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Existence of periodic solutions and Sokes coefficients for ODE

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Summary. We consider one dimensional Shrödinger equation with analytic periodic potential. We show that the existence of periodic solutions is completely characterized by the special values of Stokes coefficients, which can be written down explicitly.

1. Notations and results. In this note we consider one dimensional Shrödinger equation on **R**

$$Hu \equiv -(\frac{d}{dt})^2u - 2ic\frac{d}{dt}u + V(t)u + c^2u = \lambda u, \tag{1.1}$$

where c is a complex number and where $V(t)$ is π periodic and is analytic function of the form

$$V(t) = \sum_{n=-1}^{\infty} a_{-n}e^{-2nit},\ a_0 = 0,\ a_1 \neq 0, \tag{1.2}$$

where we assume that the series converges for all complex value of $t\in\mathbf{C}$.

The following fact is well-known (cf.[3]). There exists $\theta\in\mathbf{R}$ such that for every $k = \pm 1, \pm 2, \ldots$, there exists in the domain $S_{2k} = \{t \in \mathbf{C};\ -\theta + 2k\pi - 3\pi/2 < \mathrm{Re}\ t < -\theta + 2k\pi + 3\pi/2\}$, a unique solution u_{2k} of (1.1) which decreases exponentially when $\mathrm{Re}\ t = -\theta + 2k\pi$, $\mathrm{Im}\ t \to -\infty$. Similarly, there exists a unique solution u_{2k+1} of (1.1) in the domain $S_{2k+1} = \{t\in\mathbf{C};\ -\theta + (2k+1)\pi-3\pi/2 < \mathrm{Re}\ t < -\theta + (2k+1)\pi+3\pi/2\}$ which decreases exponentially when $\mathrm{Re}\ t = -\theta+(2k+1)\pi$, $\mathrm{Im}\ t \to -\infty$ for every $k=\pm 1,\pm 2,\ldots$ The systems of solutions (u_{2k+1}, u_{2k}), (u_{2k+1}, u_{2k+2}) form fundamental systems of solutions of (1.1) in $S_{2k} \cap S_{2k+1}$ and $S_{2k+1} \cap S_{2k+2}$ respectively. Hence, in the domain $S_{2k} \cap S_{2k+1} \cap S_{2k+1} \cap S_{2k+2} = S_{2k} \cap S_{2k+2}$ there is a linear relation between these systems.

(1.3) $$(u_{2k+1}, u_{2k+2}) = (u_{2k+1}, u_{2k})\begin{pmatrix} 1 & s(2k) \\ 0 & 1 \end{pmatrix}.$$

Similarly, there are linear relations between two fundamental systems (u_{2k+2}, u_{2k+1}), (u_{2k+2}, u_{2k+3}) of solutions in the domain $S_{2k+1} \cap S_{2k+3}$.

(1.4) $$(u_{2k+2}, u_{2k+3}) = (u_{2k+2}, u_{2k+1})\begin{pmatrix} 1 & s(2k+1) \\ 0 & 1 \end{pmatrix}.$$

We call the coefficients $s(2k)$ and $s(2k+1)$ in (1.3) and (1.4) Stokes coefficients. Our result is the following

Theorem 1.1. *Under the assumptions above the equation (1.1) has a* π *periodic solution if and only if* $s(2k) = s(2k+1) = -i(e^{\pi ic}+e^{-\pi ic})$ *for all* $k = \pm 1, \pm 2, \ldots$.

We give some consequences of Theorem 1.1. Let us consider the equation

$$(1.5) \qquad Pu \equiv -\left(\frac{d}{dt}\right)^2 u + V(t)u = \lambda u.$$

Let $\Delta(\lambda)$ be the Hill's discriminant of (1.5), and let λ be any solution of the equation $\Delta(\lambda) = e^{\pi ic} + e^{-\pi ic}$. We know that this equation has infinite number of solutions λ.(cf.[2]). Then we have

Corollary 1.2. *Let λ be any solution of the equation $\Delta(\lambda) = e^{\pi ic}+e^{-\pi ic}$. Then we have $s(2k) = s(2k+1) = -i\Delta(\lambda)$ for all $k = \pm 1, \pm 2, \ldots$*

Corollary 1.3. *The equation (1.5) has a π periodic solution if and only if $s(2k) = s(2k+1) = -2i$ for all $k = \pm 1, \pm 2, \ldots$ The equation (1.5) has a solution such that $u(t+\pi) = -u(t)$ for all t if and only if $s(2k) = s(2k+1) = 2i$ for all $k = \pm 1, \pm 2, \ldots$*

We shall give an application of our results. We denote by $C^\infty(T^2)$ the set of smooth functions on $\mathbf{T}^2 = \mathbf{R}^2/2\pi\mathbf{Z}^2$. We say that a differential operator Q on $\mathbf{T}^2$ is locally hypoelliptic if, for any subdomain $\Omega \subset \mathbf{T}^2$ and any distribution u in Ω such that Qu is smooth in Ω it follows that u is smooth in Ω. On the other hand, we say that Q is globally hypoelliptic if, for any distribution u in $\mathbf{T}^2$ such that Qu is smooth in $\mathbf{T}^2$ it follows that u is smooth in $\mathbf{T}^2$. Then we have

