An algorithm of constructing the integral of a module
— an infinite dimensional analog of Gröbner basis

NOBUKI TAKAYAMA

Department of Mathematics, Kobe University
Rokko, Kobe, 657, Japan

(Febraury 8, 1990)

Abstract.
Let $K$ be a field of characteristic zero. The Weyl algebra:

$$K\{x_{1},\cdots,x_{n},\partial_{1},\cdots,\partial_{n}\}$$

is denoted by $A_{n}$. We have:

$$[x_{i},\partial_{j}] = x_{i}\partial_{j} - \partial_{j}x_{i} = \begin{cases} -1, & i = j, \\ 0, & i \neq j, \end{cases}$$

in the Weyl algebra. Let $\mathfrak{a}$ be a left ideal of $A_{n}$. We put $M = A_{n}/\mathfrak{a}$. $M$ is a left $A_{n}$ module. The purpose of this paper is an explicit construction of the left $A_{n-1}$ module:

$$\int Mdx_{n} := M/\partial_{n}M$$

by introducing an analog of Gröbner basis of a submodule of a kind of infinite dimensional free module. We call $M/\partial_{n}M$ the integral of the module $M$. The non-commutativity of $A_{n}$ prevents us from using the usual Buchberger algorithm to construct $M/\partial_{n}M$. (If $A_{n}$ is commutative, then $M/\partial_{n}M \simeq A_{n}/(\partial_{n},\mathfrak{a})$. There is no problem.) We must consider a sum of left and right ideal of $A_{n}$. We overcome this difficulty by using an infinite dimensional analog of Gröbner basis.

The algorithm of constructing the integral of a module is not only important to mathematicians, but also has many impacts on the classical fields of computer algebra. It plays central roles in mathematical formula verification [Zeil], [Tak2], computation of a definite integral [AZ], [Tak2] and an asymptotic expansion of a definite integral with respect to parameters. However, a complete algorithm of obtaining $M/\partial_{n}M$ had not been known. We give a complete algorithm in this paper. The algorithm is an answer to the research problem of the paper [AZ].

We refer to [Buch1], [Buch2], [MM], [FSK] for the Gröbner basis of a polynomial ideal and free module, to [Gal], [Cas], [Tak1], [Nou], [UT] for the Gröbner basis of the ideal of Weyl algebra, to [Ber], [Bjo] for holonomic system and Weyl algebra. We remark that [Ber] also considered infinite set of reduction systems.

Submitted to ISSAC'90.
§1. A simple example

We explain an idea by using a simple example.

Example 1.1 We compute a differential operator that annihilates the function:

\[ I(x) = \int_{-\infty}^{\infty} f(x, t) dt, \quad f(x, t) = e^{-xt^2}. \]

We have \( \ell_1 f = \ell_2 f = 0 \) where

\[ \ell_1 = \partial_x + t^2, \quad \ell_2 = \partial_t + 2xt. \]

We put \( x_2 = t \) and \( x_1 = x \). Let \( \mathfrak{A} \) be the left ideal of \( A_2 \) generated by \( \ell_1 \) and \( \ell_2 \). The Gröbner basis of \( \mathfrak{A} \) by the lexicographic order \( t \succ \partial_t \succ x \succ \partial_x \) is

\[ G = \{ t^2 + \partial_x, 2xt + \partial_t, -(t\partial_t - 2x\partial_x), \partial_t^2 + 4x^2\partial_x + 2x \}. \]

The Hilbert function of \( A_2/\mathfrak{A} \) is \( k^2 + 3k + 1 \). We put \( R = \mathbb{C}(x, \partial_x, \partial_t) \) and define a map

\[ \psi : \mathbb{C}(t, x, \partial_t, \partial_x) \ni \sum_{k=0}^{m-1} t^k f_k \mapsto (f_0, \cdots, f_{m-1}) \in R^m \]

where \( f_k \in R \). We have

\[ \psi(G) = \{(\partial_x, 0, 1, \cdots), (\partial_t, 0, 1, \cdots), (-2x\partial_x, \partial_t, 0, \cdots), (\partial_t, 4x^2\partial_x + 2x, 0, \cdots)\} \]

\[ \psi(tG) = \{(0, \partial_x, 0, 1, \cdots), (0, \partial_t, 2x, 0, \cdots), \cdots\} \]

\[ \psi(t^{m-3}G) = \{(0, \cdots, 0, \partial_x, 0, 1), \cdots\}, \]

and we use

\[ \psi^{-1}(1+2x\partial_x) = (1, \partial_t, 0, \cdots) \]

\[ \psi^{-1}(\partial_x) = (0, \cdots, 0, \partial_x, 0, 1), \cdots \]

(1.1) is obtained from

\[ \partial_t t^k = t^k \partial_t + kt^{k-1}. \]

\( R^m \) has a left \( R \) module structure defined in the beginning of the section two.

Let \( M_m \) be a left \( R \) submodule of \( R^m \) generated by (1.1) and (1.2). We remark that

\[ \psi^{-1}(M_m) \subseteq \partial_t A_2 + \mathfrak{A}. \]

We apply the algorithm of Gröbner basis of submodule ([MM], [FSK]) to \( M_m \). Then we obtain, for example,

\[ \text{sp}(1, \partial_t, 0, \cdots, 0, \partial_x, 0, \cdots) = (1 + 2x\partial_x, 0, \cdots, \cdots). \]

The above equation is equivalent to

\[ \partial_t t - (t\partial_t - 2x\partial_x) = 1 + 2x\partial_x, \quad \partial_t t \in \partial_t A_2, t\partial_t - 2x\partial_x \in \mathfrak{A}. \]

Therefore we have

\[ \partial_t tf - (t\partial_t - 2x\partial_x)f = \partial_t(tf) = (1 + 2x\partial_x)f. \]

We conclude that

\[ (1 + 2x\partial_x)I(x) = \int_{-\infty}^{\infty} \partial_t(tf) dt = 0 \quad (x > 0). \]
§2. Gröbner basis for submodule of \( R_{\infty} = \lim_{\rightarrow} R^{m} \)

Let us consider the general situation. We put

\[
R = K(x_{1}, \cdots, x_{n-1}, \partial_{1}, \cdots, \partial_{n}) = A_{n-1}[\partial_{n}].
\]

We define a left \( R \) module structure to \( R^{m} \) in the following way. For

\[
R^{m} \ni \vec{f} = (f(0), \cdots, f(m-1)), \ a \in A_{n-1},
\]

we put

\[
(2.1) \quad a\vec{f} = (af(0), \cdots, af(m-1))
\]

and for \( a = \partial_{n}, \)

\[
(2.2) \quad a\vec{f} = (af(0) + f(1), \cdots, af(k) + (k+1)f(k+1), \cdots, af(m-1)).
\]

The Weyl algebra \( A_{n} \) has a left \( R \) module structure in the standard way. The map

\[
\varphi : R^{m} \ni \vec{f} \longmapsto \sum_{k=0}^{m-1} x_{n}^{k}f(k) \in A_{n}
\]

is homomorphism of left \( R \) module.

