \beta \" \), it holds that

\[ p_1 \vdash \beta + 1 \" L_f \beta \cap (C_\delta \triangle C_\eta) \) is infinite \".

But this contradicts that

\[ p_0 \vdash L_f \beta \cap C_\delta \sim X_\beta \sim L_f \beta \cap C_\eta. \quad \text{QED. of Claim.} \]

Take \( \delta \in S' \) such that \( \beta < \delta \). By Claim, since \( S' \) is cofinal in \( \kappa \), it holds that

\[ p_0 \vdash C_\delta \) separates \{ X_\alpha ; \alpha < \kappa \} \) and \{ Y_\alpha ; \alpha < \kappa \} \).

But, this contradicts (17). \( \Box \)
Appendix A. We start some definitions. Let \( T = \langle (a_\alpha, b_\alpha) \mid a < \omega_1 \rangle \) be an \((\omega_1, \omega_1)\)-gap. For each \( \alpha < \omega_1 \), set \( b_\alpha' = b_\alpha \setminus a_\alpha \). Define the poset \( P_T \) by

\[
P_T = \{ (s,u) \mid \exists n < \omega (s : n \to 2) \land u \subseteq \omega_1 \land |u| < \omega \land u \subseteq \bigcup_{\alpha \in u} a_\alpha \land u \subseteq \bigcup_{\alpha \in u} b_\alpha' \subseteq \text{dom}(s) \},
\]

\((s,u) \leq (t,v) \) iff \( t \subseteq s \) & \( v \subseteq u \) & \( \forall k \in \text{dom}(s \setminus t) \)

\[
[ \big( k \in \bigcup_{\alpha \in v} a_\alpha \Rightarrow s(k) = 1 \big) \land \big( k \in \bigcup_{\alpha \in v} b_\alpha' \Rightarrow s(k) = 0 \big) ].
\]

For each \( \alpha < \omega_1 \), set \( p_\alpha = (\emptyset, \{ \alpha \}) \). Define the poset \( Q_T \) by

\[
Q_T = \{ u \subseteq \omega_1 \mid |u| < \omega \land \{ p_\alpha ; \alpha \in u \} \text{ is an antichain of } P_T \}
\]

\( u \preceq v \) iff \( v \subseteq u \).

The following theorem is due to Kunen (see [B, p.931 Theorem 4.2]).

Theorem A. Let \( T \) be an \((\omega_1, \omega_1)\)-gap. Set \( P = P_T \) and \( Q = Q_T \).

(a) If \( T \) is filled, then \( P \) satisfies the countable chain condition.

(b) If \( T \) is unfilled, then

(b.1) \( q \equiv q'' \) " \( P \) has an uncountable antichain " , for some \( q \in Q \),

(b.2) \( Q \) satisfies the countable chain condition.

We shall show

Theorem B. Let \( n < \omega \) and \( T_i \) be an unfilled \((\omega_1, \omega_1)\)-gap, for each \( i < n \). Then, the product of \( \langle Q_{T_i} \mid i < n \rangle \) satisfies the countable chain condition.

Remark. Let \( T \) be an unfilled \((\omega_1, \omega_1)\)-gap. Then, under the assumption of MA + \( \neg CH \), Theorem B is a trivial consequence of Theorem A , because any poset which satisfies the countable chain condition also satisfies Knaster’s condition. The next theorem claims that the assumption of MA + \( \neg CH \) (or some assumption as this) is necessary to show that \( Q_T \) satisfies Knaster’s condition.
Theorem C. There are a poset $R$ and an $R$-name $X$ such that

1. $R$ satisfies the countable chain condition and $\text{tr} \alpha = \omega_1$,

2. $\text{H}_R$ " $X$ is an unfilled $(\omega_1, \omega_1)$-gap and $Q_X$ doesn't satisfy Knaster's condition."

Theorems B, C shall be proved in Appendix B, C (respectively).

The rest of this appendix is Proof of Lemma 3.3. For each unfilled $(\omega_1, \omega_1)$-gap $T$, by using

Theorem A (b.1), take a $q_T \in Q_T$ such that

$q_T \models " P_T$ has an uncountable antichain",

and set $Q_T = \{ q \in Q_T \mid q \leq q_T \}$. Set $Q = \text{the finite support product of } \langle Q_T \mid T$ is an unfilled $(\omega_1, \omega_1)$-gap $\rangle$. Then, by theorem B, $Q$ is as required. \qed

Appendix B. We first show the following combinatorial lemma.

