数理解析研究所講究録
第 729 巻 1990 年 15-31

An algorithm of constructing the integral of a module
— an infinite dimensional analog of Gröbner basis

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(*February 8, 1990*)

Abstract.
Let K be a field of characteristic zero. The Weyl algebra:

$$K\langle x_1, \cdots, x_n, \partial_1, \cdots, \partial_n\rangle$$

is denoted by A_n. We have

$$[x_i, \partial_j] = x_i\partial_j - \partial_j x_i = \begin{cases} -1, & i = j, \\ 0, & i \neq j, \end{cases}$$

in the Weyl algebra. Let $\mathfrak{A}$ be a left ideal of A_n. We put $M = A_n/\mathfrak{A}$. M is a left A_n module. The purpose of this paper is an explicit construction of the left A_{n-1} module:

$$\int M dx_n := M/\partial_n M$$

by introducing an analog of Gröbner basis of a submodule of a kind of infinite dimensional free module. We call $M/\partial_n M$ the *integral* of the module M. The non-commutativity of A_n prevents us from using the usual Buchberger algorithm to construct $M/\partial_n M$. (If A_n is commutative, then $M/\partial_n M \simeq A_n/(\partial_n, \mathfrak{A})$. There is no problem.) We must consider a sum of left and right ideal of A_n. We overcome this difficulty by using an infinite dimensional analog of Gröbner basis.

The algorithm of constructing the integral of a module is not only important to mathematicians, but also has many impacts on the classical fields of computer algebra. It plays central roles in mathematical formula verification [Zeil], [Tak2], computation of a definite integral [AZ], [Tak2] and an asymptotic expansion of a definite integral with respect to parameters. However, a complete algorithm of obtaining $M/\partial_n M$ had not been known. We give a complete algorithm in this paper. The algorithm is an answer to the research problem of the paper [AZ].

We refer to [Buch1], [Buch2], [MM], [FSK] for the Gröbner basis of a polynomial ideal and free module, to [Gal], [Cas], [Tak1], [Nou] , [UT] for the Gröbner basis of the ideal of Weyl algebra, to [Ber], [Bjo] for holonomic system and Weyl algebra. We remark that [Berg] also considered infinite set of reduction systems.

Submitted to ISSAC'90.

§1. A simple example

We explain an idea by using a simple example.

EXAMPLE 1.1 We compute a differential operator that annihilates the function:

$$I(x) = \int_{-\infty}^{\infty} f(x,t)dt, \quad f(x,t) = e^{-xt^2}.$$

We have $\ell_1 f = \ell_2 f = 0$ where

$$\ell_1 = \partial_x + t^2, \quad \ell_2 = \partial_t + 2xt.$$

We put $x_2 = t$ and $x_1 = x$. Let $\mathfrak{A}$ be the left ideal of A_2 generated by ℓ_1 and ℓ_2. The Gröbner basis of $\mathfrak{A}$ by the lexicographic order $t \succ \partial_t \succ x \succ \partial_x$ is

$$G = \left\{t^2 + \partial_x, 2xt + \partial_t, t\partial_t - 2x\partial_x, {\partial_t}^2 + 4x^2\partial_x + 2x\right\}.$$

The Hilbert function of $A_2/\mathfrak{A}$ is $k^2 + 3k + 1$. We put $R = \mathbf{C}\langle x, \partial_x, \partial_t \rangle$ and define a map

$$\psi : \mathbf{C}\langle t, x, \partial_t, \partial_x \rangle \ni \sum_{k=0}^{m-1} t^k f_k \longmapsto (f_0, \cdots, f_{m-1}) \in R^m$$

where $f_k \in R$. We have

$$(1.1) \qquad \begin{aligned} \psi(G) &= \left\{(\partial_x, 0, 1, 0, \cdots), (\partial_t, 2x, 0, \cdots), (-2x\partial_x, \partial_t, 0, \cdots), ({\partial_t}^2 + 4x^2\partial_x + 2x, 0, \cdots)\right\} \\ \psi(tG) &= \{(0, \partial_x, 0, 1, 0, \cdots), (0, \partial_t, 2x, 0, \cdots), \cdots\} \\ &\cdots \\ \psi(t^{m-3}G) &= \{(0, \cdots, 0, \partial_x, 0, 1), \cdots\}, \end{aligned}$$

and we use

$$(1.2) \qquad \{(\partial_t, 0, \cdots), (1, \partial_t, 0, \cdots), (0, 2, \partial_t, 0, \cdots), (0, \cdots, m-1, \partial_t)\}.$$

(1.2) is obtained from

$$\partial_t t^k = t^k \partial_t + k t^{k-1}.$$

R^m has a left R module structure defined in the beginning of the section two.

Let $\mathcal{M}_m$ be a left R submodule of R^m generated by (1.1) and (1.2). We remark that

$$\psi^{-1}(\mathcal{M}_m) \subseteq \partial_t A_2 + \mathfrak{A}.$$

We apply the algorithm of Gröbner basis of submodule ([MM], [FSK]) to $\mathcal{M}_m$. Then we obtain, for example,

$$\mathrm{sp}\left((1, \partial_t, 0, \cdots), (-2x\partial_x, \partial_t, 0, \cdots)\right) = (1 + 2x\partial_x, 0, \cdots), \ \cdots \text{etc.}$$

The above equation is equivalent to

$$\partial_t t - (t\partial_t - 2x\partial_x) = 1 + 2x\partial_x, \quad \partial_t t \in \partial_t A_2, t\partial_t - 2x\partial_x \in \mathfrak{A}.$$

Therefore we have

$$\partial_t(tf) - (t\partial_t - 2x\partial_x)f = \partial_t(tf) = (1 + 2x\partial_x)f.$$

We conclude that

$$(1 + 2x\partial_x)I(x) = \int_{-\infty}^{\infty} \partial_t(tf)dt = 0 \quad (x > 0).$$

§2. Gröbner basis for submodule of $R_\infty = \varinjlim R^m$

Let us consider the general situation. We put

$$R = K\langle x_1, \cdots, x_{n-1}, \partial_1, \cdots, \partial_n\rangle = A_{n-1}[\partial_n].$$

We define a left R module structure to R^m in the following way. For

$$R^m \ni \vec{f} = (f(0), \cdots, f(m-1)), \ a \in A_{n-1},$$

we put

$$a\vec{f} = (af(0), \cdots, af(m-1)) \tag{2.1}$$

and for $a = \partial_n$,

$$a\vec{f} = (af(0) + f(1), \cdots, af(k) + (k+1)f(k+1), \cdots, af(m-1)). \tag{2.2}$$

The Weyl algebra A_n has a left R module structure in the standard way. The map

$$\varphi : R^m \ni \vec{f} \longmapsto \sum_{k=0}^{m-1} x_n^k f(k) \in A_n$$

is homomorphism of left R module.