We can define the notions of addmissible order, reducible, \( S \)-polynomial( \( sp \) ) and Gröbner basis of the ring \( R \) in a similar way to the case of the polynomial ring. Let us explain some of them to avoid misunderstandings. We define an order \( \prec_{1} \) between monomials of \( R \) by

\[
(2.3) \quad x_{1}^{\alpha_{1}} \cdots x_{n-1}^{\alpha_{n-1}} \partial_{1}^{\beta_{1}} \cdots \partial_{n}^{\beta_{n}} \prec_{1} x_{1}^{\gamma_{1}} \cdots x_{n-1}^{\gamma_{n-1}} \partial_{1}^{\delta_{1}} \cdots \partial_{n}^{\delta_{n}}
\]

\[
\Leftrightarrow \quad (\alpha_{1}, \cdots, \alpha_{n-1}, \beta_{1}, \cdots, \beta_{n}) \prec_{2} (\gamma_{1}, \cdots, \gamma_{n-1}, \delta_{1}, \cdots, \delta_{n})
\]

where \( \prec_{2} \) is the total degree order in \( \mathbb{N}^{2n-1} \). We use the order in the sequel. Let \( r \) and \( s \) be elements of \( R \). We put

\[
\text{head}(r) = \text{head term of } r \text{ by the order } (2.3) = cx^\alpha \partial^\beta, \ c \in K
\]

and \( \text{head}(s) = dx^\gamma \partial^\delta, \ d \in K \). We define

\[
lcm(\alpha, \gamma) = (\max\{\alpha_{1}, \gamma_{1}\}, \cdots, \max\{\alpha_{n-1}, \gamma_{n-1}\}),
\]

\[
lcm(\beta, \delta) = (\max\{\beta_{1}, \delta_{1}\}, \cdots, \max\{\beta_{n}, \delta_{n}\}).
\]

If \( \text{lcm}(\alpha, \gamma) = \alpha \) and \( \text{lcm}(\beta, \delta) = \beta \), \( r \) is reducible by \( s \). Put \( \xi = \text{lcm}(\alpha, \gamma) \) and \( \eta = \text{lcm}(\gamma, \delta) \). We define

\[
sp(r, s) = x^{\xi - \alpha} \partial^{\eta - \beta} r - \frac{c}{d} x^{\xi - \gamma} \partial^{\eta - \delta} s.
\]

Let \( r \) be reducible by \( s \) and \( t = sp(r, s) \), then the situation is denoted by \( "r \mapsto t \) by \( s" \). Let \( \rightarrow \) be a transitive closure of \( \mapsto \). A finite subset \( G \) of \( R \) is called Gröbner basis of an ideal \( \mathfrak{a} \) if \( \forall r_{i}, r_{j} \in G, \ sp(r_{i}, r_{j}) \rightarrow 0 \) by \( G \) and \( \mathfrak{a} = RG \). It is well known that every left ideal of \( R \) has a Gröbner basis [Gal], [Cas], [Tak1], [Nou], [UT].

Consider \( R^{m} \). [MM] and [FSK] extended the notion of Gröbner basis to free modules. We can apply their extension to \( R^{m} \). Let us review their extension (See [Tak1] for proofs in the case of a free module over a non-commutative ring). We define

\[
\text{topIndex}(\vec{f}) = k, \ \vec{f} \in R^{m},
\]

if \( f(i) = 0 (k < i \leq m-1) \) and \( f(k) \neq 0 \). Let \( \vec{f} \) and \( \vec{g} \) be elements of \( R^{m} \).
DEFINITION 2.1 \( \vec{f} \) is reducible by \( \vec{g} \) iff \( k = \text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g}) \) and \( f(k) \) is reducible by \( g(k) \) in \( R \).

DEFINITION 2.2
\[
\text{sp}(\vec{f}, \vec{g}) := \begin{cases} 
0, & \text{if } \text{topIndex}(\vec{f}) \neq \text{topIndex}(\vec{g}) \\
c_1\vec{f} - c_2\vec{g}, & \text{if } k = \text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g})
\end{cases}
\]
where \( \text{sp}(f(k), g(k)) = c_1f(k) - c_2g(k) \).

DEFINITION 2.3
\[
\text{topIndex}(\vec{f}) > \text{topIndex}(\vec{g}) \iff \left( \text{topIndex}(\vec{f}) = \text{topIndex}(\vec{g}) = k \text{ and } f(k) \succ g(k) \right)
\]

We use the order (2.4) of \( R^m \) in the sequel. We remark that other order in \( R^m \) can be used in our theory. The use of good orders leads us to a fast termination of Buchberger algorithm.

We put
\[
\mathcal{G} = \{\vec{g}_1, \ldots, \vec{g}_p\}
\]

DEFINITION 2.4 If there exists an element \( \vec{h} \) of \( \mathcal{G} \) such that \( \vec{f} \) is reducible by \( \vec{h} \), then we write the situation as follows.
\[
\vec{f} \rightarrow^* \vec{h} \text{ by } \mathcal{G}
\]
where \( \vec{h} = \text{sp}(\vec{f}, \vec{g}) \).

We remark \( \vec{h} \prec \vec{f} \). Let \( \rightarrow^* \) be transitive closure of \( \rightarrow \). Suppose \( \vec{f} \rightarrow^* \vec{h} \). We remark that \( \vec{h} \) is not uniquely determined by \( \vec{f} \). It depends on the sequence of reductions.

DEFINITION 2.5 \( \mathcal{G} \) is a Gröbner basis of a left \( R \) submodule \( \mathcal{M} \) of \( R^m \) iff
(1) \( \forall i, j \), \( \text{sp}(\vec{g}_i, \vec{g}_j) \rightarrow^* 0 \) by \( \mathcal{G} \).
(2) \( \mathcal{G} \) generates \( \mathcal{M} \) over \( R \).

Any left submodule \( \mathcal{M} \) of \( R^m \) has a Gröbner basis ([MM], [FSK], [Takl]).
\( \vec{e}_i \) denotes \( i \)-th unit vector, i.e.,
\[
\vec{e}_0 = (1, 0, \ldots, 0), \quad \vec{e}_1 = (0, 1, 0, \ldots, 0), \ldots
\]
\( \vec{g}_i \) can be decomposed into a sum of (monomial of \( R \) \times (unit vector)) which is written as
\[
\vec{g}_i = \sum_j c^j_i \vec{e}_k, \quad c^j_i \text{ is a monomial of } R.
\]
\( \mathcal{G} \) is a reduced Gröbner basis of \( \mathcal{M} \) iff \( \mathcal{G} \) is a Gröbner basis of \( \mathcal{M} \),
\[
\forall i, j \quad c^j_i \vec{e}_k \rightarrow^* c^j_i \vec{e}_k \text{ by } \mathcal{G} \setminus \{\vec{g}_i\}
\]
and the head coefficients of \( \vec{g}_i \) is 1.
We define these notions on 
\[ R_{\infty} = \lim_{\rightarrow} R^{m} \simeq K[x_{n}] \otimes_{K} R. \]

Any element \( f \) of \( R_{\infty} \) can be written as 
\[ f = (f(0), f(1), \cdots), \quad \exists k, i > k \Rightarrow f(i) = 0 \text{ and } f(k) \neq 0. \]

The number \( k \) is denoted by \( \text{topIndex}(f) \). Therefore we can consider \( f \) as the element of \( R^{m}, \ m \geq k \). We define the notions of reducibility, s-polynomial and order \( \prec \) identifying the element \( f \) of \( R_{\infty} \) with the element \( (f(0), \cdots, f(m-1)) \) of \( R^{m} \), \( m \geq k \).