Lemma B.1. Let $n < \omega$ and $\langle (a^i, b^i) \mid i < \omega_1 \rangle$ be an unfilled $(\omega_1, \omega_1)$-gap, for each $i < n$. Then, there are $\alpha, \beta < \omega_1$ such that

$a^i \cap b^i \neq \emptyset$, for all $i < n$.

To show Lemma B.1, we need the following definition.

Definition. For each $\mathcal{A} = \langle a^i \mid i < \omega_1 \rangle$ and $U \subseteq \omega_1$, set

$\text{lim}_U \mathcal{A} = \bigcap_{\alpha < \omega_1} \bigcup_{\beta \in U \setminus \alpha} a^\beta$.

Sublemma. Let $\mathcal{A}$ be an $\omega_1$-tower and $U$ a cofinal subset of $\omega_1$. Then, it holds that $\mathcal{A} \ll \{ \text{lim}_U \mathcal{A} \}$.

Proof. Let $\mathcal{A} = \langle a^i \mid i < \omega_1 \rangle$ be an $\omega_1$-tower and $U$ a cofinal
subset of $\omega_1$. Set $x = \bigcup_{\gamma} a_{\alpha \setminus x}$. To get a contradiction, assume that $a_{\alpha \setminus x}$ is infinite, for some $\alpha < \omega_1$. Since

$$\forall n \in \omega \setminus x \exists \beta < \omega_1( n \notin U_{\gamma} a_{\gamma}, \gamma \in U_{\beta} a_{\gamma})$$

take a $\beta < \omega_1$ such that

$$\alpha < \beta \quad \text{and} \quad (a_{\alpha \setminus x}) \cap (\bigcup_{\gamma \in U_{\beta}} a_{\gamma}) = \emptyset.$$ 

Since $U$ is cofinal in $\omega_1$, take a $\gamma \in U \setminus \beta$. Then, it holds that $(a_{\alpha \setminus x}) \cap a_{\gamma} = \emptyset$. But, this contradicts that

$$a_{\alpha \setminus x} \subset a_{\gamma}$$

and $a_{\alpha \setminus x}$ is infinite. \[\square\]

Proof of Lemma B.1. Let $n < \omega$ and $T_i = \langle(a_i, b_i^i) \mid \alpha < \omega_1\rangle$ an unfilled $(\omega_1, \omega_1)$-gap, for each $i < n$. Set $\text{Seq} = \bigcup_{i < n} i \omega$ and

$$\text{Seq}^* = \bigcup_{i < n} i \omega.$$ 

Define $U_s \subset \omega_1$ (for $s \in \text{Seq}^*$) and $x_s, y_s \subset \omega$, $i < n$.

$\gamma_s < \omega_1$ (for $s \in \text{Seq}$) by induction on length(s) as follows:

Set $U_0 = \omega_1$.

Assume that $s \in \text{Seq}$ and $U_s$ is defined. Set $i = \text{the length of } s$.

(\text{the definition of } x_s, y_s \text{ and } \gamma_s)

Case 1. $U_s$ is not cofinal in $\omega_1$.

Set $x_s = y_s = \emptyset$ and $\gamma_s = 0$.

Case 2. otherwise.

Set $x_s = \lim_{U_s} (a^{i}_{\alpha} \mid \alpha < \omega_1)$. Since $\{a^{i}_{\alpha} \mid \alpha < \omega_1\} \ll \{x_s\}$,

take a $\gamma_s < \omega_1$ such that

$\gamma_t < \gamma_s$, for any $t \subset s (t \neq s)$ & $x_s \cap b_{\gamma_s}^i$ is infinite.

Set $y_s = x_s \cap b_{\gamma_s}^i$.

(\text{the definition of } U_s^{<k} (\text{for } k < \omega))

Set $U_s^{<k} = \{ \alpha \in U_s \mid k \in a^{i}_{\alpha} \}$.

Set $\beta = \sup \{ \gamma_t \mid t \in \text{Seq} \}$.

Claim. There are $k_j < \omega$ (for $j < n$) such that
\( k_j \in b^j_{\beta} \cap y_{k_0, \ldots, k_{j-1}} \), for each \( j < n \).

Proof of Claim. By induction on \( j < n \). Suppose that \( j < n \) and \( k_m \) \((m < j)\) are chosen which satisfy (1). Set \( t = \langle k_0, \ldots, k_{j-1} \rangle \). Then, it holds that \( U_t \) is cofinal in \( \omega_1 \). So, \( y_t \) is infinite.

From this and the fact that \( y_t \subseteq b^j_{\gamma_t} \subseteq b^j_{\beta} \), we can take a \( k_j \) which satisfies (1).

QED of Claim.