We can define the notions of addmissible order, reducible, S-polynomial(sp) and Gröbner basis of the ring R in a similar way to the case of the polynomial ring. Let us explain some of them to avoid misunderstandings. We define an order $\prec_1$ between monomials of R by

$$x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \prec_1 x_1^{\gamma_1} \cdots x_{n-1}^{\gamma_{n-1}} \partial_1^{\delta_1} \cdots \partial_n^{\delta_n} \tag{2.3}$$

$\Longleftrightarrow$

$$(\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_n) \prec_2 (\gamma_1, \cdots, \gamma_{n-1}, \delta_1, \cdots, \delta_n)$$

where $\prec_2$ is the total degree order in $\mathbb{N}_0^{2n-1}$. We use the order in the sequel. Let r and s be elements of R. We put

$$\text{head}(r) = \text{ head term of } r \text{ by the order (2.3)} = cx^\alpha \partial^\beta, \quad c \in K$$

and $\text{head}(s) = dx^\gamma \partial^\delta$, $d \in K$. We define

$$\text{lcm}(\alpha, \gamma) = (\max\{\alpha_1, \gamma_1\}, \cdots, \max\{\alpha_{n-1}, \gamma_{n-1}\}),$$

$$\text{lcm}(\beta, \delta) = (\max\{\beta_1, \delta_1\}, \cdots, \max\{\beta_n, \delta_n\}).$$

Iff $\text{lcm}(\alpha, \gamma) = \alpha$ and $\text{lcm}(\beta, \delta) = \beta$, r is reducible by s. Put $\xi = \text{lcm}(\alpha, \gamma)$ and $\eta = \text{lcm}(\gamma, \delta)$. We define

$$\text{sp}(r, s) = x^{\xi-\alpha}\partial^{\eta-\beta} r - \frac{c}{d} x^{\xi-\gamma}\partial^{\eta-\delta} s.$$

Let r be reducible by s and $t = \text{sp}(r, s)$, then the situation is denoted by "$r \longrightarrow t$ by s". Let $\to^*$ be a transitive closure of $\to$. A finite subset G of R is called Gröbner basis of an ideal $\mathfrak{A}$ if $\forall r_i, r_j \in G, \text{sp}(r_i, r_j) \longrightarrow^* 0$ by G and $\mathfrak{A} = RG$. It is well known that every left ideal of R has a Gröbner basis [Gal], [Cas], [Takl], [Nou], [UT].

Consider R^m. [MM] and [FSK] extended the notion of Gröbner basis to free modules. We can apply their extension to R^m. Let us review their extension (See [Takl] for proofs in the case of a free module over a non-commutative ring). We define

$$\text{topIndex}(\vec{f}) = k, \quad \vec{f} \in R^m,$$

iff $f(i) = 0$ $(k < i \le m-1)$ and $f(k) \neq 0$. Let $\vec{f}$ and $\vec{g}$ be elements of R^m.

DEFINITION 2.1 $\vec{f}$ is reducible by $\vec{g}$ iff $k = \mathrm{topIndex}(\vec{f}) = \mathrm{topIndex}(\vec{g})$ and $f(k)$ is reducible by $g(k)$ in R.

DEFINITION 2.2

$$\mathrm{sp}(\vec{f},\vec{g}) := \begin{cases} 0, & \text{if } \mathrm{topIndex}(\vec{f}) \neq \mathrm{topIndex}(\vec{g}) \\ c_1\vec{f} - c_2\vec{g}, & \text{if } k = \mathrm{topIndex}(\vec{f}) = \mathrm{topIndex}(\vec{g}), \end{cases}$$

where $\mathrm{sp}(f(k), g(k)) = c_1 f(k) - c_2 g(k)$.

DEFINITION 2.3

$$\vec{f} \succ \vec{g} \Longleftrightarrow \begin{array}{l} \mathrm{topIndex}(\vec{f}) > \mathrm{topIndex}(\vec{g}) \\ \text{or } \left(\mathrm{topIndex}(\vec{f}) = \mathrm{topIndex}(\vec{g}) = k \text{ and } f(k) \succ_1 g(k)\right) \end{array} \tag{2.4}$$

We use the order (2.4) of R^m in the sequel. We remark that other order in R^m can be used in our theory. The use of good orders leads us to a fast termination of Buchberger algorithm.

We put

$$\mathcal{G} = \{\vec{g}_1, \cdots, \vec{g}_p\}$$

DEFINITION 2.4 If there exists an element $\vec{g}_i$ of $\mathcal{G}$ such that $\vec{f}$ is reducible by $\vec{g}_i$, then we write the situation as follows.

$$\vec{f} \longrightarrow \vec{h} \text{ by } \mathcal{G}$$

where $\vec{h} = \mathrm{sp}(\vec{f}, \vec{g}_i)$.

We remark $\vec{h} \prec \vec{f}$. Let $\rightarrow^*$ be transitive closure of $\rightarrow$. Suppose $\vec{f} \rightarrow^* \vec{h}$. We remark that $\vec{h}$ is not uniquely determined by $\vec{f}$. It depends on the sequence of reductions.

DEFINITION 2.5 $\mathcal{G}$ is a Gröbner basis of a left R submodule $\mathcal{M}$ of R^m iff
(1) $\forall i, j,\ \mathrm{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$.
(2) $\mathcal{G}$ generates $\mathcal{M}$ over R.

Any left submodule $\mathcal{M}$ of R^m has a Gröbner basis ([MM], [FSK], [Tak1]).

$\vec{e}_i$ denotes i-th unit vector, i.e.,

$$\vec{e}_0 = (1, 0, \cdots, 0),\ \vec{e}_1 = (0, 1, 0, \cdots, 0), \cdots.$$

$\vec{g}_i$ can be decomposed into a sum of (monomial of R) $\times$ (unit vector) which is written as

$$\vec{g}_i = \sum_j c_i^j \vec{e}_{k_j}, \quad c_i^j \text{ is a monomial of } R.$$

$\mathcal{G}$ is a reduced Gröbner basis of $\mathcal{M}$ iff $\mathcal{G}$ is a Gröbner basis of $\mathcal{M}$,

$$\forall i, j \ \ c_i^j \vec{e}_{k_j} \longrightarrow^* c_i^j \vec{e}_{k_j} \ \text{ by } \mathcal{G} \setminus \{\vec{g}_i\}$$

and the head coefficients of $\vec{g}_i$ is 1.

We define these notions on

$$R_\infty = \lim_{\rightarrow} R^m \simeq K[x_n] \otimes_K R.$$

Any element $\vec{f}$ of R_∞ can be written as

$$\vec{f} = (f(0), f(1), \cdots), \quad \exists k, i > k \Rightarrow f(i) = 0 \text{ and } f(k) \neq 0.$$

The number k is denoted by topIndex($\vec{f}$). Therefore we can consider $\vec{f}$ as the element of R^m, $m \geq k$. We define the notions of reducibility, s-polynomial and order $\prec$ identifying the element $\vec{f}$ of R_∞ with the element $(f(0), \cdots, f(m-1))$ of R^m, $(m \geq k)$.

Put

$$\mathcal{G} = \{\vec{g}_1, \vec{g}_2, \cdots\}, \quad \vec{g}_i \in R_\infty.$$

We do not assume that $\mathcal{G}$ is finite set. Put

$$\mathcal{G}(k) = \{\vec{g} \in \mathcal{G} | \text{topIndex}(\vec{g}) \leq k\}.$$

Assumption 2.1

$$\forall k, \ \#\mathcal{G}(k) < +\infty.$$

We consider the existence of a Gröbner basis under Assumption 2.1 in the sequal.

Definition 2.6

$$\vec{f} \longrightarrow \vec{h} \text{ by } \mathcal{G} \Longleftrightarrow \exists i, \exists m, \vec{g}_i \in \mathcal{G}, \text{topIndex}(g_i) \leq m, \text{topIndex}(f) \leq m \text{ and } \vec{f} \longrightarrow \vec{h} \text{ by } \vec{g}_i \text{ in } R^m.$$

$\vec{f} \longrightarrow \vec{h}$ is called reduction of $\vec{f}$.

Proposition 2.1 *For any element $\vec{f} \in R_\infty$, any sequence of reduction of $\vec{f}$ by $\mathcal{G}$ terminates in finite steps.*

Proof. Put m = topIndex($\vec{f}$). Note that any sequence of reduction of $\vec{f}$ uses the elements of $\mathcal{G}(m)$. Since $\mathcal{G}(m)$ is the finite set, the sequence terminates in finite steps. ∎

It follows from Proposition 2.1 that we can take a transitive closure of $\rightarrow$ in finite steps. The transitive closure is denoted by $\rightarrow^*$.