Put 
\[ G = \{g_1, g_2, \cdots\}, \quad g_i \in R_{\infty}. \]

We do not assume that \( G \) is finite set. Put 
\[ G(k) = \{g \in G \mid \text{topIndex}(g) \leq k\}. \]

**Assumption 2.1**

\[ \forall k, \ #G(k) < +\infty. \]

We consider the existence of a Gröbner basis under Assumption 2.1 in the sequel.

**Definition 2.6**

\[ f \rightarrow h \text{ by } G \iff \exists i, m, g_i \in G, \text{topIndex}(g_i) \leq m, \text{topIndex}(f) \leq m \text{ and } f \rightarrow h \text{ by } g_i \text{ in } R^{m}. \]

\[ f \rightarrow h \text{ is called reduction of } f. \]

**Proposition 2.1** For any element \( f \in R_{\infty} \), any sequence of reduction of \( f \) by \( G \) terminates in finite steps.

**Proof.** Put \( m = \text{topIndex}(f) \). Note that any sequence of reduction of \( f \) uses the elements of \( G(m) \). Since \( G(m) \) is the finite set, the sequence terminates in finite steps. \( \blacksquare \)

It follows from Proposition 2.1 that we can take a transitive closure of \( \rightarrow \) in finite steps. The transitive closure is denoted by \( \rightarrow^{*} \).

**Definition 2.7** \( G \) is a Gröbner basis of a left \( R \) submodule \( M \) of \( R_{\infty} \) iff

1. \( \forall i, j, \ \text{sp}(\bar{g}_i, \bar{g}_j) \rightarrow^{*} 0 \text{ by } G. \)
2. \( G \) generates \( M \) over \( R \), i.e., \( \forall f \in M, \exists I \subset \mathbb{N}, \exists a_i \in R \) such that \( \#I < \infty \) and

\[ f = \sum_{i \in I} a_i \bar{g}_i. \]

3. (local finiteness)

\[ \forall m, \ #G(m) < +\infty. \]

**Proposition 2.2** If \( G \) is a Gröbner basis of an \( R \) submodule \( M \subset R_{\infty} \), then

\[ \forall \bar{g}_i, \bar{g}_j \in G(m), \ \text{sp}(\bar{g}_i, \bar{g}_j) \rightarrow^{*} 0 \text{ by } G(m). \]
Proof. We have $\text{sp}(\bar{g}_i, \bar{g}_j) \rightarrow^* 0$ by $\mathcal{G}$. Since $\text{topIndex}(\text{sp}(\bar{g}_i, \bar{g}_j)) \leq m$, we have $\text{sp}(\bar{g}_i, \bar{g}_j) \rightarrow^* 0$ by $\mathcal{G}(m)$.

**Theorem 2.1** Let $\mathcal{M}$ be a left $R$ submodule of $R_\infty$ and $\mathcal{G}$ be a Gröbner basis of $\mathcal{M}$. If $\bar{f} \in \mathcal{M}$, then $\bar{f} \rightarrow^* 0$ by $\mathcal{G}$.

Proof. Since $\mathcal{G}$ is a set of generators of $\mathcal{M}$, there exist an index set $I$ and elements $a_i \in R$, $i \in I$ such that $\# I < +\infty$ and $\bar{f} = \sum_{i \in I} a_i \bar{g}_i$. Put $m = \max_{i \in I} \{\text{topIndex}(\bar{g}_i)\}$. We can consider $\bar{f}$ as an element of $R^m$. It follows from Proposition 2.2 that $\mathcal{G}(m)$ is a Gröbner basis of $R\mathcal{G}(m)$ in $R^m$. Since $\bar{f} \in R\mathcal{G}(m)$, we have $\bar{f} \rightarrow^* 0$ by $\mathcal{G}(m)$ for any sequence of reduction.

Let $\mathcal{H}_m$, $m = 0, 1, 2, \cdots$ be subsets of $R_\infty$ that satisfy the conditions:

\[ \cdots \subseteq \mathcal{H}_m \subseteq \mathcal{H}_{m+1} \subseteq \cdots \]

\[ \# \mathcal{H}_m < +\infty, \forall \bar{f} \in \mathcal{H}_m, \text{topIndex}(\bar{f}) \leq m. \]

Suppose that $\mathcal{M}_\infty$ is the left $R$ submodule generated by $\bigcup_{m=0}^\infty \mathcal{H}_m$. We have

$\mathcal{M}_\infty = \bigcup_{m=0}^\infty R\mathcal{H}_m = \lim_{\rightarrow} R\mathcal{H}_m$.

**Theorem 2.2** Let $\mathcal{G}_m$ be the reduced Gröbner basis of $R\mathcal{H}_m$ in $R^m$.

$\mathcal{G}_\infty = \bigcup_{m=0}^\infty \mathcal{G}_m$

is a Gröbner basis of $\mathcal{M}_\infty$.

Proof. We prove the local finiteness condition: $\# \mathcal{G}_\infty(m) < +\infty$. We remark that $\mathcal{G}_\infty(m) \neq \mathcal{G}_m$ in general. Put

$\mathcal{G}_k(m) = \{ \bar{f} \in \mathcal{G}_k | \text{topIndex}(\bar{f}) \leq m \}$.

$\mathcal{G}_k(m)$ is a Gröbner basis of $R\mathcal{G}_k(m)$ in $R^m$. Since $\cdots \subseteq R\mathcal{G}_k(m) \subseteq R\mathcal{G}_{k+1}(m) \subseteq \cdots$ in $R^m$, there exists $k_0$ such that $\forall k \geq k_0$, $R\mathcal{G}_k(m) = R\mathcal{G}_{k_0}(m)$. $\mathcal{G}_k$ is the reduced Gröbner basis, then we have $\forall k \geq k_0$, $\mathcal{G}_k(m) = \mathcal{G}_{k_0}(m)$. Hence $\# \mathcal{G}_\infty(m) < +\infty$.

Other conditions are easily verified.

**Corollary 2.1** For any $m$, we can obtain $\mathcal{G}_\infty(m)$ in finite steps.

**Algorithm 2.1**

INPUT: $\mathcal{H}_m$ : generators of a submodule that satisfy the condition (2.5).
OUTPUT: $\mathcal{G}_m$ : $m$-th approximation of Gröbner basis $\mathcal{G}_\infty$ of the submodule $\mathcal{M}_\infty$.

(1) $\mathcal{G}_m :=$ the reduced Gröbner basis of $R\mathcal{H}_m$ in $R^m$.

**Remark 2.1** If $m$ is large number in Algorithm 2.1, then it follows from Corollary 2.1 that we have $\mathcal{G}_m(k) = \mathcal{G}_\infty(k)$ for small number $k$ where $\mathcal{G}_\infty$ is a Gröbner basis of $\mathcal{M}_\infty$. However, we do not have an algorithm of deciding $\mathcal{G}_m(k) = \mathcal{G}_\infty(k)$ or not.
§3. Computation of the integral of $A_n$ module

Let $\mathfrak{a}$ be a left ideal of $A_n$ and $M$ be $A_n/\mathfrak{a}$. We have

$$M/\partial_n M \simeq A_n/(\partial_n A_n + \mathfrak{a})$$
as $A_{n-1}$ module.