Let \( s = \langle k_0, \ldots, k_{n-1} \rangle \) be as in Claim. Since \( U_s \) is cofinal in \( \omega_1 \), take an \( \alpha \in U_s \) such that \( \beta < \alpha \). Then, for each \( i < n \), since \( \alpha \in U_{\langle k_0, \ldots, k_i \rangle} \), it holds that \( k_i \in a^i_{\alpha} \).

So, \( k_i \in a^i_{\alpha} \cap b^i_{\beta} \), for each \( i < n \).

Now we are ready to prove Theorem B. The proof is similar to the proof of Theorem A (b,2) \((\text{in } \text{B, p.932})\) except we need Lemma B.1.

Let \( \langle w, \alpha \rangle < \omega \) and \( T_i = \langle a^i_{\alpha}, b^i_{\alpha} \rangle \) an unfilled \((\omega_1, \omega_1)\)-gap, for \( i < n \). Set \( Q \) the product of \( \langle Q_{T_i}, i < n \rangle \). To get a contradiction, suppose that \( \langle w, \alpha \rangle \) is an antichain of \( Q \).

For each \( \alpha < \omega_1 \), let \( w_\alpha = (w^0_\alpha, \ldots, w^{n-1}_\alpha) \). By using \( \Delta \)-system argument, we may assume that there are \( k_0, \ldots, k_{n-1} \in \omega \setminus \{0\} \) such that

1. \( w^i_\alpha = k_i \), for each \( \alpha < \omega_1 \),
2. if \( \alpha < \beta \), then \( w^i_\alpha \cap w^i_\beta = \emptyset \) and \( \max(w^i_\alpha) < \min(w^i_\beta) \).

For each \( i < n \) and \( \alpha < \omega_1 \), take \( m_i, \alpha < \omega \) such that

\[
\begin{align*}
& a^i_\xi \setminus m_i, \alpha \subseteq a^i_\eta \setminus m_i, \alpha \quad \text{and} \quad b^i_\xi \setminus m_i, \alpha \subseteq b^i_\eta \setminus m_i, \alpha ,
\end{align*}
\]

if \( \xi, \eta \in w^i_\alpha \) and \( \xi < \eta \). Again without loss of generality, we may assume that \( m_i, \alpha = m_i \) for all \( i < n \) and all \( \alpha < \omega_1 \).

For each \( i < n \) and \( \alpha < \omega_1 \), set
\[ c^i_\alpha = s^i_\xi \setminus \emptyset \quad \text{and} \quad d^i_\alpha = b^i_\xi \setminus \emptyset, \text{where} \ \xi = \min(w^i_\alpha). \]

Then, it holds that
\[ \langle (c^i_\alpha, d^i_\alpha) \mid \alpha < \omega_1 \rangle \] is an unfilled \((\omega_1, \omega_1)\)-gap, for \(i < n\).

So, by Lemma B.1, there are \(\alpha, \beta < \omega_1\) such that
\[ c^i_\alpha \cap d^i_\beta \neq \emptyset, \text{ for all } i < n. \]

So, \(w_\alpha\) and \(w_\beta\) are compatible, a contradiction. \(\Box\)

Appendix C. A poset \(P\) is said to satisfy Knaster's condition if
for any uncountable \(X \subseteq P\) there is an uncountable \(Y \subseteq X\) such that
any two members of \(Y\) are compatible. The following facts are well-known.

1. If \(P\) satisfies Knaster's condition, then \(P\) satisfies the
countable chain condition.
2. If \(P\) satisfies Knaster's condition and \(Q\) satisfies the countable
chain condition, then \(P \times Q\) satisfies the countable chain condition.
3. \(\text{MA} + \neg \text{CH}\) implies the reverse implication of (1).

There are several examples of a poset which satisfies the countable
chain condition but does not satisfy Knaster's condition, under some
set theoretical assumption (see e.g., [W] section 3). Theorem C

gives another such example.

We turn to a proof of Theorem C.

Lemma C.1. Let \(R\) be a poset and \(X\) an \(R\)-name such that
\(R satisifies\) the countable chain condition and \(|R| = \omega_1\).
\(V^R = "X\ is\ an\ unfilled\ \(\omega_1, \omega_1\)-gap."\)

Suppose that there exists an \(R\)-name \(Y\) such that, in \(V^R\),
\(Y\ is\ a\ poset\ and\ satisfies\ the\ countable\ chain\ condition,
\(W^Y = "X\ is\ filled\".\)

Then, it holds that, in \(V^R\), \(Q_Y\ doesn't\ satisfy\ Knaster's\ condition.