Definition 2.7 $\mathcal{G}$ is a Gröbner basis of a left R submodule $\mathcal{M}$ of R_∞ iff
(1) $\forall i, j$, $\text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$.
(2) $\mathcal{G}$ generates $\mathcal{M}$ over R, i.e., $\forall \vec{f} \in \mathcal{M}, \exists I \subset \mathbb{N}, \exists a_i \in R$ such that $\#I < \infty$ and

$$\vec{f} = \sum_{i \in I} a_i \vec{g}_i.$$

(3) (local finiteness)

$$\forall m, \ \#\mathcal{G}(m) < +\infty.$$

Proposition 2.2 *If $\mathcal{G}$ is a Gröbner basis of an R submodule $\mathcal{M} \subset R_\infty$, then*

$$\forall \vec{g}_i, \vec{g}_j \in \mathcal{G}(m), \ \text{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0 \text{ by } \mathcal{G}(m).$$

Proof. We have $\mathrm{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}$. Since $\mathrm{topIndex}(\mathrm{sp}(\vec{g}_i, \vec{g}_j)) \leq m$, we have $\mathrm{sp}(\vec{g}_i, \vec{g}_j) \longrightarrow^* 0$ by $\mathcal{G}(m)$. ∎

THEOREM 2.1 *Let $\mathcal{M}$ be a left R submodule of R_∞ and $\mathcal{G}$ be a Gröbner basis of $\mathcal{M}$. If $\vec{f} \in \mathcal{M}$, then $\vec{f} \longrightarrow^* 0$ by $\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is a set of generators of $\mathcal{M}$, there exist an index set I and elements $a_i \in R$, $i \in I$ such that $\#I < +\infty$ and $\vec{f} = \sum_{i \in I} a_i \vec{g}_i$. Put $m = \max_{i \in I}\{\mathrm{topIndex}(\vec{g}_i)\}$. We can consider $\vec{f}$ as an element of R^m. It follows from Proposition 2.2 that $\mathcal{G}(m)$ is a Gröbner basis of $R\mathcal{G}(m)$ in R^m. Since $\vec{f} \in R\mathcal{G}(m)$, we have $\vec{f} \longrightarrow^* 0$ by $\mathcal{G}(m)$ for any sequence of reduction. ∎

Let $\mathcal{H}_m$, $m = 0, 1, 2, \cdots$ be subsets of R_∞ that satisfy the conditions:

$$\begin{aligned} &\cdots \subseteq \mathcal{H}_m \subseteq \mathcal{H}_{m+1} \subseteq \cdots \\ &\#\mathcal{H}_m < +\infty, \ \forall \vec{f} \in \mathcal{H}_m, \ \mathrm{topIndex}(\vec{f}) \leq m. \end{aligned} \tag{2.5}$$

Suppose that $\mathcal{M}_\infty$ is the left R submodule generated by $\bigcup_{m=0}^{\infty} \mathcal{H}_m$. We have

$$\mathcal{M}_\infty = \bigcup_{m=0}^{\infty} R\mathcal{H}_m = \varinjlim R\mathcal{H}_m.$$

THEOREM 2.2 *Let $\mathcal{G}_m$ be the reduced Gröbner basis of $R\mathcal{H}_m$ in R^m.*

$$\mathcal{G}_\infty = \bigcup_{m=0}^{\infty} \mathcal{G}_m$$

is a Gröbner basis of $\mathcal{M}_\infty$.

Proof. We prove the local finiteness condition: $\#\mathcal{G}_\infty(m) < +\infty$. We remark that $\mathcal{G}_\infty(m) \neq \mathcal{G}_m$ in general. Put

$$\mathcal{G}_k(m) = \{\vec{f} \in \mathcal{G}_k | \mathrm{topIndex}(\vec{f}) \leq m\}.$$

$\mathcal{G}_k(m)$ is a Gröbner basis of $R\mathcal{G}_k(m)$ in R^m. Since $\cdots \subseteq R\mathcal{G}_k(m) \subseteq R\mathcal{G}_{k+1}(m) \subseteq \cdots$ in R^m, there exists k_0 such that $\forall k \geq k_0$, $R\mathcal{G}_k(m) = R\mathcal{G}_{k_0}(m)$. $\mathcal{G}_k$ is the reduced Gröbner basis, then we have $\forall k \geq k_0$, $\mathcal{G}_k(m) = \mathcal{G}_{k_0}(m)$. Hence $\#\mathcal{G}_\infty(m) < +\infty$.

Other conditions are easily verified. ∎

COROLLARY 2.1 *For any m, we can obtain $\mathcal{G}_\infty(m)$ in finite steps.*

ALGORITHM 2.1
INPUT: $\mathcal{H}_m$: generators of a submodule that satisfy the condition (2.5).
OUTPUT: $\mathcal{G}_m$: m-th approximation of Gröbner basis $\mathcal{G}_\infty$ of the submodule $\mathcal{M}_\infty$.

(1) $\mathcal{G}_m :=$ the reduced Gröbner basis of $R\mathcal{H}_m$ in R^m.

REMARK 2.1 If m is large number in Algorithm 2.1, then it follows from Corollary 2.1 that we have $\mathcal{G}_m(k) = \mathcal{G}_\infty(k)$ for small number k where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty$. However, we do not have an algorithm of deciding $\mathcal{G}_m(k) = \mathcal{G}_\infty(k)$ or not.

§3. Computation of the integral of A_n module

Let $\mathfrak{A}$ be a left ideal of A_n and M be $A_n/\mathfrak{A}$. We have

$$M/\partial_n M \simeq A_n/(\partial_n A_n + \mathfrak{A}) \quad \text{as } A_{n-1} \text{ module.}$$

The set $\partial_n A_n + A_n \mathfrak{A} = \partial_n A_n + \mathfrak{A}$ is not left A_n module. Let us note that R is a subalgebra which is commutative to ∂_n. Therefore $\partial_n A_n + \mathfrak{A}$ has a left R module structure. We will show that $\partial_n A_n + \mathfrak{A}$ is the left R submodule of R_∞, prove the existence of a Gröbner basis (with the local finiteness property) of the module and present a construction algorithm of the basis.

Let

$$G = \{g_1 \cdots, g_p\} \tag{3.1}$$

be generators of the left ideal $\mathfrak{A}$ of A_n. g_k can be written as

$$g_k = \sum_{j=0}^{s_k} x_n^j g_{kj}, \quad g_{kj} \in R.$$

We put

$$\psi(\partial_n x_n^k) = (0, \cdots, 0, k, \partial_n, 0, \cdots, 0) \in R^m,$$

and

$$\psi(x_n^i g_k) = (0, \cdots, 0, g_{k0}, g_{k1}, \cdots, g_{ks_k}, 0, \cdots, 0) \in R^m.$$

Let $\mathcal{H}_m \subset R^m$ be

$$\left(\bigcup_{k=0}^{m-1} \{\psi(\partial_n x_n^k)\}\right) \bigcup \left(\bigcup_{k=1}^{p} \bigcup_{i=0}^{m-s_k-1} \{\psi(x_n^i g_k)\}\right). \tag{3.2}$$

We have $\cdots \subseteq \mathcal{H}_m \subseteq \mathcal{H}_{m+1} \subseteq \cdots$ and $\#\mathcal{H}_m < +\infty$. $\mathcal{M}_\infty = \bigcup_{m=0}^\infty R\mathcal{H}_m$ is the left R submodule of R_∞. It follows from Theorem 2.2 that $\mathcal{M}_\infty$ has a Gröbner basis $\mathcal{G}_\infty$. We can compute approximations of $\mathcal{G}_\infty$ by Algorithm 2.1.