The set $\partial_n A_n + A_n \mathfrak{a} = \partial_n A_n + \mathfrak{a}$ is not left $A_n$ module. Let us note that $R$ is a subalgebra which is commutative to $\partial_n$. Therefore $\partial_n A_n + \mathfrak{a}$ has a left $R$ module structure. We will show that $\partial_n A_n + \mathfrak{a}$ is the left $R$ submodule of $R_\infty$, prove the existence of a Gröbner basis (with the local finiteness property) of the module and present a construction algorithm of the basis.

Let

$$G = \{g_1, \ldots, g_p\}$$
be generators of the left ideal $\mathfrak{a}$ of $A_n$. $g_k$ can be written as

$$g_k = \sum_{j=0}^{s_k} x_n^j g_{kj}, \quad g_{kj} \in R.$$

We put

$$\psi(\partial_n x_n^k) = (0, \cdots, 0, k, \partial_n, 0, \cdots, 0) \in R^m,$$
and

$$\psi(x_n^i g_k) = (0, \cdots, 0, g_{k0}, g_{k1}, \cdots, g_{ks_k}, 0, \cdots, 0) \in R^m.$$

Let $H_m \subset R^m$ be

$$(3.2)\quad \left(\bigcup_{k=0}^{m-1}\{\psi(\partial_n x_n^k)\}\right) \cup \left(\bigcup_{k=1}^{p}\bigcup_{i=0}^{m-s_k-1}\{\psi(x_n^i g_k)\}\right).$$

We have $\cdots \subseteq H_m \subseteq H_{m+1} \subseteq \cdots$ and $\# H_m < +\infty$. $M_\infty = \bigcup_{m=0}^{\infty} R H_m$ is the left $R$ submodule of $R_\infty$.

It follows from Theorem 2.2 that $M_\infty$ has a Gröbner basis $G_\infty$. We can compute approximations of $G_\infty$ by Algorithm 2.1.

**Theorem 3.1.**

$$R_\infty/M_\infty \simeq A_n/(\partial_n A_n + \mathfrak{a}) = \int M dx_n$$
as left $A_{n-1}$ module.

**Proof.** We define a map:

$$\theta : R_\infty \ni \vec{f} = (f(0), f(1), \ldots, f(m), 0, \cdots) \mapsto \quad f(0) + x_n f(1) + \cdots + (x_n)^m f(m) \in A_n$$

where $m = \text{topIndex}(\vec{f})$.

We prove if $\vec{f} \in M_\infty$, then $\theta(\vec{f}) \in \partial_n A_n + \mathfrak{a}$. Since $\vec{f} \in M_\infty$, there exists $a_j, b_j \in R$ such that

$$\vec{f} = \sum_j a_j \psi(\partial_n x_n^k_j) + \sum_j b_j \psi(x_n^j g_k_j)$$

where $\sum_j$ is a finite sum. Then we have

$$\theta(\vec{f}) = \sum_j a_j \partial_n x_n^k_j + \sum_j b_j x_n^j g_k_j$$

$$= \sum_j \partial_n (a_j x_n^k_j) + \sum_j (b_j x_n^j) g_k_j \in \partial_n A_n + \mathfrak{a}.$$
Therefore we can define a map: 
\[ \hat{\theta} : \mathcal{R}_{\infty}/\mathcal{M}_{\infty} \rightarrow A_{n}/(\partial_{n}A_{n} + \mathfrak{k}), \]
by \( \hat{\theta}(\vec{f}) = [\theta(\vec{f})] \).

It is easily verified that \( \hat{\theta} \) is an \( A_{n-1} \) homomorphism and surjective.

We will show that \( \hat{\theta} \) is injective. We assume that \( \theta(\vec{f}) = \theta(\vec{g}) \), \( \vec{f}, \vec{g} \in \mathcal{R}_{\infty}/\mathcal{M}_{\infty}, \ h \in A_{n}, \ g \in \mathfrak{k} \). \( h \) can be written as \( h = \sum_{k} h_{k}x_{n}^{k} \), \( h_{k} \in \mathcal{R} \). Then we have \( \partial_{n}h = \sum_{k} h_{k}(\partial_{n}x_{n}^{k}) \). \( g \) can be written as \( g = \sum_{k} c_{k}g_{k} \), \( c_{k} \) has an expression of the form \( c_{k} = \sum_{j} b_{kj}x_{n}^{j} \), \( b_{kj} \in \mathcal{R} \). Since \( \theta \) is injective, then we have
\[ \vec{f} - \vec{g} = \sum_{k} h_{k}(\partial_{n}x_{n}^{k}) + \sum_{k,j} b_{kj}\psi(x_{n}'g_{k}) \in \mathcal{M}_{\infty}. \]

Therefore \( \hat{\theta} \) is injective. \( \blacksquare \)

**Corollary 3.1** If \( M \) is holonomic, then there exists a number \( m \) such that
\[ \mathcal{R}^{m}/\mathcal{R}\mathcal{G}_{\infty}(m) \cong \int Mdx_{n} \]
as an \( A_{n-1} \) module where \( \mathcal{G}_{\infty} \) is a Gröbner basis of \( \mathcal{M}_{\infty} = \bigcup \mathcal{R}\mathcal{H}_{m} \) of (3.2).

**Proof.** If \( M \) is holonomic, then \( \mathcal{R}_{\infty}/\mathcal{M}_{\infty} \) is finitely generated. Let \( \vec{h}_{1}, \ldots, \vec{h}_{p} \) be generators. Assume \( \vec{h}_{i}, i = 1, \ldots, p \) are irreducible by \( \mathcal{M}_{\infty} \). Put \( m = \max_{i=1}^{p} \{ \text{topIndex}(\vec{h}_{i}) \} \). Let us consider a map \( \rho : \mathcal{R}^{m}/\mathcal{R}\mathcal{G}_{\infty}(m) \rightarrow \mathcal{R}_{\infty}/\mathcal{M}_{\infty} \)
where \( \mathcal{G}_{\infty} \) is a Gröbner basis of \( \mathcal{M}_{\infty} \). If an element \( \vec{f} \) of \( \mathcal{R}^{m} \) is irreducible by \( \mathcal{G}_{\infty}(m) \), then \( \vec{f} \) is irreducible by \( \mathcal{G}_{\infty} \). Therefore the map \( \rho \) is injective. Since \( \vec{h}_{1}, \ldots, \vec{h}_{p} \) are generators, then the map \( \rho \) is surjective. \( \blacksquare \)

The lexic-total degree order is
\[ \{ x_{n} \} \succ \{ x_{1}, \ldots, x_{n-1}, \partial_{1}, \ldots, \partial_{n} \} \tag{3.3} \]
and the lexicographic order is
\[ x_{n} \succ \partial_{n} \succ x_{n-1} \succ \cdots \succ x_{1} \succ \partial_{1}. \tag{3.4} \]

We will show an application of our theory to the zero recognition problem \([\text{Zei}]\) \([\text{Tak}2]\) and the study of definite integral with parameters \([\text{AZ}]\) \([\text{Tak}2]\). Algorithm 3.1 can be used in Algorithm 1.2 of \([\text{Tak}2]\) and is "correct" algorithm in the sense of \([\text{Tak}2]\).