So, \(R\) and \(X\ satisfy\ (1)\ and\ (2)\ in\ Theorem\ C.\)

Proof. Set \(W = V^R\) and \(W^* = W^V.\) By (c.4) and Theorem A (a),

it holds that
\[ W^* = P_X \] satisfies the countable chain condition.

Since \( \omega W^* = \omega W \), it holds that

(c.5) \[ W \models P_X \] satisfies the countable chain condition.

Since

\[ W \models \exists q \in Q_X ( q \not\subseteq Q_X \ " P_X \ has \ an \ uncountable \ antichain \ " ) \],

it holds that

(c.6) \[ W \models Q_X \times P_X \] doesn't satisfy the countable chain condition.

By (c.5) and (c.6),

\[ W \models Q_X \] doesn't satisfy Knaster's condition. \( \Box \)

We shall construct a poset \( R \) and \( R \)-names \( X \) and \( Y \) which satisfy (c.1) \(-\) (c.4). The method for doing this is due to Hechler [H] and Dordal [D]. Hechler used it for adjoining a tower in a generic extension and later Dordal generalized it for adjoining an arbitrary partially order type of \( \mathcal{P}(\omega)/\text{finite} \) in a generic extension.

Definition (Hechler and Dordal). Let \( A = (A, <_A) \) be a partial order type. Define the poset \( P(A) \) by

\[ P(A) = \{ p : \exists u \subseteq A \exists n < \omega ( \ \forall \nu < \omega \ & p : u \times n \to 2 ) \} , \]

and for any \( p, q \in P(A) \) such that \( p : u \times n \to 2 \) and \( q : v \times m \to 2 \),

\[ p \subseteq q \iff q \subseteq p \ & \forall a, b \in v \forall k \in [m, n) ( a <_A b \Rightarrow p(a, k) \leq p(b, k) ) \]

For each \( a \in A \), define \( P(A) \)-name \( H_a \) by

\[ \ Relief \ " H_a \subseteq \omega " , \]

\[ \| n \in H_a \| = \{ p \in P(A) ; p(a, n) = 1 \} , \] for each \( n < \omega \).

The following lemma is due to P. Dordal ([D, Lemma 5.4, p. 45]).

Lemma C.2. Let \( A = (A, <_A) \) be a linear order type and \( B \) is a sub-order type of \( A \).

(1) \( P(A) \) satisfies the countable chain condition.

(2) If \( G \) is \( V \)-generic on \( P(A) \), then \( G \cap P(B) \) is \( V \)-generic on \( P(B) \).
(3) If $x$ is a $P(A)$-name such that $\models " x \subseteq \omega ", \text{ then there exists a countable }\text{ subset } C \text{ of } A \text{ such that } x \text{ is a } P(C)\text{-name.}\n
(4) \text{ For any } a, b \in A, a <_A b \text{ if and only if } \models " H_a \subseteq^* H_b ". \text{ (i.e., } \langle H_a, a \in A \rangle \text{ is a chain of } \mathcal{P}(\omega)/\text{finite}.\text{)}

Let $Q$ denote the set of rationals. Set $A = Q \times \omega_1 \times 2$ and $B = A \cup \{0\}$. Define the linear ordering $<_B$ on $B$ by

- $(q, \alpha, 0) <_B (q, \alpha, 1)$, for any $q \in Q$ and any $\alpha < \omega_1$.
- $(q, \alpha, 0) <_B (r, \beta, 0)$, if $\alpha < \beta$ or $(\alpha = \beta$ and $q < r)$.
- $(q, \alpha, 1) <_B (r, \beta, 1)$, if $\alpha > \beta$ or $(\alpha = \beta$ and $q < r)$.

We regard $B$ as the linear order type $(B, <_B)$ and $A$ it's sub-order type.

Set the poset $R = P(A)$. Define $R$-names $a_\alpha, b_\alpha$ (for $\alpha < \omega_1$) by

- $a_\alpha = H(0, \alpha, 0)$ and $b_\alpha = \omega \setminus H(0, \alpha, 1)$, for each $\alpha < \omega_1$.

Set $W = V^R$. In $W$, set $X = \langle (a_\alpha, b_\alpha) | \alpha < \omega_1 \rangle$ and take the poset $Y$ such that $W^Y = V^P(B)$. Then, by Lemma C.2 (4), it holds that $W \models " X \text{ is an } (\omega_1, \omega_1)\text{-gap } " \text{ and } W^Y \models " X \text{ is filled } " \).

So, the next lemma completes a proof of Theorem C. The lemma is proved by the same way in the proof of Theorem 5.3 in [D]. So, we omit a proof.

Lemma C.3. $W \models " X \text{ is unfilled } " .\n
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