THEOREM 3.1.

$$R_\infty/\mathcal{M}_\infty \simeq A_n/(\partial_n A_n + \mathfrak{A}) = \int M dx_n$$

as left A_{n-1} module.

Proof. We define a map:

$$\theta : R_\infty \ni \vec{f} = (f(0), f(1), \cdots, f(m), 0, \cdots) \mapsto f(0) + x_n f(1) + \cdots + (x_n)^m f(m) \in A_n$$

where $m = \text{topIndex}(\vec{f})$.

We prove if $\vec{f} \in \mathcal{M}_\infty$, then $\theta(\vec{f}) \in \partial_n A_n + \mathfrak{A}$. Since $\vec{f} \in \mathcal{M}_\infty$, there exists $a_j, b_j \in R$ such that

$$\vec{f} = \sum_j a_j \psi(\partial_n x_n^{k_j}) + \sum_j b_j \psi(x_n^{i_j} g_{k_j})$$

where $\sum_j$ is a finite sum. Then we have

$$\begin{aligned}\theta(\vec{f}) &= \sum_j a_j \partial_n x_n^{k_j} + \sum_j b_j x_n^{i_j} g_{k_j} \\ &= \sum_j \partial_n (a_j x_n^{k_j}) + \sum_j (b_j x_n^{i_j}) g_{k_j} \in \partial_n A_n + \mathfrak{A}.\end{aligned}$$

Therefore we can define a map:

$$\hat{\theta} : R_\infty/\mathcal{M}_\infty \longrightarrow A_n/(\partial_n A_n + \mathfrak{A}),$$

by $\hat{\theta}([\vec{f}]) = [\theta(\vec{f})]$.

It is easily verified that $\hat{\theta}$ is A_{n-1} homomorphism and surjective.

We will show that $\hat{\theta}$ is injective. We assume that $\theta(\vec{f}) = \partial_n h + g \in \partial_n A_n + \mathfrak{A}$, $h \in A_n, g \in \mathfrak{A}$. h can be written as $h = \sum_k h_k x_n^k$, $h_k \in R$. Then we have $\partial_n h = \sum_k h_k(\partial_n x_n^k)$. g can be written as $g = \sum_k c_k g_k$. c_k has an expression of the form $c_k = \sum_j b_{kj} x_n^j$, $b_{kj} \in R$. Then we have $g = \sum_{k,j} b_{kj} x_n^j g_k$. Since θ is injective, then we have

$$\vec{f} = \sum_k h_k \psi(\partial_n x_n^k) + \sum_{k,j} b_{kj} \psi(x_n^j g_k) \in \mathcal{M}_\infty.$$

Therefore $\hat{\theta}$ is injective. ∎

COROLLARY 3.1 *If M is holonomic, then there exists a number m such that*

$$R^m/R\mathcal{G}_\infty(m) \simeq \int M dx_n$$

as A_{n-1} module where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty = \bigcup R\mathcal{H}_m$ of (3.2).

Proof. If M is holonomic, then $R_\infty/\mathcal{M}_\infty$ is finitely generated. Let $\vec{h}_1, \cdots, \vec{h}_p$ be generators. Assume $\vec{h}_i, i = 1, \cdots, p$ are irreducible by $\mathcal{M}_\infty$. Put $m = \max_{i=1}^p\{\text{topIndex}(\vec{h}_i)\}$. Let us consider a map ρ :

$$\rho : \ R^m/R\mathcal{G}_\infty(m) \longrightarrow R_\infty/\mathcal{M}_\infty$$

where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty$. If an element $\vec{f}$ of R^m is irreducible by $\mathcal{G}_\infty(m)$, then $\vec{f}$ is irreducible by $\mathcal{G}_\infty$. Therefore the map ρ is injective. Since $\vec{h}_1, \cdots, \vec{h}_p$ are generators, then the map ρ is surjective. ∎

The lexico-total degree order is

(3.3) $$\{x_n\} \succ \{x_1, \cdots, x_{n-1}, \partial_1, \cdots, \partial_n\}$$

and the lexicographic order is

(3.4) $$x_n \succ \partial_n \succ x_{n-1} \succ \cdots \succ x_1 \succ \partial_1.$$

We will show an application of our theory to the zero recognition problem [Zeil] [Tak2] and the study of definite integral with parameters [AZ] [Tak2]. Algorithm 3.1 can be used in Algorithm 1.2 of [Tak2] and is "correct " algorithm in the sense of [Tak2].

ALGORITHM 3.1 (Computation of differential equations for a definite integral with parameters)
INPUT: $G = \{g_k\}$, generators (3.1) of a left ideal $\mathfrak{A}$ of A_n. We assume that $M = A_n/\mathfrak{A}$ is holonomic.
OUTPUT: $\mathcal{G}(0)$, a Gröbner basis in R such that $R/R\mathcal{G}(0)$ is holonomic A_{n-1} module, i.e. , $\mathcal{G}(0)$ is a very large system of differential equations such that $\mathcal{G}(0) \subseteq \partial_n A_n + \mathfrak{A}$.
(1) $G :=$ a Gröbner basis of G in A_n by the lexicographic order (3.4) or the lexico-total degree order (3.3).
(2) $m := \max\{s_k + 1\}$; $\mathcal{G} := \emptyset$;
(3) repeat
(4) $\quad \mathcal{H}_m :=$(3.2);
(5) $\quad \mathcal{G} := \mathcal{G} \cup \{$ reduced Gröbner basis of $R\mathcal{H}_m$ in R^m by the order (2.4) $\}$;
(6) $\quad m := m + 1$;
(7) until ($R/R\mathcal{G}(0)$ is holonomic)

For the computation of the Gröbner basis of the step (1), it is fast to call Algorithm 4.3 of [Tak2] and construct a Gröbner basis from the output of Algorithm 4.3 by pure Buchberger algorithm by the order (3.4) or (3.3).

THEOREM 3.2 *Algorithm 3.1 stops.*

Proof. It follows from Corollary 2.1 that we can obtain $\mathcal{G}_\infty(0)$ by finite iterations where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty$. Since $R/R\mathcal{G}_\infty(0)$ is A_{n-1} submodule of $R_\infty/\mathcal{M}_\infty = \int M dx_n$, then $R/R\mathcal{G}_\infty(0)$ is holonomic A_{n-1} module. ∎

THEOREM 3.3 *Assume a function f of $x_1, \cdots, x_n$ is rapidly decreasing with respect to x_n. Let $\mathfrak{A}$ be an ideal of A_n such that $\mathfrak{A}f = 0$. If $A_n/\mathfrak{A}$ is holonomic, then the integral*

$$\int_{-\infty}^{\infty} f dx_n$$

is annihilated by differential operators $\mathcal{G}(0)$ where $\mathfrak{A}$ is the input of Algorithm 3.1 and $\mathcal{G}(0)$ is the output. $R/R\mathcal{G}(0)$ is holonomic A_{n-1} module.