**Algorithm 3.1** (Computation of differential equations for a definite integral with parameters)
**INPUT:** \( G = \{ g_{k} \} \), generators (3.1) of a left ideal \( \mathfrak{k} \) of \( A_{n} \). We assume that \( M = A_{n}/\mathfrak{k} \) is holonomic.
**OUTPUT:** \( \mathcal{G}(0) \), a Gröbner basis in \( R \) such that \( R/\mathcal{R}\mathcal{G}(0) \) is holonomic \( A_{n-1} \) module, i.e., \( \mathcal{G}(0) \) is a very large system of differential equations such that \( \mathcal{G}(0) \subseteq \partial_{n}A_{n} + \mathfrak{k} \).
(1) \( G := \) a Gröbner basis of \( G \) in \( A_{n} \) by the lexicographic order (3.4) or the lexic-total degree order (3.3).
(2) \( m := \max\{ s_{k} + 1 \}; \ \mathcal{G} := \emptyset ; \)
(3) repeat
(4) \( \mathcal{H}_{m} := (3.2); \)
(5) \( \mathcal{G} := \mathcal{G} \cup \{ \text{reduced Gröbner basis of } \mathcal{R}\mathcal{H}_{m} \text{ in } \mathcal{R}^{m} \text{ by the order (2.4) } \}; \)
(6) \( m := m + 1; \)
(7) until ( \( R/\mathcal{R}\mathcal{G}(0) \) is holonomic)

For the computation of the Gröbner basis of the step (1), it is fast to call Algorithm 4.3 of \([\text{Tak}2]\) and construct a Gröbner basis from the output of Algorithm 4.3 by pure Buchberger algorithm by the order (3.4) or (3.3).
**Theorem 3.2** Algorithm 3.1 stops.

**Proof.** It follows from Corollary 2.1 that we can obtain $G_{\infty}(0)$ by finite iterations where $G_{\infty}$ is a Gröbner basis of $M_{\infty}$. Since $R/RG_{\infty}(0)$ is $A_{n-1}$ submodule of $R_{\infty}/M_{\infty} = \int M dx_{n}$, then $R/RG_{\infty}(0)$ is holonomic $A_{n-1}$ module. ■

**Theorem 3.3** Assume a function $f$ of $x_{1}, \cdots, x_{n}$ is rapidly decreasing with respect to $x_{n}$. Let $\mathcal{U}$ be an ideal of $A_{n}$ such that $\mathcal{U}f = 0$. If $A_{n}/\mathcal{U}$ is holonomic, then the integral

$$\int_{-\infty}^{\infty} f dx_{n}$$

is annihilated by differential operators $G(0)$ where $\mathcal{U}$ is the input of Algorithm 3.1 and $G(0)$ is the output. $R/RG(0)$ is holonomic $A_{n-1}$ module.

**Example 3.1** This is continuation of Example 1.1. We consider in $R^{m}$, $m = 3$. Put

$h_{1} = (\partial_{x}, 0, 1)$
$h_{2} = (\partial_{t}, 2x, 0)$
$h_{3} = (-2x\partial_{x}, \partial_{t}, 0)$
$h_{4} = (p_{0}, 0, 0)$
$h_{5} = (\partial_{t}, 0, 0)$
$h_{6} = (1, \partial_{t}, 0)$
$h_{7} = (0, 2, \partial_{t})$

where $p_{0} = \partial_{t}^{2} + 2x + 4x^{2}\partial_{x}$. We set $H_{3} = \{h_{i}| i = 1, \cdots 7\}$. We compute Gröbner basis of $RH_{3}$. We have, for example,

$$\text{sp}(h_{1}, h_{7}) = \partial_{t}h_{1} - h_{7} = (\partial_{t}, 2, \delta_{t}\delta_{x}) - (\partial_{t}, 2, 0) \rightarrow 0$$

by $h_{5}$.

We have the reduced Gröbner basis:

$$\{h_{1}, (0, 2x, 0), (1 + 2x\partial_{x}, 0, 0), h_{5}, h_{6}\}.$$

Therefore we have

$$R^{3}/RH_{3} \simeq A_{1}/(1 + 2x\partial_{x}) + tA_{1}/(x)$$

as $A_{1}$ module. The output of Algorithm 3.1 is $\{1 + 2x\partial_{x}\}$ which is a differential equation for $I(x)$.

§ 4. A fast algorithm of obtaining differential operators for a definite integral

We must compute a Gröbner basis with lexicographic order or lexico total degree order in step (1) of Algorithm 3.1 and we must use the lexicographic order (2.4) in $R^{m}$, but the orders spend much time and very large memory. We will state an efficient algorithm. Put

$$R = K(x_{1}, \cdots, x_{n-1})(\partial_{1}, \cdots, \partial_{n}).$$

The theory of the first part of the section two is valid in this case. Let us define the notions of reducibility, s-polynomial etc. to avoid misunderstandings. We define a left $R$ module structure to $R^{m}$ in the following way. For $f \in R^{m}$ and $a \in K(x_{1}, \cdots, x_{n-1})(\partial_{1}, \cdots, \partial_{n-1})$, we define $af$ the right hand side of (2.1) and for $a = \partial_{x}$, $a^{f}$ the right hand side of (2.2). A monomial of $R$ can be written as $c\partial^{\alpha}$, $c \in K(x_{1}, \cdots, x_{n-1})$. Monomials of $R$ are ordered as

$$\partial^{\alpha} \prec_{3} \partial^{\beta} \iff \alpha \prec \beta$$

by the total degree order in $\mathbb{N}_{0}^{n}$.

Let $r$ and $s$ be elements of $R$. We put

$$\text{head}(r) = \text{head term of } r \text{ by the above order } \prec_{3} = c\partial^{\alpha}$$
and head$(s) = d \partial^\beta$ where $c, d \in K(x_1, \ldots, x_{n-1})$. $r$ is reducible by $s$ iff $\text{lcm}(\alpha, \beta) = \alpha$. Put $\xi = \text{lcm}(\alpha, \beta)$ and assume $c = c_1/c_2, d = d_1/d_2$, $c, d \in K[x_1, \ldots, x_{n-1}]$. We define
\[ \text{sp}(r, s) = \frac{d_1 d_2 c_2 d_2 \xi^\alpha r - c_1 c_2 d_2 \xi^\beta s}{e} \]
where $e = \gcd(d_1 d_2 c_2, c_1 c_2 d_2)$. Let $r$ be reducible by $s$ and $t = \text{sp}(r, s)$. The situation is denoted by "$r \longrightarrow t$ by $s$".
Consider $R^m$. Let $c, d \in K(x_1, \ldots, x_{n-1})$. We define
\[ k + |\alpha| > \ell + |\beta| \]
(4.1) \[ c\partial^\alpha \xi^k \succ d\partial^\beta \xi^\ell \iff \text{or } k + |\alpha| = \ell + |\beta| \text{ and } k > \ell \]
(4.1) \[ \text{or } k + |\alpha| = \ell + |\beta| \text{ and } k = \ell \text{ and } \partial^\alpha \succ_3 \partial^\beta. \]
Let $f$ and $g$ be elements of $R^m$. We put
\[ \text{head}(f) = \text{head term of } f \text{ by the above order (4.1) } = c\partial^\alpha \xi^k \]
and head$(g) = d\partial^\beta \xi^\ell$. In the above situation, we put topIndex$(f) = k$ and lpp$(f) = \partial^\alpha$. We define reducibility, s-polynomial, reduction and Gröbner basis by using Definition 2.1, 2.2, 2.4 and 2.5. There exists a Gröbner basis for any submodule of $R^m$.
We state our improvement of Algorithm 3.1. The improved algorithm is based on solving systems of linear equations (see [Bach2] method 6.11) and is a modification of Algorithm 4.3 of [Tak2]. We refer to [Tak2] for the notations $k(I)$ and $e_Q$. It is not proved that Algorithm 4.1 stops. Therefore if Algorithm 4.1 fails, we must call Algorithm 3.1 that always stops.

