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 involving an integro-differential operator

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On the existence of viscosity solutions to nonlinear problems involving an integro-differential operator

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1. Introduction

This is a part of the joint work [11] with Suzanne M. Lenhart at University of Tennessee, Knoxville.

In this note we consider the existence of viscosity solutions for an obstacle problem involving an integro-differential operator associated with piecewise-deterministic processes.

Let

$$Lu(x) = -g(x) \cdot \nabla u(x) + \alpha(x)u(x) - \lambda(x) \int_{\Omega} (u(y) - u(x)) Q(dy, x),$$

where \( \cdot \) is the inner product in \( \mathbb{R}^n \), \( \nabla u \) is the gradient vector of \( u \) and \( Q(\cdot, z) \) is a probability measure.

We consider the following obstacle problem:

$$(1.1) \quad \min\{Lu - f, u - \psi \} = 0 \quad \text{in } \Omega,$$

with the boundary condition

$$(1.2) \quad u(x) = \int_{\Omega} u(z)Q(dz, x) \quad \text{on } \partial \Omega.$$

The operator \( L \) arises as a generalized infinitesimal generator of a piecewise-deterministic (PD in short) process. These PD processes have deterministic dynamics \( g \) between
random jumps. The jump distribution is represented by transition probability measure $Q(\cdot, x)$. See Davis [4] for the detail of PD processes.

In the case that $L$ is an infinitesimal generator of a diffusion process, it is well known that the unilateral obstacle problem (1.1) with the Dirichlet boundary condition arises as a dynamic programming equation associated with an appropriate optimal control problem (see Bensoussan and Lions [1]).

The equation (1.1) is also the dynamic programming equation associated with an optimal control problem in which the underlying process is a PD process.

In the case that the domain $\Omega$ is a bounded domain in $\mathbb{R}^n$, the PD process jumps back into the interior upon hitting the boundary which leads to the boundary condition (1.2) (see Davis [4]).

The obstacle problem (1.1), (1.2) is first treated by Lenhart and Liao [9], [10] by using singular perturbation method. After introduction of the notion of viscosity solution by Crandall and Lions [2], Lenhart [8] has proved the existence and uniqueness of viscosity solution for a system of obstacle problems.

In these articles, it is commonly assumed that

$$\alpha(x) \geq \alpha_0 > 0 \text{ for sufficiently large } \alpha_0.$$

The purpose of this note is to eliminate the condition of largeness for the zero-th order term by using Perron's method which is introduced by Ishii [6].

In section 2, we state the notion of viscosity solutions and assumptions. We also give a brief review of Perron's method. In section 3, we shall explain how to apply the Perron's method to get a viscosity solution of (1.1) satisfying the boundary condition (1.2). To show the existence of super- and subsolution, which are needed to apply Perron's method, we consider also a linear first order PDE with the boundary condition (1.2). Our main result is Theorem 3.3.
2. Assumptions and Perron's method

Let

\[(2.1) \quad Lu(x) = -g(x) \cdot \nabla u(x) + \alpha(x) u(x) - \lambda(x) \int_{\Omega} (u(y) - u(x)) Q(dy, x),\]

where \(\cdot\) is the usual inner product in \(\mathbb{R}^n\), \(\nabla u\) is the gradient vector of \(u\) and \(Q(\cdot, x)\) is a probability measure.

We consider the following obstacle problem.

\[(2.2) \quad \min\{Lu - f, u - \psi\} = 0 \quad \text{in} \ \Omega,\]
\[(2.3) \quad u(x) = \int_{\Omega} u(y) Q(dy, x) \quad \text{on} \ \partial\Omega\]

We assume the following conditions.