EXAMPLE 3.1 This is continuation of Example 1.1. We consider in R^m, $m = 3$. Put

$$\begin{aligned} h_1 &= (\partial_x, 0, 1) & h_5 &= (\partial_t, 0, 0) \\ h_2 &= (\partial_t, 2x, 0) & h_6 &= (1, \partial_t, 0) \\ h_3 &= (-2x\partial_x, \partial_t, 0) & h_7 &= (0, 2, \partial_t) \\ h_4 &= (p_0, 0, 0) \end{aligned}$$

where $p_0 = \partial_t^2 + 2x + 4x^2\partial_x$. We set $\mathcal{H}_3 = \{h_i | i = 1, \cdots 7\}$. We compute Gröbner basis of $R\mathcal{H}_3$. We have, for example,

$$\mathrm{sp}(h_1, h_7) = \partial_t h_1 - h_7 = (\partial_t, 2, \partial_t\partial_x) - (\partial_t, 2, 0) \to 0 \text{ by } h_5.$$

We have the reduced Gröbner basis:

$$\{h_1, (0, 2x, 0), (1 + 2x\partial_x, 0, 0), h_5, h_6\}.$$

Therefore we have

$$R^3/R\mathcal{H}_3 \simeq A_1/(1 + 2x\partial_x) + tA_1/(x)$$

as A_1 module. The output of Algorithm 3.1 is $\{1 + 2x\partial_x\}$ which is a differential equation for $I(x)$.

§4. A fast algorithm of obtaining differential operators for a definite integral

We must compute a Gröbner basis with lexicographic order or lexico total degree order in step (1) of Algorithm 3.1 and we must use the lexicographic order (2.4) in R^m, but the orders spend much time and very large memory. We will state an efficient algorithm. Put

$$R = K(x_1, \cdots, x_{n-1})\langle\partial_1, \cdots, \partial_n\rangle.$$

The theory of the first part of the section two is valid in this case. Let us define the notions of reducibility, s-polynomial etc. to avoid misunderstandings. We define a left R module structure to R^m in the following way. For $\vec{f} \in R^m$ and $a \in K(x_1, \cdots, x_{n-1})\langle\partial_1, \cdots, \partial_{n-1}\rangle$, we define $a\vec{f}$ = the right hand side of (2.1) and for $a = \partial_n$, $a\vec{f}$ = the right hand side of (2.2). A monomial of R can be written as $c\partial^\alpha$, $c \in K(x_1, \cdots, x_{n-1})$. Monomials of R are ordered as

$$\partial^\alpha \prec_3 \partial^\beta \iff \alpha \prec \beta \text{ by the total degree order in } \mathbb{N}_0^n.$$

Let r and s be elements of R. We put

$$\mathrm{head}(r) = \text{ head term of } r \text{ by the above order } \prec_3 = c\partial^\alpha$$

and head(s) $= d\partial^\beta$ where $c, d \in K(x_1, \cdots, x_{n-1})$. r is reducible by s iff $\mathrm{lcm}(\alpha, \beta) = \alpha$. Put $\xi = \mathrm{lcm}(\alpha, \beta)$ and assume $c = c_1/c_2, d = d_1/d_2,\ c_i, d_i \in K[x_1, \cdots, x_{n-1}]$. We define

$$\mathrm{sp}(r, s) = \frac{d_1 d_2 c_2}{e}\partial^{\xi-\alpha} r - \frac{c_1 c_2 d_2}{e}\partial^{\xi-\beta} s$$

where $e = \gcd(d_1 d_2 c_2, c_1 c_2 d_2)$. Let r be reducible by s and $t = \mathrm{sp}(r, s)$. The situation is denoted by "$r \longrightarrow t$ by s".

Consider R^m. Let $c, d \in K(x_1, \cdots, x_{n-1})$. We define

$$c\partial^\alpha \vec{e}_k \succ d\partial^\beta \vec{e}_\ell \iff \begin{array}{l} k + |\alpha| > \ell + |\beta| \\ \text{or } k + |\alpha| = \ell + |\beta| \text{ and } k > \ell \\ \text{or } k + |\alpha| = \ell + |\beta| \text{ and } k = \ell \text{ and } \partial^\alpha \succ_3 \partial^\beta. \end{array} \tag{4.1}$$

Let $\vec{f}$ and $\vec{g}$ be elements of R^m. We put

$$\mathrm{head}(\vec{f}) = \text{head term of } \vec{f} \text{ by the above order (4.1)} = c\partial^\alpha \vec{e}_k$$

and head($\vec{g}$) $= d\partial^\beta \vec{e}_\ell$. In the above situation, we put topIndex($\vec{f}$) $= k$ and lpp($\vec{f}$) $= \partial^\alpha$. We define reducibility, s-polynomial, reduction and Gröbner basis by using Definition 2.1, 2.2, 2.4 and 2.5. There exists a Gröbner basis for any submodule of R^m.

We state our improvement of Algorithm 3.1. The improved algorithm is based on solving systems of linear equations (see [Buch2] method 6.11) and is a modification of Algorithm 4.3 of [Tak2]. We refer to [Tak2] for the notations $k(I)$ and e_Q. It is not proved that Algorithm 4.1 stops. Therefore if Algorithm 4.1 fails, we must call Algorithm 3.1 that always stops.

ALGORITHM 4.1 (Finding differential equations for an integral of a module)
INPUT: A left ideal $\mathfrak{A}$ of A_n such that $A_n/\mathfrak{A}$ is holonomic.
OUTPUT: G, generators of a zero dimensional ideal of $K(x_1, \cdots, x_{n-1})\langle\partial_1, \cdots, \partial_{n-1}\rangle$ such that

$$G \subset \partial_n A_n + \mathfrak{A}.$$

(1) $G = \{g_1, \cdots, g_p\}$:= a Gröbner basis of $\mathfrak{A}$ constructed in the ring

$$K(x_1, \cdots, x_{n-1})\langle x_n, \partial_1, \cdots, \partial_n\rangle$$

by the total degree order in the varibles $x_n, \partial_1, \cdots, \partial_n$.
(2) $m := \max_k\{$degree of g_k with respect to $x_n\} + 1$
(3) Do
(4) $\mathcal{H}_m := (3.2)$
(5) $\mathcal{G}$:= reduced Gröbner basis of $R\mathcal{H}_m$ by the order (4.1)
(6) for $k := 1$ to $n-1$ do
(7) $d_k := \#\left(\mathbb{N}_0^{n-1} \setminus \bigcup_{\vec{f}\in\mathcal{G},\mathrm{topIndex}(\vec{f})=k,\mathrm{lpp}(\vec{f})=\partial^\alpha}(\alpha + \mathbb{N}_0^{n-1})\right)$
(8) endfor
(9) $d := \sum_{k=1}^{n-1} d_k$
(10) if $d < \infty$ then
(11) Select a monoideal $I \subseteq \mathbb{N}_0^{n-1}$ such that $k(I) \leq d$
(12) $Q := \mathbb{N}_0^{n-1} \setminus I$
(13) for all $\gamma \in G(I)$ do
(14) Reduce ∂^k, $k \in Q \cup \{\gamma\}$ by the Gröbner basis $\mathcal{G}$ and obtain a similar equation of (4.4) of [Tak2] and solve it.
(15) if $d_\gamma = 0$ then goto (11)
(16) else obtain $e_{Q\cup\{\gamma\}}\vec{e}_0$ of (4.5) or (4.6) of [Tak2]
(17) endfor
(18) endif
(19) $m := m + 1$
(20) while $(d = \infty)$
(21) $G := \{e_{Q\cup\{\gamma\}} | \gamma \in G(I)\}$

Acknowledgement. The author thanks to Prof. Noumi for his encouragement of considering the problem of the paper.

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Appendix: An implementation

(*March 2, 1990*)

The algorithms of the paper "An algorithm of constructing the integral of a module" is implemented by the language C. The implementation is the first software of the NMA(ma)thematical libraries for C, C++ or other object oriented languages. The NMA project aims at a compiler oriented computer algebra system that has high performance (NMA's not MAthematica or REDUCE).

The program "lexgrob" computes the Gröbner basis of an input by the lexico-total degree order (3.3). The program "modulegrob -rank m" computes the Gröbner basis in R^m by the order (2.4). From an input, "modulegrob" obtains generators $\mathcal{H}_m$ of (3.2) and computes the Gröbner basis of $\mathcal{H}_m$.