**Algorithm 4.1** (Finding differential equations for an integral of a module)
**INPUT:** A left ideal $\mathfrak{U}$ of $A_n$ such that $A_n/\mathfrak{U}$ is holonomic.
**OUTPUT:** $G$, generators of a zero dimensional ideal of $K(x_1, \ldots, x_{n-1})(\partial_1, \ldots, \partial_{n-1})$ such that $G \subset \partial_n A_n + \mathfrak{U}$.

1. $G = \{g_1, \cdots, g_p\} := \text{a Gröbner basis of } \mathfrak{U} \text{ constructed in the ring } K(x_1, \cdots, x_{n-1})(x_n, \partial_1, \cdots, \partial_n)$ by the total degree order in the variables $x_n, \partial_1, \ldots, \partial_n$.
2. $m := \max_k \{\text{degree of } g_k \text{ with respect to } x_n\} + 1$
3. Do
4. $\mathcal{H}_m := (3.2)$
5. $G := \text{reduced Gröbner basis of } RH_m$ by the order (4.1)
6. for $k := 1 \to n - 1$
7. $d_k := \# (\mathbb{N}_0^{n-1} \setminus \bigcup_{f \in G, \text{topIndex}(f) = k, \text{lpp}(f) = \partial^\alpha} (\alpha + \mathbb{N}_0^{n-1}))$
8. endfor
9. $d := \sum_{k=1}^{n-1} d_k$
10. if $d < \infty$ then
11. Select a monomial $I \subseteq \mathbb{N}_0^{n-1}$ such that $k(I) \leq d$
12. $Q := \mathbb{N}_0^{n-1} \setminus I$
13. for all $\gamma \in G(I)$ do
14. Reduce $\partial^\gamma, k \in Q \cup \{\gamma\}$ by the Gröbner basis $G$ and obtain a similar equation of (4.4) of [Tak2] and solve it.
15. if $d_{\gamma} = 0$ then goto (11)
16. else obtain $e_{Q\cup\{\gamma\}} \xi^{\ell_0}$ of (4.5) or (4.6) of [Tak2]
17. endfor
18. endif
19. $m := m + 1$
20. while ($d = \infty$)
21. $G := \{e_{Q\cup\{\gamma\}} | \gamma \in G(I)\}$
Acknowledgement. The author thanks to Prof. Noumi for his encouragement of considering the problem of the paper.

References


[Zei] Zeilberger,D., A holonomic systems approach to special functions identities. to appear, Drexel Univ.
Appendix: An implementation

(March 2, 1990 )

The algorithms of the paper “An algorithm of constructing the integral of a module” is implemented by the language C. The implementation is the first software of the NMA(ama)thematical libraries for C, C++ or other object oriented languages. The NMA project aims at a compiler oriented computer algebra system that has high performance (NMA’s not MATHematica or REDUCE).

The program “lexgrob” computes the Gröbner basis of an input by the lexicos-total degree order (3.3). The program “modulegro rank m” computes the Gröbner basis in $R^m$ by the order (2.4). From an input, “modulegrob” obtains generators $H_m$ of (3.2) and computes the Gröbner basis of $H_m$.

The data structure of a polynomial that is used in the programs is described by the figure 5.1.

We show examples of computations.

**Example 5.1** Put

\[(5.2) \quad \ell_1 = \partial_x + t^2 \text{ and } \ell_2 = \partial_x + 2tx.\]

We have $\ell_1 f = \ell_2 f = 0$ where

\[f = e^{-xt^2}.\]

Let the input of “lexgrob” be (5.2). The figure 5.5 is a program readable form of (5.2) where $x_4 = t, \partial_4 = \partial_t, x_3 = x$ and $\partial_3 = \partial_x$. The order is

\[\{x_4\} \succ \{\partial_4, x_3, \partial_3, x_2, \partial_2, x_1, \partial_1, x_0, \partial_0\}.

The output of “lexgrob” is the figure 5.6. Let the input of “modulegrob -rank 3” is the figure 5.6. The output is the figure 5.7.

**Example 5.2** Put

\[(5.3) \quad \ell_1 = (1 + xt + t^p)\partial_x - \lambda t \text{ and } \ell_2 = (1 + xt + t^p)\partial_t - \lambda(x + pt^{p-1}).\]

We have $\ell_1 f = \ell_2 f = 0$ where

\[f = (1 + xt + t^p)^\lambda.\]

Put $p = 3$. Let the input of “lexgrob” be (5.3). The figure 5.8 is a program readable form of (5.3) where $x_4 = t, \partial_4 = \partial_t, x_3 = x, \partial_3 = \partial_x$ and $x_2 = \lambda$. We cannot obtain an output by four minutes. However, we can obtain an annihilator of the integral of $f$ by “modulegrob -rank 7” without preprocessing the input (5.3) by “lexgrob”. The output of “modulegrob -rank 7” is the figure 5.9. It is remarkable that “modulegrob” runs faster than “lexgrob | modulegrob” where | denotes a pipe. We remark that the output of “modulegrob -rank 5” does not contains an annihilator of the integral of $f$. There exists the variable $x_4$ in all expressions of the output.
**Example 5.3** (Gauss hypergeometric differential equation) Put

\[
\ell_1 = (1 - xy)y(1 - y)\partial_y - \alpha xy(1 - y) - (\beta - 1)(1 - xy)(1 - y) + (\gamma - \beta - 1)(1 - xy)y
\]
\[
\ell_2 = (1 - xy)\partial_x - \alpha y
\]

We have \( \ell_1 f = \ell_2 f = 0 \) where

\[
f = (1 - xy)^{-\alpha}y^{\beta-1}(1 - y)^{\gamma-\beta-1}.
\]

Let the input of "lexgrob" be (5.4). The figure 5.10 is a program readable form of (5.4) where \( x_4 = y, \partial_4 = \partial_y, x_3 = x, \partial_3 = \partial_x, x_2 = \gamma, x_1 = \beta \) and \( x_0 = \alpha \). The output is the figure 5.11. We can obtain Gauss hypergeometric equation using "modulegrob -rank 5" where the input is the figure 5.11. The output of "modulegrob" is the figure 5.12.

We can also obtain an annihilator of the integral of \( f \) that is hypergeometric operator by "modulegrob -rank 5" without preprocessing the input (5.4) by "lexgrob". The output of "modulegrob -rank 5" is the figure 5.13. It is remarkable that "modulegrob" runs faster than "lexgrob | modulegrob".