Corollary 1.4. *The operator $P-\lambda$ given by (1.5) is not (locally) hypoelliptic on $C^\infty(\mathbf{T}^2)$. $P-\lambda$ is globally hypoelliptic if and only if* *either $s(2k) \neq \pm 2i$ or $s(2k+1) \neq \pm 2i$ is satisfied for some integer k. The last condition is satisfied except for countable number of λ.*

Remark 1.5. By Corollary 1.4 we obtain infinitely many globally hypoelliptic but not (locally) hypoelliptic operators. We also see the existence of not globally hypoelliptic operators. In any case these operators do not satisfy a vector field condition of Hörmander type or its general version because these equations depend only on one variable. (cf. [1],[4]).

§2. Proof of Theorems.

Proof of Theorem 1.1. We set $z = e^{it}$ in (1.1). Then (1.1) is written

$$(2.1)\quad z^2\left(\frac{d}{dz}\right)^2 u + 2c\frac{d}{dz}u + \tilde{V}(z)u = \lambda u,\quad \tilde{V}(z) = \sum_{n=-1}^{\infty} a_{-n}\, z^{-2n}.$$

We set $a_{-1} = \mu^2$. Let k be an integer. Then, by the general theory in [3] and [5] we have a solution $\tilde{u}_{2k}(z)$ of (2.1) in the sector $\tilde{S}_{2k} = \{z \in \mathbf{C};\ -\theta+2k\pi-3\pi/2 < \operatorname{Arg} z < -\theta+2k\pi+3\pi/2\}$. We have an asymptotic expansion $\tilde{u}_{2k}(z) \sim e^{-i\mu z} z^{-c-1/2}$ for $z \in \tilde{S}_{2k}$, $z\to\infty$, where $\operatorname{Arg}(i\mu) = \theta$. The solution is unique except for a constant factor. Hence, we normalize $\tilde{u}_{2k}(z)$ such that $\tilde{u}_{2k}(z)e^{i\mu z}z^{c1/2} \to 1$ when $z \in \tilde{S}_{2k}$, $z \to \infty$. Similary, in the sector $\tilde{S}_{2k+1} = \{z \in \mathbf{C};\ -\theta+(2k+1)\pi-3\pi/2 < \operatorname{Arg} z < -\theta+(2k+1)\pi+3\pi/2\}$ there exists a unique solution $\tilde{u}_{2k+1}(z)$ of (2.1) which has an asymptotic expansion $\tilde{u}_{2k+1}(z)e^{-i\mu z}z^{c+1/2} \sim 1$ for $z\in\tilde{S}_{2k+1}$, $z \to \infty$. Since $z = e^{it}$ we may take the branch of t so that Re t = Arg z, Im $t = -\log|z|$. Hence we have the solutions $u_{2k}(t)$ and $u_{2k+1}(t)$ in the domains S_{2k} and S_{2k+1}, respectively.

Suppose that there exists a π periodic solution $v(t)$ to (1.1). Because the equation is analytic in t, $v(t)$ is analytic and π periodic in **C**, $v(t+\pi) = v(t)$ for all $t\in\mathbf{C}$. In $S_{2k} \cap S_{2k+1}$ we have an expression

(2.2) $\quad v(t) = a\, u_{2k}(t) + b\, u_{2k+1}(t)$ for some a and b.

Similarly, in $S_{2k+2} \cap S_{2k+3}$ we have

(2.3) $\quad v(t) = a^{*}u_{2k+2}(t) + b^{*}u_{2k+3}(t)$ for some a^{*} and b^{*}.

Because $v(it-\theta+2k\pi) = v(it-\theta+2k\pi+\pi)$ for all $t<0$, it follows from (2.2) and the asymptotic expansions that

(2.4) $$b = ae^{-\pi(c+1/2)i}.$$

Similarly, by comparing the asymptotic behavior of both sides of the identity $v(it-\theta+2k\pi+2\pi) = v(it-\theta+2k\pi+3\pi)$ for all $t<0$ we have

(2.5) $$b^{*} = a^{*}e^{-\pi(c+1/2)i}.$$

On the other hand, since $v(it-\theta+2k\pi) = v(it-\theta+2k\pi+2\pi)$ for all $t<0$ by assumption it follows from (2.2) and (2.3) that

$$(2.6)\qquad b = b^{*}e^{-2\pi(c+1/2)i}.$$

Hence it follows from (2.2), (2.3), (2.4), (2.5) and (2.6) that

$$(2.7)\quad u_{2k}(t) + e^{-\pi(c+1/2)i}\, u_{2k+1}(t)$$

$$= e^{2\pi(c+1/2)i}u_{2k+2}(t) + e^{\pi(c+1/2)i}u_{2k+3}(t).$$

We note that (2.7) is valid for all t in **C** by analytic continuation. By (1.3) and (1.4) which are valid for all t in **C** by analytic continuation we have

$$(2.8)\qquad (u_{2k+2}, u_{2k+3}) = (u_{2k+2}, u_{2k+1})\begin{pmatrix} 1 & s(2k+1) \\ 0 & 1 \end{pmatrix}$$

$$= (u_{2k}, u_{2k+1})\begin{pmatrix} 1 & 0 \\ s(2k) & 1 \end{pmatrix}\begin{pmatrix} 1 & s(2k+1) \\ 0 & 1 \end{pmatrix}$$

$$= (u_{2k}, u_{2k+1})\begin{pmatrix} 1 & s(2k+1) \\ s(2k) & s(2k)s(2k+1)+1 \end{pmatrix}.$$