The data structure of a polynomial that is used in the programs is described by the figure 5.1.

We show examples of computations.

EXAMPLE 5.1 Put

$$\ell_1 = \partial_x + t^2 \text{ and } \ell_2 = \partial_x + 2tx. \tag{5.2}$$

We have $\ell_1 f = \ell_2 f = 0$ where

$$f = e^{-xt^2}.$$

Let the input of "lexgrob" be (5.2). The figure 5.5 is a program readable form of (5.2) where $x_4 = t, \partial_4 = \partial_t, x_3 = x$ and $\partial_3 = \partial_x$. The order is

$$\{x_4\} \succ \{\partial_4, x_3, \partial_3, x_2, \partial_2, x_1, \partial_1, x_0, \partial_0\}.$$

The output of "lexgrob" is the figure 5.6. Let the input of "modulegrob -rank 3" is the figure 5.6. The output is the figure 5.7.

EXAMPLE 5.2 Put

$$\ell_1 = (1 + xt + t^p)\partial_x - \lambda t \text{ and } \ell_2 = (1 + xt + t^p)\partial_t - \lambda(x + pt^{p-1}). \tag{5.3}$$

We have $\ell_1 f = \ell_2 f = 0$ where

$$f = (1 + xt + t^p)^\lambda.$$

Put $p = 3$. Let the input of "lexgrob" be (5.3). The figure 5.8 is a program readable form of (5.3) where $x_4 = t, \partial_4 = \partial_t, x_3 = x, \partial_3 = \partial_x$ and $x_2 = \lambda$. We cannot obtain an output by four minutes. However, we can obtain an annihilator of the integral of f by "modulegrob -rank 7" without preprocessing the input (5.3) by "lexgrob". The output of "modulegrob -rank 7" is the figure 5.9. It is remarkable that "modulegrob" runs faster than "lexgrob | modulegrob" where | denotes a pipe. We remark that the output of "modulegrob -rank 5" does not contains an annihilator of the integral of f. There exists the variable x_4 in all expressions of the output.

EXAMPLE 5.3 (Gauss hypergeometric differential equation) Put

(5.4)

$$\ell_1 = (1-xy)y(1-y)\partial_y - \alpha xy(1-y) - (\beta-1)(1-xy)(1-y) + (\gamma-\beta-1)(1-xy)y$$
$$\ell_2 = (1-xy)\partial_x - \alpha y$$

We have $\ell_1 f = \ell_2 f = 0$ where

$$f = (1-xy)^{-\alpha} y^{\beta-1}(1-y)^{\gamma-\beta-1}.$$

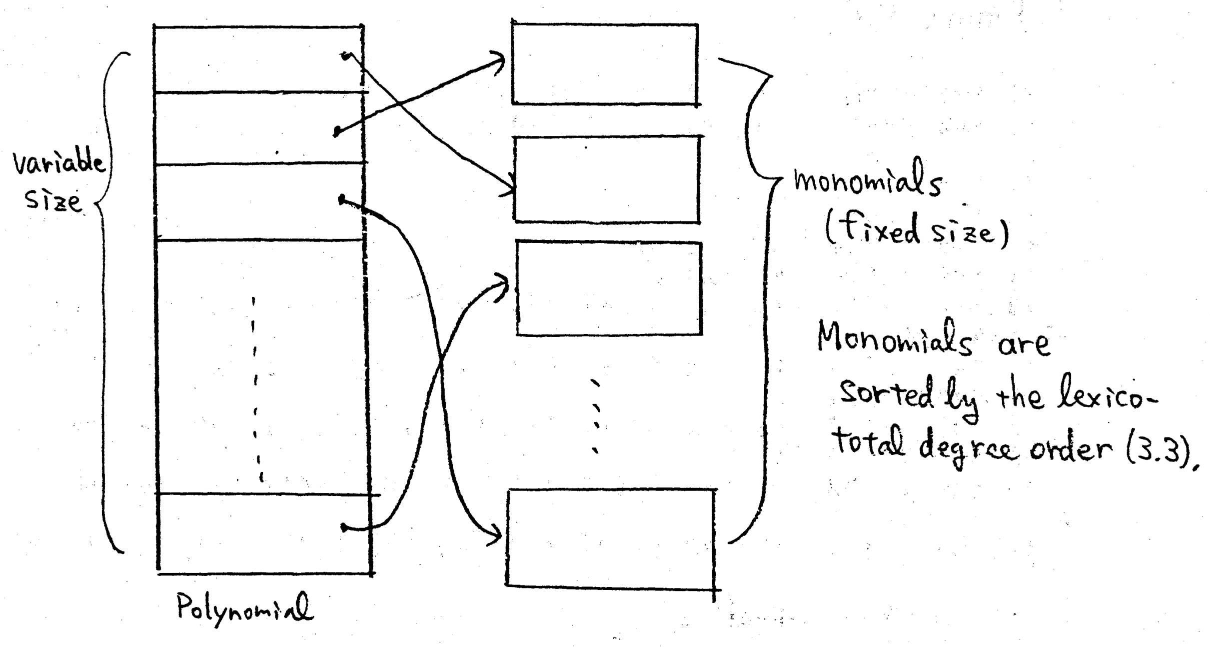
Let the input of "lexgrob" be (5.4). The figure 5.10 is a program readable form of (5.4) where $x_4 = y, \partial_4 = \partial_y, x_3 = x, \partial_3 = \partial_x, x_2 = \gamma, x_1 = \beta$ and $x_0 = \alpha$. The output is the figure 5.11. We can obtain Gauss hypergeometric equation using "modulegrob -rank 5" where the input is the figure 5.11. The output of "modulegrob" is the figure 5.12.

We can also obtain an annihilator of the integral of f that is hypergeometric operator by "modulegrob -rank 5" without preprocessing the input (5.4) by "lexgrob". The output of "modulegrob -rank 5" is the figure 5.13. It is remarkable that "modulegrob" runs faster than "lexgrob | modulegrob".

Timing data

Problem and algorithm	Input	Output	Time on SUN3 260C
Example 5.1, lexgrob|modulegrob -rank 3	Fig 5.5	Fig 5.7	0.7s
Example 5.2, lexgrob|modulegrob -rank 5	Fig 5.8		more than 210s
Example 5.3, lexgrob|modulegrob -rank 5	Fig 5.10	Fig 5.12	182.4s
Example 5.2, modulegrob -rank 5	Fig 5.8		0.8s
Example 5.2, modulegrob -rank 7	Fig 5.8	Fig 5.9	4.9s
Example 5.3, modulegrob -rank 5	Fig 5.10	Fig 5.13	11.4s

Figure 5.1 ↓



```
d4+2*x3*x4 ;
d3+x4^2;
0;
```

← $\partial_4 + 2x_3x_4$

← $\partial_3 + x_4^2$

↑Figure 5.5

```
  +2 *x3 *x4  +1 *D4  ;
  +1 *x4^2  +1 *D3  ;
  +1 *x4 *D4  -2 *x3 *D3  ;
  +4 *x3^2 *D3  +1 *D4^2  +2 *x3  ;
0;
```

← cf. Example 1.1, G

↑Figure 5.6

```
  +2 *x3 *x4  +1 *D4  ;
  +1 *x4^2  +1 *D3  ;
  -2 *x3 *D3  -1  ;
  +1 *D4  ;
  +1 *x4 *D4  +1  ;
0;
```

← cf. Example 3.1

↑Figure 5.7

```
(1+x3*x4+x4^3)*d3-x2*x4;
(1+x3*x4+x4^3)*d4-x2*(x3+3*x4^2);
0;
```