<table>
<thead>
<tr>
<th>Problem and algorithm</th>
<th>Input</th>
<th>Output</th>
<th>Time on SUN3 260C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 5.1, lexgrob</td>
<td>modulegrob -rank 3</td>
<td>Fig 5.5</td>
<td>Fig 5.7</td>
</tr>
<tr>
<td>Example 5.2, lexgrob</td>
<td>modulegrob -rank 5</td>
<td>Fig 5.8</td>
<td>Fig 5.10</td>
</tr>
<tr>
<td>Example 5.3, lexgrob</td>
<td>modulegrob -rank 5</td>
<td>Fig 5.10</td>
<td>Fig 5.12</td>
</tr>
<tr>
<td>Example 5.2, modulegrob -rank 5</td>
<td>Fig 5.8</td>
<td>Fig 5.8</td>
<td>Fig 5.9</td>
</tr>
<tr>
<td>Example 5.3, modulegrob -rank 5</td>
<td>Fig 5.10</td>
<td>Fig 5.13</td>
<td>11.4s</td>
</tr>
</tbody>
</table>

Timing data

\[
\text{Figure 5.1} \Rightarrow \text{Variable size} \Rightarrow \text{Polynomial} \Rightarrow \text{Monomials (fixed size)} \Rightarrow \text{Monomials are sorted by the lexicototal degree order (3.3)}.
\]
\[ d_4 + 2x_3x_4; \quad \leftarrow d_4 + 2x_3x_4 \]
\[ d_3 + x_4^2; \quad \leftarrow d_3 + x_4^2 \]

\[ \uparrow \text{Figure 5.5} \]
\[ +2 \times x_3 \times x_4 + 1 \times D_4 ; \]
\[ +1 \times x_4^2 + 1 \times D_3 ; \]
\[ +1 \times x_4 \times D_4 - 2 \times x_3 \times D_3 ; \]
\[ +4 \times x_3^2 \times D_3 + 1 \times D_4^2 + 2 \times x_3 \quad \leftarrow \text{cf. Example 1.1, 6} \]

\[ \uparrow \text{Figure 5.6} \]
\[ +2 \times x_3 \times x_4 + 1 \times D_4 ; \]
\[ +1 \times x_4^2 + 1 \times D_3 ; \]
\[ +1 \times D_4 ; \]
\[ -2 \times x_3 \times D_3 - 1 \quad \leftarrow \text{cf. Example 3.1} \]
\[ +1 \times x_4 \times D_4 + 1 ; \]
\[ 0; \]

\[ \uparrow \text{Figure 5.7} \]
\[ (1+3x+4+x_4^3) \times d_3-x_2 \times x_4; \]
\[ (1+3x+4+x_4^3) \times d_4-x_2 \times (x_3+3x \times x_4^2); \]
\[ 0; \]

\[ \uparrow \text{Figure 5.8} \]
\[ +1 \times x_4^3 \times D_3 + 1 \times x_3 \times x_4 \times D_3 - 1 \times x_2 \times x_4 + 1 \times D_3 ; \]
\[ -3 \times x_2 \times x_4^2 - 3 \times x_4^2 + 1 \times x_3 \times x_4 \times D_4 - 1 \times x_2 \times x_3 + 1 \times D_4 ; \]
\[ +1 \times D_4 ; \]
\[ +1 \times x_4 \times D_4 + 1 ; \]
\[ +1 \times x_4^2 \times D_4 + 2 \times x_4 ; \]
\[ +1 \times x_4^3 \times D_4 + 3 \times x_4^2 ; \]
\[ +3 \times x_2 \times x_4^3 + 4 \times x_4^3 - 1 \times x_3 \times x_4^2 \times D_4 + 1 \times x_2 \times x_3 \times x_4 \]
\[ -1 \times x_4 \times D_4 ; \]
\[ +3 \times x_2 \times x_4^4 + 5 \times x_4^4 - 1 \times x_3 \times x_4^3 \times D_4 + 1 \times x_2 \times x_3 \times x_4^2 \]
\[ -1 \times x_4^2 \times D_4 ; \]
\[ +3 \times x_2 \times x_4^5 + 6 \times x_4^5 - 1 \times x_3 \times x_4^4 \times D_4 + 1 \times x_2 \times x_3 \times x_4^3 \]
\[ -1 \times x_4^3 \times D_4 ; \]
\[ +1 \times x_4^4 \times D_3 + 1 \times x_3 \times x_4^2 \times D_3 - 1 \times x_2 \times x_4^2 + 1 \times x_4 \times D_3 ; \]
\[ +1 \times x_4^5 \times D_3 + 1 \times x_3 \times x_4^3 \times D_3 - 1 \times x_2 \times x_4^3 + 1 \times x_4^2 \times D_3 ; \]

\text{to be continued.}
\begin{align*}
+1 \times x^4 \times 6 \times D_3 &+1 \times x_3 \times x^4 \times 4 \times D_3 -1 \times x_2 \times x^4 \times 4 +1 \times x^4 \times 3 \times D_3 \\
-1 \times x_4 \times D_4 &-3 \times x_2 \times x^4 \times 3 +1 \times x_3 \times x^4 \times 2 \times D_4 -1 \times x_2 \times x_3 * \\
+x_4 +1 \times x^4 \times D_4 &; \\
+1 \times x^4 \times 5 \times D_4 &-3 \times x_2 \times x^4 \times 4 +1 \times x_3 \times x^4 \times 3 \times D_4 -1 \times x_2 \times x_3 * \\
x^4 +1 \times D_4 &+1 \times x^4 \times 6 \times D_4 -3 \times x_2 \times x^4 \times 5 +1 \times x_3 \times x^4 \times 4 \times D_4 -1 \times x_2 \times x_3 * \\
x^4 +1 \times x^4 \times 3 \times D_4 &; \\
-27 \times x_2 \times x_4 &-45 \times x_2 \times x_4 -18 \times x_4 +4 \times x_2 * x_3 \mathcal{3} \times D_3 +4 \\
\times x^3 \mathcal{3} \times D_3 &+2 \times x_3 \mathcal{3} \times D_3 \times D_4 -6 \times x_2 \times x_3 \mathcal{3} \times 2 -8 \times x_2 * x_3 \mathcal{3} \times 2 + \\
6 \times x^2 \times x_3 \times D_4 &-2 \times x_3 \times 2 +6 \times x_3 \times D_4 +27 \times x_2 \times D_3 +27 \times D_3 ; \\
+9 \times x_2 \times x_4 &* D_3 +9 \times x_4 \times D_3 -2 \times x_2 \times x_3 \mathcal{3} \times 2 \times D_3 -2 \times x_3 \mathcal{3} \times 3 \times D_3 -3 \times x_2 \times 3 \times x_3 \times D_4 +1 \times x_3 -3 \times D_4 ; \\
+4 \times x_2 \times x_3 \mathcal{3} \times D_3 \times D_3 \times D_3 -1 &+2 \times x_3 \mathcal{3} \times 3 \times D_3 \times D_4 \times D_4 -1 \\
2 \times x_2 \mathcal{3} \times x_3 \mathcal{3} \times D_3 -6 \times x_2 \times x_3 \mathcal{3} \times 2 \times D_3 +3 \times x_2 * x_3 \mathcal{3} \times D_3 \times D_4 +9 \\
\times x_2 \times x_3 \mathcal{3} \times x_6 &+6 \times x_3 \mathcal{3} \times D_3 \times D_3 \times D_4 +6 \times x_2 \times x_3 \mathcal{3} \times 3 +27 \times \\
\times x_2 \times D_3 \times 2 &-9 \times x_2 \times x_4 \times D_4 -5 \times x_2 \times x_3 +27 \times D_3 \times D_3 \times 2 -9 \times x_2 \times D_4 - \\
2 &\times x_3 ; \\
0 ;
\end{align*}

\text{This is an annihilation of } \int f \, dt.