(H.1) \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\partial\Omega\).
(H.2) \(g(x) : \Omega \rightarrow \mathbb{R}^n\) is Lipschitz continuous, \(\alpha(x), \lambda(x) : \overline{\Omega} \rightarrow \mathbb{R}\) are continuous.
(H.3) There exists \(\alpha_0 > 0\) such that \(\alpha(x) \geq \alpha_0\) for \(x \in \overline{\Omega}\).
(H.4) \(\lambda(x) > 0\) for \(x \in \Omega\).
(H.5) \(Q(\cdot, x)\) satisfies:
   (i) \(Q(\cdot, x)\) is a probability measure on \(\Omega\) for \(x \in \overline{\Omega}\) such that
   \[\left| \int_{\Omega} v(y) Q(dy, x) \right| \leq C ||v||_{L^1(\Omega)} \quad \text{for all} \ v \in L^1(\Omega).\]
   (ii) The function
   \[x \rightarrow \int_{\Omega} v(y) Q(dy, x),\]
   is continuous with respect to \(x \in \overline{\Omega}\), uniformly on \(v \in L^\infty(\Omega)\).
(H.6) \(g(x) \cdot \eta(x) > 0\) for \(x \in \partial\Omega\), where \(\eta(x)\) is the outward unit normal at \(x \in \partial\Omega\).
(H.7) \(f, \psi\) are continuous on \(\overline{\Omega}\).

We denote that

\[F(x, u, p, r) = \min\{-g(x) \cdot p + (\alpha(x) + \lambda(x))u - \lambda(x)r - f(x), u - \psi(x)\}.\]
for $x \in \Omega, u \in R, p \in R^n, r \in R$. Notice that if we fix $v \in L^\infty(\Omega)$, then the equation

$$F \left(x, u(x), \nabla u(x), \int_{\Omega} v(y)Q(dy, x)\right) = 0 \quad \text{in } \Omega$$

is an obstacle problem with a first order Hamiltonian.

We give some notation necessary to state the definition of viscosity solution. For bounded functions, we set

$$u^*(x) = \lim \sup_{r \to 0} \{u(y) | |x - y| < r\} \quad \text{upper semi-continuous envelope of } u$$

and

$$u_*(x) = \lim \inf_{r \to 0} \{u(y) | |x - y| < r\} \quad \text{lower semi-continuous envelope of } u.$$ 

Now we state the definition of viscosity solutions.

**Definition.** Let $u$ be a bounded measurable function.

(i) $u$ is a viscosity subsolution of (2.2) if

$$F \left(x, u^*(x), \nabla \phi(x), \int_{\Omega} u^*(y)Q(dy, x)\right) \leq 0$$

wherever $u^* - \phi$ attains its maximum for $\phi \in C^1(\Omega)$.

(ii) $u$ is a viscosity supersolution of (2.2) if

$$F \left(x, u_*(x), \nabla \phi(x), \int_{\Omega} u_*(y)Q(dy, x)\right) \geq 0$$

wherever $u_* - \phi$ attains its minimum for $\phi \in C^1(\Omega)$.

(iii) $u$ is a viscosity solution if $u$ is a viscosity sub- and supersolution.

In the following, "(sub/super) solution" means "viscosity (sub/super) solution".

Assume that there exists a supersolution $W$ of (2.2) such that

$$W(x) \geq \int_{\Omega} W(y)Q(dy, x) \quad \text{on } \partial\Omega.$$
Define

\[ S = \{ v \mid v \text{ is a subsolution of (2.2) such that} \] \[
    v \leq W \text{ in } \Omega \text{ and} \]
\[
    v(x) \leq \int_{\Omega} v(y) Q(dy, x) \text{ on } \partial \Omega \}. \]

We put

\[ u_0(x) = \sup\{v(x) \mid v \in S \}. \]

Perron's method consists of the following two propositions:

**Proposition 2.1.** Assume that \( S \) is not empty, then \( u_0 \in S \).

**Proposition 2.2.** Assume \( S \neq \emptyset \). If \( v \in S \) is not a supersolution, then there exists \( w \in S \) such that \( v(y) < w(y) \) at some \( y \in \Omega \).

These two Propositions can be proved by the same idea of Ishii [6]. So we omit the proofs. See [11] for the detail.

Note that \( u_0 \) is a viscosity solution of (2.2).