↑Figure 5.8

```
  +1 *x4^3 *D3  +1 *x3 *x4 *D3  -1 *x2 *x4  +1 *D3  ;
  -3 *x2 *x4^2  -3 *x4^2  +1 *x3 *x4 *D4  -1 *x2 *x3  +1 *D4
  ;
  +1 *D4  ;
  +1 *x4 *D4  +1  ;
  +1 *x4^2 *D4  +2 *x4  ;
  +1 *x4^3 *D4  +3 *x4^2  ;
  +3 *x2 *x4^3  +4 *x4^3  -1 *x3 *x4^2 *D4  +1 *x2 *x3 *x4
-1 *x4 *D4  ;
  +3 *x2 *x4^4  +5 *x4^4  -1 *x3 *x4^3 *D4  +1 *x2 *x3 *x4^2
  -1 *x4^2 *D4  ;
  +3 *x2 *x4^5  +6 *x4^5  -1 *x3 *x4^4 *D4  +1 *x2 *x3 *x4^3
  -1 *x4^3 *D4  ;
  +1 *x4^4 *D3  +1 *x3 *x4^2 *D3  -1 *x2 *x4^2  +1 *x4 *D3
;
  +1 *x4^5 *D3  +1 *x3 *x4^3 *D3  -1 *x2 *x4^3  +1 *x4^2 *D3
  ;
```

to be continued.

```
  +1 *x4^6 *D3  +1 *x3 *x4^4 *D3  -1 *x2 *x4^4  +1 *x4^3 *D3
  ;
  +1 *x4^4 *D4  -3 *x2 *x4^3  +1 *x3 *x4^2 *D4  -1 *x2 *x3 *
x4  +1 *x4 *D4  ;
  +1 *x4^5 *D4  -3 *x2 *x4^4  +1 *x3 *x4^3 *D4  -1 *x2 *x3 *
x4^2  +1 *x4^2 *D4  ;
  +1 *x4^6 *D4  -3 *x2 *x4^5  +1 *x3 *x4^4 *D4  -1 *x2 *x3 *
x4^3  +1 *x4^3 *D4  ;
  -27 *x2^2 *x4  -45 *x2 *x4  -18 *x4  +4 *x2 *x3^3 *D3  +4
*x3^3 *D3  +2 *x3^2 *D3 *D4  -6 *x2^2 *x3^2  -8 *x2 *x3^2  +
6 *x2 *x3 *D4  -2 *x3^2  +6 *x3 *D4  +27 *x2 *D3  +27 *D3  ;

  +9 *x2 *x4 *D3  +9 *x4 *D3  -2 *x2 *x3^2 *D3  -2 *x3^2 *D3
  -1 *x3 *D3 *D4  +3 *x2^2 *x3  +4 *x2 *x3  -3 *x2 *D4  +1 *
x3  -3 *D4  ;
  +4 *x2 *x3^3 *D3^2  +4 *x3^3 *D3^2  +2 *x3^2 *D3^2 *D4  -1
2 *x2^2 *x3^2 *D3  -6 *x2 *x3^2 *D3  +3 *x2 *x3 *D3 *D4  +9
*x2^3 *x3  +6 *x3^2 *D3  +8 *x3 *D3 *D4  +6 *x2^2 *x3  +27 *
x2 *D3^2  -9 *x2^2 *D4  -5 *x2 *x3  +27 *D3^2  -9 *x2 *D4  -
2 *x3  ;
0;
```

↖ Figure 5.9

This is an annihilator of $\int f dt$.

```
-(1-x3*x4)*x4*(1-x4)*d4+x0*x3*x4*(1-x4)+(x1-1)*(1-x3*x4)*(1-
x4)-(x2-x1-1)*(1-x3*x4)*x4 ;
-(1-x3*x4)*d3+x0*x4 ;
0 ;
```

↖ Figure 5.10

```
  -1 *x3 *x4^3 *D4  +1 *x2 *x3 *x4^2  -1 *x0 *x3 *x4^2  +1 *
x3 *x4^2 *D4  -2 *x3 *x4^2  +1 *x4^2 *D4  -1 *x1 *x3 *x4  +1
 *x0 *x3 *x4  +1 *x3 *x4  -1 *x2 *x4  -1 *x4 *D4  +2 *x4  +1
 *x1  -1  ;
  +1 *x3 *x4 *D3  +1 *x0 *x4  -1 *D3  ;
  -1 *x0 *x4^2 *D4  +1 *x4^2 *D4  +1 *x1 *x3^2 *x4 *D3  -1 *
x0 *x3^2 *x4 *D3  +1 *x0 *x3 *x4 *D3  +1 *x0 *x1 *x3 *x4  -1
 *x0^2 *x3 *x4  -1 *x3 *x4 *D3  +1 *x0 *x2 *x4  +1 *x0 *x4 *
D4  -1 *x2 *x4  -2 *x0 *x4  -1 *x4 *D4  +2 *x4  -1 *x1 *x3 *
D3  +1 *x3 *D3  -1 *x0 *x1  +1 *x1  +1 *x0  -1  ;
  -1 *x0 *x4 *D3 *D4  +1 *x3 *x4 *D3  +1 *x4 *D3 *D4  +1 *x0
 *x4  -1 *x0 *x3^2 *D3^2  +1 *x3^2 *D3^2  +1 *x0 *x3 *D3^2
-1 *x0 *x1 *x3 *D3  -1 *x0^2 *x3 *D3  -1 *x3 *D3^2  +1 *x1 *
x3 *D3  +1 *x0 *x2 *D3  +1 *x0 *D3 *D4  -1 *x0^2 *x1  +1 *x3
 *D3  -1 *x2 *D3  -1 *x0 *D3  -1 *D3 *D4  +1 *x0 *x1  ;
```

to be continued.

```
  -1 *x0^2 *x4 *D4  +1 *x0 *x4 *D4  +1 *x0 *x3^3 *D3^2  -1 *
x3^3 *D3^2  -1 *x0 *x3^2 *D3^2  +1 *x0 *x1 *x3^2 *D3  +1 *x0
^2 *x3^2 *D3  +1 *x3^2 *D3^2  -1 *x1 *x3^2 *D3  -1 *x0 *x2 *
x3 *D3  -1 *x0 *x3 *D3 *D4  +1 *x0^2 *x1 *x3  -1 *x3^2 *D3
+1 *x2 *x3 *D3  +1 *x3 *D3 *D4  -1 *x0 *x1 *x3  +1 *x0 *D3 *
D4  -1 *D3 *D4  -1 *x0^2  +1 *x0  ;
  +1 *x0 *x3^3 *D3^3  -1 *x3^3 *D3^3  -1 *x0 *x3^2 *D3^3  +1
 *x0 *x1 *x3^2 *D3^2  +2 *x0^2 *x3^2 *D3^2  +1 *x3^2 *D3^3
-1 *x1 *x3^2 *D3^2  +2 *x0 *x3^2 *D3^2  -1 *x0 *x2 *x3 *D3^2
  -1 *x0^2 *x3 *D3^2  -1 *x0 *x3 *D3^2 *D4  +2 *x0^2 *x1 *x3
 *D3  +1 *x0^3 *x3 *D3  -4 *x3^2 *D3^2  +1 *x2 *x3 *D3^2  -1
 *x0 *x3 *D3^2  +1 *x3 *D3^2 *D4  +2 *x0^2 *x3 *D3  +1 *x0 *
D3^2 *D4  -1 *x0^2 *x2 *D3  -1 *x0^2 *D3 *D4  +1 *x0^3 *x1
+2 *x3 *D3^2  -2 *x1 *x3 *D3  -1 *x0 *x3 *D3  -1 *D3^2 *D4
-2 *x3 *D3  +1 *x2 *D3  +1 *D3 *
D4  -1 *x0 *x1  ;
0;
```