It follows from (2.7) and (2.8) that

$$u_{2k}(t) + e^{-\pi(c+1/2)i}\, u_{2k+1}(t) = e^{2\pi(c+1/2)i}\left(u_{2k}(t)+s(2k)u_{2k+1}(t)\right)$$

$$+ e^{\pi(c+1/2)i}\left(s(2k+1)u_{2k}(t) + (s(2k)s(2k+1)+1)u_{2k+1}(t)\right).$$

Hence we have

$$1 = e^{2\pi(c+1/2)i} + e^{\pi(c+1/2)i}s(2k+1),$$

(2.9)

$$e^{-\pi(c+1/2)i} = e^{2\pi(c+1/2)i}s(2k) + e^{\pi(c+1/2)i}(s(2k)s(2k+1)+1).$$

Therefore we have

$$s(2k) = s(2k+1) = -i(e^{\pi ci} + e^{-\pi ci}). \tag{2.10}$$

This proves the sufficiency.

Conversely, let us assume that (2.10) is satisfied. By definition the functions $u_1(t)$ and $u_1(t+\pi)$ are solutions of (1.1) in the domain $A_0 = \{t \in \mathbf{C};\ -\theta -\pi/2 < \mathrm{Re}\ t < -\theta +\pi/2\}$. Since $u_1(t+\pi)$ and $u_0(t)$ are decreasing in A_0 as $t \to \infty$ we get, from the uniqueness that $u_1(t+\pi) = ku_0(t)$ in A_0 for some k. This identity is valid if $t \in A_0+\pi$. Hence we have that $u_1(t+2\pi) = ku_0(t+\pi)$ in A_0. By comparing the asymptotic behavior as $t \to \infty$ we have that $k=e^{-\pi(c+1/2)i}$. Hence we have

$$u_1(t+\pi) = e^{-\pi(c+1/2)i}\ u_0(t). \tag{2.11}$$

We note that (2.11) is valid for all t in $\mathbf{C}$ because both sides of (2.11) are analytic functions of t.

Next we show that

$$u_2(t+\pi) = e^{-\pi(c+1/2)i}\, u_1(t), \quad t \in \mathrm{C}. \tag{2.12}$$

Indeed the functions $u_2(t+\pi)$ and $u_1(t)$ are decreasing solutions of (1.1) in the domain $A_0+\pi$. Hence there exists k such that $u_2(t+\pi) = k\, u_1(t)$ in $A_0+\pi$. Since this is valid in $A_0+2\pi$ we have that $u_2(t+2\pi) = k\, u_1(t+\pi)$ in $A_0+\pi$. By comparing the asymptotic behavior as $t\to\infty$ we have that $k = e^{-\pi(c+1/2)i}$. Hence we have (2.12).

It follows from (1.3) with $k=0$ and (2.10) that $u_2(t)=s(0)u_1(t)+u_0(t) = -i(e^{\pi ci} + e^{-\pi ci})u_1(t) + u_0(t)$. This implies that $u_2(t+\pi)= -i(e^{\pi ci} + e^{-\pi ci})u_1(t+\pi) + u_0(t+\pi)$. Hence it follows from (2.11) and (2.12) that

$$u_0(t+\pi) = u_2(t+\pi) + i(e^{\pi ci} + e^{-\pi ci})u_1(t+\pi) = e^{-\pi(c+1/2)i}\, u_1(t) +$$

$$i(e^{\pi ci}+ e^{-\pi ci})e^{-\pi(c+1/2)i}\, u_0(t) = e^{-\pi(c+1/2)i}\, u_1(t) + (1 + e^{-2\pi ci})u_0(t).$$

In view of (2.11) this implies that

$$u_0(t+\pi) + e^{-\pi(c+1/2)i}\, u_1(t+\pi) = e^{-\pi(c+1/2)i}\, u_1(t) + (1 + e^{-2\pi ci})u_0(t) +$$
$$e^{-\pi(c+1/2)i}e^{-\pi(c+1/2)i}\, u_0(t) = u_0(t) + e^{-\pi(c+1/2)i}\, u_1(t),$$

showing that the function $v(t) \equiv u_0(t) + e^{-\pi(c+1/2)i}\, u_1(t)$ is π periodic in t. Hence we have proved the sufficiency. This proves Theorem 1.1.

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