\begin{align*}
-(1-x^3 \times x_4) \times x_4 \times (1-x^4) \times d_4 + x_0 \times x_3 \times x_4 \times (1-x^4) + (x_1-1) \times (1-x^3 \times x_4) \times (1-x^4) - (x_2-x_1-1) \times (1-x_3 \times x_4) \times x_4 ; \\
-(1-x^3 \times x_4) \times d_3 + x_0 \times x_4 ; \\
0 ;
\end{align*}

\text{to be continued.}

\text{Figure 5.9}

-1 \times x_3 \times x_4 \times 3 \times D_4 +1 \times x_2 \times x_3 \times x_4 \times 2 -1 \times x_0 \times x_3 \times x_4 \times 2 +1 \times \\
x_3 \times x_4 \times 2 \times D_4 -2 \times x_3 \times x_4 \times 2 +1 \times x_4 \times 2 \times D_4 +1 \times x_1 \times x_3 \times x_4 +1 \\
\times x_0 \times x_3 \times x_4 +1 \times x_3 \times x_4 -1 \times x_2 \times x_4 -1 \times x_4 \times D_4 +2 \times x_4 +1 \\
\times x_1 -1 ; \\
+1 \times x_3 \times x_4 \times D_3 +1 \times x_0 \times x_4 -1 \times D_3 ; \\
-1 \times x_0 \times x_4 \times 2 \times D_4 +1 \times x_4 \times 2 \times D_4 +1 \times x_1 \times x_3 \times x_4 \times D_3 -1 \times \\
x_0 \times x_3 \times x_4 \times D_3 +1 \times x_0 \times x_3 \times x_4 \times D_3 +1 \times x_0 \times x_1 \times x_3 \times x_4 -1 \\
\times x_0 \times 2 \times x_3 \times x_4 -1 \times x_3 \times x_4 \times D_3 +1 \times x_0 \times x_2 \times x_4 +1 \times x_0 \times x_4 \times D_4 -1 \times x_2 \times x_4 -1 \times x_4 \times D_4 +2 \times x_4 -1 \times x_1 \times x_3 \times D_3 +1 \times x_1 \times x_0 -1 ; \\
-1 \times x_0 \times x_4 \times D_3 \times D_4 +1 \times x_3 \times x_4 \times D_3 +1 \times x_4 \times D_3 \times D_4 +1 \times x_0 \times x_4 \times D_4 -1 \times x_0 \times x_4 \times D_4 +1 \times x_0 \times x_3 \times D_3 \times 2 +1 \times x_0 \times x_3 \times x_0 \times x_3 \times x_3 \times D_3 \times 2 -1 \times x_0 \times x_1 \times x_3 \times D_3 +1 \times x_0 \times x_0 \times x_2 \times x_3 \times D_3 -1 \times x_3 \times D_3 \times 2 +1 \times x_1 \times x_3 \times D_3 +1 \times x_0 \times x_2 \times x_3 \times D_3 -1 \times x_0 \times x_0 \times x_1 \times x_3 \times D_3 -1 \times x_2 \times x_3 \times D_3 -1 \times x_0 \times D_3 -1 \times D_3 \times D_3 -1 \times x_0 \times D_3 +1 \times x_0 \times x_1 \times x_3 \times D_3 -1 \times x_0 \times D_3 +1 \times x_0 \times x_1 ; \\
-x_3 \times D_3 +1 \times x_0 \times x_2 \times D_3 +1 \times x_0 \times D_3 \times D_4 -1 \times x_0 \times x_2 \times x_1 +1 \times x_3 \times D_3 -1 \times x_2 \times x_3 \times D_3 -1 \times D_3 \times D_4 +1 \times \text{Figure 5.10}
This is an annihilator of $\int f dy$, but $F$ is a third order operator.

This is hypergeometric differential operator (11) by $D_4 \to 0$. 

---

\textbf{Figure 5.11}

\textbf{Figure 5.12}
+1 *x2 *x3 *x4^2 -1 *x0 *x3 *x4^2 +1 *x3 *x4^2 *D4 +1 *

x3 *x4^2 +1 *x4^2 *D4 -1 *x1 *x3 *x4 +1 *x0 *x3 *x4 +1 *

x3 *x4 -1 *x2 *x4 *D4 -1 *x0 *x4 *D4 +2 *x4 +1 *x1 -1 ;
+1 *x3 *x4 *D3 +1 *x0 *x4 -1 *D3 ;
+1 *D4 ;
+1 *x4 *D4 +1 ;
+1 *x4^2 *D4 +2 *x4 ;
+1 *x4^3 *D4 +3 *x4^2 ;
+1 *x4^4 *D4 +4 *x4^3 ;
+1 *x2 *x3 *x4^3 -1 *x0 *x3 *x4^3 +1 *x3 *x4^3 *D4 +2 *

x3 *x4^3 +1 *x4^3 *D4 -1 *x1 *x3 *x4^2 +1 *x0 *x3 *x4^2
+1 *x3 *x4^2 -1 *x2 *x4^2 -1 *x4^2 -1 *x4^2 +1 *x1 *

x4 -1 *x4 ;
+1 *x3 *x4^2 *D3 +1 *x0 *x4^2 -1 *x4 *D3 ;
+1 *x3 *x4^3 *D3 +1 *x0 *x4^3 -1 *x4^2 *D3 ;
+1 *x3 *x4^4 *D3 +1 *x0 *x4^4 -1 *x4^3 *D3 ;
-1 *x0 *x2 *x4^2 +1 *x0^2 *x4^2 -1 *x0 *x4^2 *D4 +1 *x2

x4^2 -2 *x0 *x4^2 +1 *x4^2 *D4 +1 *x4^2 -1 *x1 *x3 *x4

x4 *D3 +1 *x0 *x3 *x4 *D3 -1 *x3 *x4 *D3 -1 *x0 *x4 *D4 +1 *x1 *x3 *x4

x4 *D3 +1 *x1 *x4 -1 *x0 *x4 +1 *x4 +1 *x1 *D3 ;
-1 *x0 *x2 *x4^3 +1 *x0^2 *x4^3 -1 *x0 *x4^3 *D4 +1 *x2

x4 *D3 +1 *x0 *x3 *x4^2 *D3 -2 *x3 *x4^2 *D3 -1 *x0 *x4^2

x4 *D3 +1 *x1 *x4^2 *D3 -1 *x1 *x4^2 -2 *x0 *x4^2 +1 *x4^2 +1

x1 *x4 *D3 +1 *x4 *D3 ;
+1 *x0 *x2 *x4 -1 *x0^2 *x4 +1 *x0 *x4 *D4 -1 *x2 *x4
+1 *x0 *x4 -1 *x4 *D4 -1 *x0 *x3 *D3 +1 *x3 *D3 +1 *x0 *

D3 -1 *x0 *x1 -1 *D3 +1 *x1 +1 *x0 -1 ;
+1 *x0 *x2 *D3^2 -1 *x3^2 *D3^2 -1 *x0 *x3 *D3^2 +1 *
x0 *x1 *x3 *D3 +1 *x0^2 *x3 *D3 +1 *x3 *D3^2 -1 *x1 *x3 *

D3 -1 *x0 *x2 *D3 -1 *x0 *D3 *D4 +1 *x0^2 *x1 -1 *x3 *D3
+1 *x2 *D3 +1 *D3 *D4 -1 *x0 *x1 ;

0;

\[ (-\chi_0) \left[ x_3 (1 - x_3) \partial_3^2 + \left\{ x_2 - (1 + x_0 + x_1) x_3 \right\} \partial_3 - x_0 x_1 \right] \]