↑ Figure 5.11

This is an annihilator of $\int f dy$, but ~~it~~ is a third order operator.

```
  +1 *x2 *x3 *x4^2  -1 *x0 *x3 *x4^2  +1 *x3 *x4^2 *D4  +1 *
x3 *x4^2  +1 *x4^2 *D4  -1 *x1 *x3 *x4  +1 *x0 *x3 *x4  +1 *
x3 *x4  -1 *x2 *x4  -1 *x4 *D4  +2 *x4  +1 *x1  -1  ;
  +1 *x3 *x4 *D3  +1 *x0 *x4  -1 *D3  ;
  +1 *x0 *x2 *x4  -1 *x0^2 *x4  +1 *x0 *x4 *D4  -1 *x2 *x4
+1 *x0 *x4  -1 *x4 *D4  -1 *x0 *x3 *D3  +1 *x3 *D3  +1 *x0 *
D3  -1 *x0 *x1  -1 *D3  +1 *x1  +1 *x0  -1  ;
  -1 *x0 *x3^2 *D3^2  +1 *x3^2 *D3^2  +1 *x0 *x3 *D3^2  -1 *
x0 *x1 *x3 *D3  -1 *x0^2 *x3 *D3  -1 *x3 *D3^2  +1 *x1 *x3 *
D3  +1 *x0 *x2 *D3  +1 *x0 *D3 *D4  -1 *x0^2 *x1  +1 *x3 *D3
  -1 *x2 *D3  -1 *D3 *D4  +1 *x0 *x1  ;
  +1 *D4  ;
  +1 *x4 *D4  +1  ;
  +1 *x4^2 *D4  +2 *x4  ;
  +1 *x4^3 *D4  +3 *x4^2  ;
  +1 *x4^4 *D4  +4 *x4^3  ;
  +1 *x2 *x3 *x4^3  -1 *x0 *x3 *x4^3  +1 *x3 *x4^3 *D4  +2 *
x3 *x4^3  +1 *x4^3 *D4  -1 *x1 *x3 *x4^2  +1 *x0 *x3 *x4^2
+1 *x3 *x4^2  -1 *x2 *x4^2  -1 *x4^2 *D4  +2 *x4^2  +1 *x1 *
x4  -1 *x4  ;
  +1 *x3 *x4^2 *D3  +1 *x0 *x4^2  -1 *x4 *D3  ;
  +1 *x3 *x4^3 *D3  +1 *x0 *x4^3  -1 *x4^2 *D3  ;
  +1 *x3 *x4^4 *D3  +1 *x0 *x4^4  -1 *x4^3 *D3  ;
  +1 *x0 *x2 *x4^2  -1 *x0^2 *x4^2  +1 *x0 *x4^2 *D4  -1 *x2
 *x4^2  +2 *x0 *x4^2  -1 *x4^2 *D4  -1 *x4^2  -1 *x0 *x3 *x4
 *D3  +1 *x3 *x4 *D3  +1 *x0 *x4 *D3  -1 *x0 *x1 *x4  -1 *x4
 *D3  +1 *x1 *x4  +1 *x0 *x4  -1 *x4  ;
  +1 *x0 *x2 *x4^3  -1 *x0^2 *x4^3  +1 *x0 *x4^3 *D4  -1 *x2
 *x4^3  +3 *x0 *x4^3  -1 *x4^3 *D4  -2 *x4^3  -1 *x0 *x3 *x4
^2 *D3  +1 *x3 *x4^2 *D3  +1 *x0 *x4^2 *D3  -1 *x0 *x1 *x4^2
  -1 *x4^2 *D3  +1 *x1 *x4^2  +1 *x0 *x4^2  -1 *x4^2  ;
0;
```

This is hypergeometric differential operator(*) by D4 → 0.

↑ Figure 5.12

```
  +1 *x2 *x3 *x4^2  -1 *x0 *x3 *x4^2  +1 *x3 *x4^2 *D4  +1 *
x3 *x4^2  +1 *x4^2 *D4  -1 *x1 *x3 *x4  +1 *x0 *x3 *x4  +1 *
x3 *x4  -1 *x2 *x4  -1 *x4 *D4  +2 *x4  +1 *x1  -1  ;
  +1 *x3 *x4 *D3  +1 *x0 *x4  -1 *D3  ;
  +1 *D4  ;
  +1 *x4 *D4  +1  ;
  +1 *x4^2 *D4  +2 *x4  ;
  +1 *x4^3 *D4  +3 *x4^2  ;
  +1 *x4^4 *D4  +4 *x4^3  ;
  +1 *x2 *x3 *x4^3  -1 *x0 *x3 *x4^3  +1 *x3 *x4^3 *D4  +2 *
x3 *x4^3  +1 *x4^3 *D4  -1 *x1 *x3 *x4^2  +1 *x0 *x3 *x4^2
+1 *x3 *x4^2  -1 *x2 *x4^2  -1 *x4^2 *D4  +2 *x4^2  +1 *x1 *
x4  -1 *x4  ;
  +1 *x3 *x4^2 *D3  +1 *x0 *x4^2  -1 *x4 *D3  ;
  +1 *x3 *x4^3 *D3  +1 *x0 *x4^3  -1 *x4^2 *D3  ;
  +1 *x3 *x4^4 *D3  +1 *x0 *x4^4  -1 *x4^3 *D3  ;
  -1 *x0 *x2 *x4^2  +1 *x0^2 *x4^2  -1 *x0 *x4^2 *D4  +1 *x2
 *x4^2  -2 *x0 *x4^2  +1 *x4^2 *D4  +1 *x4^2  -1 *x1 *x3 *x4
 *D3  +1 *x0 *x3 *x4 *D3  -1 *x3 *x4 *D3  -1 *x0 *x4 *D3  +1
 *x4 *D3  -1 *x1 *x4  -1 *x0 *x4  +1 *x4  +1 *x1 *D3  ;
  -1 *x0 *x2 *x4^3  +1 *x0^2 *x4^3  -1 *x0 *x4^3 *D4  +1 *x2
 *x4^3  -3 *x0 *x4^3  +1 *x4^3 *D4  +2 *x4^3  -1 *x1 *x3 *x4
^2 *D3  +1 *x0 *x3 *x4^2 *D3  -2 *x3 *x4^2 *D3  -1 *x0 *x4^2
 *D3  +1 *x4^2 *D3  -1 *x1 *x4^2  -2 *x0 *x4^2  +1 *x4^2  +1
 *x1 *x4 *D3  +1 *x4 *D3  ;
  +1 *x0 *x2 *x4  -1 *x0^2 *x4  +1 *x0 *x4 *D4  -1 *x2 *x4
+1 *x0 *x4  -1 *x4 *D4  -1 *x0 *x3 *D3  +1 *x3 *D3  +1 *x0 *
D3  -1 *x0 *x1  -1 *D3  +1 *x1  +1 *x0  -1  ;
  +1 *x0 *x3^2 *D3^2  -1 *x3^2 *D3^2  -1 *x0 *x3 *D3^2  +1 *
x0 *x1 *x3 *D3  +1 *x0^2 *x3 *D3  +1 *x3 *D3^2  -1 *x1 *x3 *
D3  -1 *x0 *x2 *D3  -1 *x0 *D3 *D4  +1 *x0^2 *x1  -1 *x3 *D3
  +1 *x2 *D3  +1 *D3 *D4  -1 *x0 *x1  ;
0;
```

↑ Figure 5.13

This is hypergeometric differential operator —(*) by $D_4 \to 0$.

(*) is

$$(1-x_0)\left[x_3(1-x_3)\partial_3^2 + \{x_2 - (1+x_0+x_1)x_3\}\partial_3 - x_0x_1\right]$$