3. **Main existence result**

First we assume that there exists a supersolution \( W \) of (2.2) satisfying (2.4). By Perron's method, there exists a solution \( u_0 \). Note that \( u_0 \) satisfies the boundary inequality

\[ u_0(x) \leq \int_{\Omega} u_0(y) Q(dy, x) \text{ on } \partial \Omega. \]

**Theorem 3.1.** Assume (H.1)–(H.7). Suppose that there exists a supersolution \( W \) of (2.2) satisfying (2.4), and a solution \( u_1 \) of

\[ F \left( x, u_1, \nabla u_1, \int_{\Omega} u_0(y) Q(dy, x) \right) = 0 \text{ in } \Omega \]
satisfying the Dirichlet boundary condition

(3.2) \[ u_1(x) = \int_{\Omega} u_0(y) Q(dy, x) \quad \text{on } \partial \Omega. \]

If \( u_1 \leq W \), then \( u_0 \) is a solution of (2.2) satisfying the boundary condition (2.3).

**Proof.** We claim \( u_1 \in S \). Let \( \phi \in C^1 \) such that \( u_1^* - \phi \) attains its maximum at \( y_0 \), then

\[ F\left(y_0, u_1^*(y_0), \nabla \phi(y_0), \int_{\Omega} u_0(y) Q(dy, y_0)\right) \leq 0. \]

Note that the comparison principle for two viscosity solutions holds for the equation of a first order Hamiltonian \( F(x, u, \nabla u, u_0) \). Since \( u_0 \) is also a subsolution of (3.1), we have \( u_0 \leq u_1 \) in \( \Omega \). Using \( u_0 \leq u_1 \) and the monotonicity of \( F \) with respect to the argument \( u \), we have

\[ F\left(y_0, u_1^*(y_0), \nabla \phi(y_0), \int_{\Omega} u_1(y) Q(dy, y_0)\right) \leq 0. \]

Also we have

\[ u_1(x) = \int_{\Omega} u_0(y) Q(dy, x) \leq \int_{\Omega} u_1(y) Q(dy, x) \quad \text{on } \partial \Omega. \]

Hence, we have the claim. By the definition of \( u_0 \) and \( u_0 \leq u_1 \), we have \( u_0 \equiv u_1 \) in \( \overline{\Omega} \).

This completes the proof.

To assure the assumptions of Theorem 3.1, we consider the equation

(3.3) \[ Lu(x) = f(x) \quad \text{in } \Omega \]

(3.4) \[ u(x) = \int_{\Omega} u(y) Q(dy, x) \quad \text{on } \partial \Omega. \]

**Theorem 3.2.** Assume (H.1)–(H.7). Then there exists a unique solution of the equation (3.3) satisfying the boundary condition (3.4).
Proof. First we note that
\[ w(x) = -\frac{\|f\|_{\infty}}{\alpha_0} \] is a subsolution,
and
\[ W(x) = \frac{\|f\|_{\infty}}{\alpha_0} \] is a supersolution.

of (3.3) satisfying (3.4).

Applying Perron's method, we have that there exists a solution \( u_0 \) of (3.3) satisfying the boundary inequality
\[ u_0(x) \leq \int_{\Omega} u_0(y)Q(dy, x) \quad \text{on } \partial\Omega. \]

Next we consider the equation
\[ -g \cdot \nabla u_1 + (\alpha + \lambda)u_1 - \lambda \int_{\Omega} u_0(y)Q(dy, x) = f \quad \text{in } \Omega \tag{3.5} \]
with the Dirichlet boundary condition
\[ u_1(x) = \int_{\Omega} u_0(y)Q(dy, x) \quad \text{on } \partial\Omega. \tag{3.6} \]

The comparison principle for this equation is well known [2,3]. By (H.6) and the method of [12], we can prove the existence of sub- and supersolutions. Then there exists a continuous solution \( u_1 \) of the equation (3.5) with (3.6). We can apply the same argument in the proof of Theorem 3.1 to yield that \( u_1 \equiv u_0 \). The uniqueness follows from Lenhart [8]. The proof is complete.

Now we can prove the main result.

**Theorem 3.3.** Assume (H.1)–(H.7). Then there exists a unique solution of the obstacle problem (2.2) satisfying the boundary condition (2.3).

Proof. It is sufficient to check the hypothesis of Theorem 3.1. To do so, we consider the obstacle problem (3.1) with (3.2).
Using the boundary inequality of $u_0$ and $u_0 \geq \psi$ in $\Omega$, the compatibility condition

$$\psi(x) \leq \int_\Omega u_0(y)Q(dy, x) \quad \text{on } \partial\Omega$$

is satisfied.

First assume

(3.7) \hspace{1cm} h(x) = \int_\Omega u_0(y)Q(dy, x) \in C^1(\Omega) \cap C(\overline{\Omega})

and

(3.8) \hspace{1cm} h(x) = \int_\Omega u_0(y)Q(dy, x) > \psi(x) \quad \text{on } \partial\Omega

In this case, problem (3.1) with (3.2) is equivalent to

(3.9) \hspace{1cm} \min\{-g \cdot \nabla w_1 + (\alpha + \lambda)w_1 - f, w_1 - \psi\} = 0 \quad \text{in } \Omega

(3.10) \hspace{1cm} w_1(x) = 0 \quad \text{on } \partial\Omega

where $f, \psi$ satisfy the same properties as $f, \psi$ in (3.1) and $\psi < 0$ on $\partial\Omega$. We show the existence of a solution to (3.9) with (3.10) by Perron’s method. Indeed, the solution of the linear equation

$$-g \cdot \nabla w + (\alpha + \lambda)w = f \quad \text{in } \Omega,$$

$$w = 0 \quad \text{on } \partial\Omega$$

is a subsolution of (3.9) with (3.10).

To construct a supersolution, we follow a barrier construction argument from Oleinik and Radkevic [12] as in Ishii and Koike [7]. Since $\psi < 0$ on $\partial\Omega$, there exists a local barrier, $\psi_z$ in $C(\Omega \cap V_z) \cap C^2(\Omega \cap V_z)$ where $z \in \partial\Omega$, $V_z$ is a sufficiently small neighborhood of $z$ satisfying

$$\psi_z(z) = 0, \quad \psi_z \geq 0 \quad \text{on } \overline{\Omega \cap V_z},$$

$$\psi_z \geq ||f||_\infty/\alpha_0 \quad \text{on } \overline{\Omega \cap \partial V_z},$$

$$-g \cdot \nabla \psi_z + (\alpha + \lambda)\psi_z \geq f \quad \text{in } \Omega \cap V_z, \quad \text{and}$$

$$\psi_z \geq \psi \quad \text{in } \Omega \cap V_z.$$
Define
\[ \hat{\psi}_z(z) = \begin{cases} 
\max\{\psi_z(x), \max\{||f||_\infty/\alpha_0, ||\psi||_\infty\}\} & \text{in } \Omega \cap V_z, \\
\max\{||f||_\infty/\alpha_0, ||\psi||_\infty\} & \text{otherwise},
\end{cases} \]
and
\[ \hat{\psi}(x) = \inf\{\hat{\psi}_z(x) | z \in \partial\Omega\}. \]

Then \( \hat{\psi} \) is a supersolution. This implies that there exists a continuous solution of (3.1) with (3.2).

For general continuous boundary value \( h \), which is not necessarily satisfy (3.7) and (3.8), we choose an approximating sequence \( \{h_n\} \) such that \( h_n \in C(\Omega) \cap C^1(\Omega) \), \( h_n > \psi \) on \( \partial\Omega \) and \( h_n \rightarrow h \) uniformly in \( \overline{\Omega} \). Let \( u_n \) be a solution of (3.1) with (3.2) associated with boundary value \( h_n \). By standard comparison argument, we have
\[ \sup_{\Omega} |u_n(x) - u_m(x)| \leq \sup_{\partial\Omega} |h_n(x) - h_m(x)|. \]
Hence \( \{u_n\} \) converges to some \( u \in C(\overline{\Omega}) \) and by stability of viscosity solutions, we have that \( u \) is a solution of (3.1) with (3.2).

By the comparison result for obstacle problems, we have \( u_1 \leq W \). Hence by Theorem 3.1, \( u_0 \) satisfies the boundary condition (3.2).

Since the uniqueness follows from the argument in Lenhart [10], the proof is completed.

References